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# Generalization of some properties of relations in the context of functional temporal×modal logic (SI-CMMSE-2006)

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In this paper, we generalize the definitions of transitivity, reflexivity, symmetry, euclidean and serial properties of relations in the context of a functional approach for temporal×modal logic. The main result is the proof of definability of these definitions which is obtained by using algebraic characterizations. As a consequence, we will have in our temporal×modal context the generalizations of modal logics $T$, $S4$, $S5$, $KD45$, etc. These new logics will allow us to establish connections among time-flows in very different ways, which enables us to carry out different relations among asynchronous systems. Our further research is focused on the construction of logics with these properties and the design of theorem provers for these logics.

*Keywords*: temporal logic, modal logic, logic in computer science, definability

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## 1. Introduction

Modal and temporal logics have been applied to interactive and multi-agent systems [9, 17] to describe the agent mental state and behaviour [13] as a suitable tool to treat parallel processes and distributed systems [14]. However, they are not capable of reasoning out the internal structure of these systems [6,10, 12]. Several extensions of propositional temporal logic have been given in order to solve this disadvantage, for example, by assuming that all the agents are synchronized [7,12,15], nevertheless, this is a very strong restriction. A combination of modal and temporal logics [2,8,11,16], could be the key to achieve a more comprehensive way to describe interactive and multi-agent systems.

In this paper, we extend the combination of modal and temporal logics presented in [3], where a new kind of frame, called *functional frame*, was introduced to manage linear time flows connected by *accessibility functions* and the definability of basic properties of functions (such as injective, surjective, increasing, decreasing, etc) was studied. Functional frames establish connections among time-flows in very different ways, which enables us to carry out different relations among not necessarily synchronized systems. This is a more general approach than others well known in the literature because the usual way about temporal×modal logics is to use equivalence relations of accessibility, for example, the Kamp-models in [16] and the reasoning about knowledge and time in asynchronous systems, in [8].

Our goal in this work is to enrich the temporal×modal logic presented in [3] by giving definitions and studying the definability of new properties. The new relations that we study, are generalizations of well known properties of relations in Kripke models [1]: transitivity, reflexivity, symmetry, euclideanity and seriality. Our framework has lead us in a natural way to give two versions for each property, which enrich the versatility of our approach. The main result of this work is the proof of the definability of these properties by using algebraic characterizations. This allows us to have in our temporal×modal context the generalizations of modal logics $T$, $S4$, $S5$, $KD45$, etc. From a practical point of view, this approach enables us a great versatility in order to design multiagent systems where time flows play a role [8] or to deal with different possibilities of communication among interactive systems which may be asynchronous [5].

The paper is organized as follows: In Section 2, the language and semantics of our temporal×modal logic is introduced. In Section 3 the different definitions, some examples and the definability of the new

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properties are studied. Finally, some conclusions and prospects of future work are presented in Section 4.

## 2. The logic $\mathcal{L}^{\mathcal{F}}_{T\times W}$

In this section we give the definitions of the language $L_{T\times W}$ and its functional semantics. The alphabet of $L_{T\times W}$ is defined as follows:

- a denumerable set, $\mathcal{V}$, of propositional variables;
- the constants $\top$ and $\perp$, and the classical connectives $\neg$ and $\rightarrow$;
- the temporal connectives $G$ and $H$, and the modal connective $\Box$.

The well-formed formulae (wffs) are generated by the construction rules of classical propositional logic adding the following rule: If $A$ is a wff, then $GA$, $HA$ and $\Box A$ are wffs.

We consider, as usual, the connectives $\wedge$, $\vee$ $F$, $P$ and $\Diamond$ to be defined connectives. The connectives $G$, $H$, $F$, and $P$ have their usual readings, but $\Box$ has the following meaning: $\Box A$ is read "$A$ is true at every accessible present". On the other hand, the notion of a *mirror image* of a formula is considered in the usual way.

**Definition 2.1** We define a **functional frame** for $L_{T\times W}$ as a tuple $(W, \mathcal{T}, \mathcal{F})$, where:

(i) $W$ is a non-empty set (set of labels for a set of temporal flows).
(ii) $\mathcal{T}$ is a non-empty set of strict linear orders, indexed by $W$. Specifically:

$$\mathcal{T} = \{(T_w, <_w) \mid w \in W\} \text{ such that } T_w \neq \varnothing \text{ for all } w \in W, \text{and } \text{ if } w \neq w', \text{ then } T_w \cap T_{w'} = \varnothing.$$

(iii) $\mathcal{F}$ is a set of non-empty **accessibility functions**, such that:
- every accessibility function is a partial function from $T_w$ to $T_{w'}$, for some pair $w, w' \in W$.
- for an arbitrary pair $w, w' \in W$, there is (in $\mathcal{F}$) at most one accessibility function from $T_w$ to $T_{w'}$, denoted by $\overset{w\ w'}{\longrightarrow}$.

We will denote $\mathcal{F}_w = \{\overset{w\ w'}{\longrightarrow} \in \mathcal{F} \mid w' \in W\}$ and $\mathcal{F} = \bigcup\limits_{w\in W} \mathcal{F}_w$.

Let $\Sigma = (W, \mathcal{T}, \mathcal{F})$ be a functional frame, the elements $t_w$ of the disjoint union $\mathcal{C}oord_\Sigma = \bigoplus_{w\in W} T_w$ are called **coordinates**.

We now introduce the following notation, which is very useful in the rest of the paper.
**Notation.** Let $\Sigma = (W, \mathcal{T}, \mathcal{F})$ be a functional frame:

(i) We will denote $R_{\mathcal{F}}$ the relation in $W$ defined by $w\, R_{\mathcal{F}}\, w'$ iff there exists $\overset{w\ w'}{\longrightarrow} \in \mathcal{F}$. In this case, we will say that $\mathcal{F}$ *induces* the relation $R_{\mathcal{F}}$.
(ii) If $X \subseteq \mathcal{C}oord_\Sigma$, we will write $\mathcal{F}(X) = \{t_{w'} \mid t_{w'} = \overset{w\ w'}{\longrightarrow} (t_w), \text{ for some } t_w \in X \text{ and } \overset{w\ w'}{\longrightarrow} \in \mathcal{F}\}$ and similarly for $\mathcal{F}_w(X)$.
(iii) If $w \in W$, we will denote by $\mathcal{C}od(\mathcal{F}_w)$ [1] the union of temporal flows reachable from $T_w$, that is:

$$\mathcal{C}od(\mathcal{F}_w) = \bigcup_{\overset{w\ w'}{\longrightarrow} \in \mathcal{F}_w} T_{w'}$$

(iv) Similarly to the previous item, $\mathcal{C}od(\mathcal{F}_{t_w})$ is the union of temporal flows reachable from a coordinate $t_w$, that is, $\mathcal{C}od(\mathcal{F}_{t_w}) = \bigcup\limits_{w'\in W'} T_{w'}$; being $W' = \{w' \in W \mid t_w \in Dom(\overset{w\ w'}{\longrightarrow})\}$ and $Dom(\overset{w\ w'}{\longrightarrow})$ the domain of accessibility function $\overset{w\ w'}{\longrightarrow}$.

**Definition 2.2** A **functional model** on $\Sigma$ is a tuple $\mathcal{M} = (\Sigma, h)$, where $\Sigma$ is a functional frame and $h : L_{T\times W} \longrightarrow 2^{\mathcal{C}oord_\Sigma}$ is a function, called a **functional interpretation**, satisfying:

[1] The notation $\mathcal{C}od$ comes from *codomain*.

$h(\bot) = \varnothing; h(\top) = \mathcal{C}oord_\Sigma;\ h(\neg A) = \mathcal{C}oord_\Sigma - h(A);$
$h(A \to B) = (\mathcal{C}oord_\Sigma - h(A)) \cup h(B)$
$h(GA) = \{t_w \in \mathcal{C}oord_\Sigma \mid (t_w, +\infty) \subseteq h(A)\}$
$h(HA) = \{t_w \in \mathcal{C}oord_\Sigma \mid (-\infty, t_w) \subseteq h(A)\}$
$h(\Box A) = \{t_w \in \mathcal{C}oord_\Sigma \mid \mathcal{F}(\{t_w\}) \subseteq h(A)\}.$

*Remark 1* If $A$ is a formula, we denote $\zeta A \equiv HA \wedge A \wedge GA$ and $\tau A \equiv PA \vee A \vee FA$. From the semantics of our connectives and the notation introduced above, for every model $(\Sigma, h)$, we have:

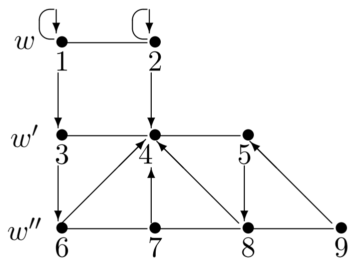
(i) $h(\zeta A) = \{t_w \in \mathcal{C}oord_\Sigma \mid T_w \subseteq h(A)\}$
(ii) $h(\tau A) = \{t_w \in \mathcal{C}oord_\Sigma \mid T_w \cap h(A) \neq \varnothing\}$
(iii) $\mathcal{C}od(\mathcal{F}_w) \subseteq h(A)$ iff $\mathcal{F}(T_w) \subseteq h(\zeta A)$
(iv) For every coordinate $t_w$, we have $\mathcal{C}od(\mathcal{F}_{t_w}) \subseteq h(A)$ iff $\mathcal{F}(\{t_w\}) \subseteq h(\zeta A)$
(v) For every $X \subseteq \mathcal{C}oord_\Sigma$, we have $X \subseteq h(\Box A)$ iff $\mathcal{F}(X) \subseteq h(A)$

This algebraic approach for the semantics will be very useful to facilitate the proof of definability for the desired properties in section 3.

**Definition 2.3** Let $A$ be a formula in $L_{T\times W}$. Then, $A$ is **true at** $t_w$ if $t_w \in h(A)$. $A$ is said to be **valid in the functional model** $(\Sigma, h)$ if $h(A) = \mathcal{C}oord_\Sigma$. If $A$ is valid in every functional model on $\Sigma$, then $A$ is said to be **valid in the functional frame** $\Sigma$, and denote it by $\models_\Sigma A$. If $A$ is valid in every functional frame, then $A$ is said to be **valid**, and denote it by $\models A$. Let $\mathbb{K}$ be a class of functional frames, if $A$ is valid in every functional frame $\Sigma$ such that $\Sigma \in \mathbb{K}$, then $A$ is said to be **valid with respect to** $\mathbb{K}$.

The following example shows a functional frame and explains the semantics of our connectives.

**Example 2.4** Consider the picture below:




We have a functional frame $\Sigma = (W, \mathcal{T}, \mathcal{F})$ such that:

- $W = \{w, w', w''\}$,
- $\mathcal{T} = \{(T_w, <_w), (T_{w'}, <_{w'}), (T_{w''}, <_{w''})\}$ where:
  $T_w = \{1_w, 2_w\}, T_{w'} = \{3_{w'}, 4_{w'}, 5_{w'}\}$ and $T_{w''} = \{6_{w''}, 7_{w''}, 8_{w''}, 9_{w''}\}$, being $<_w$, $<_{w'}$ and $<_{w''}$ the usual orderings represented intuitively in the picture.
- $\mathcal{F} = \{\stackrel{w\,w}{\longrightarrow}, \stackrel{w\,w'}{\longrightarrow}, \stackrel{w'\,w''}{\longrightarrow}, \stackrel{w''\,w'}{\longrightarrow}\}$, where:
  $\stackrel{w\,w}{\longrightarrow}: T_w \to T_w;\quad \stackrel{w\,w'}{\longrightarrow}: T_w \to T_{w'};\quad \stackrel{w'\,w''}{\longrightarrow}: T_{w'} \to T_{w''};\quad \stackrel{w''\,w'}{\longrightarrow}: T_{w''} \to T_{w'}$, defined as follows:
  $\stackrel{w\,w}{\longrightarrow} = id_{T_w}$ (identity on $T_w$); $\quad \stackrel{w\,w'}{\longrightarrow}(1_w) = 3_{w'}, \stackrel{w\,w'}{\longrightarrow}(2_w) = 4_{w'};$
  $\stackrel{w'\,w''}{\longrightarrow}(3_{w'}) = 6_{w''}, \stackrel{w'\,w''}{\longrightarrow}(5_{w'}) = 8_{w''};$
  $\stackrel{w''\,w'}{\longrightarrow}(6_{w''}) = \stackrel{w''\,w'}{\longrightarrow}(7_{w''}) = \stackrel{w''\,w'}{\longrightarrow}(8_{w''}) = 4_{w'}, \quad \stackrel{w''\,w'}{\longrightarrow}(9_{w''}) = 5_{w'}.$

Given any model on $\Sigma$, the formula $\Diamond A$ is true at 1 iff $A$ is true at some coordinate accessible from 1, that is, either in 1 or in 3. The formula $FA$ is true at 6 if $A$ is true at some coordinate in the future of 6, that is in 7, 8 or 9. Moreover, the formula $G\Box A$ is true at 7 if in every coordinate in the future, that is, $\Box A$ is true in 8 and in 9. This means that $A$ is true at every coordinate accessible from 8 and from 9, that is, in 4 and in 5. The formula $\zeta A$ is true in 4 if $A$ is true in every coordinate in the temporal flow of 4, that is in $3, 4$ and 5. On the other hand, $\tau A$ is true in 4 if $A$ is true at least in an element of $T_{w'}$, that is, either in $3, 4$ or 5.

## 3. On the new definitions for properties of accessibility relations and its definability

In this section, we study classical properties of the induced relation $R_{\mathcal{F}}$ as transitivity, reflexivity, symmetry, euclideanity and seriality and their definability. In this approach, the combination of temporal and modal aspects, has lead us in a natural way to give two versions for each property, one stronger than the other one. For example, we can define the transitivity either regarding composition of accessibility functions or not, as we will see in the following section. First of all, we give a definition which is very used in the rest of the paper.

**Definition 3.1** Let $\mathbb{J}$ be a class of functional frames and $\mathbb{K} \subseteq \mathbb{J}$. We say that $\mathbb{K}$ is **definable in** $\mathbb{J}$ if there exists a set $\Gamma$ of formulae such that for every frame $\Sigma \in \mathbb{J}$ we have that $\Sigma \in \mathbb{K}$ if and only if every formula of $\Gamma$ is valid in $\Sigma$. If $\mathbb{J}$ is the class of all functional frames, we say that $\mathbb{K}$ is **definable**.

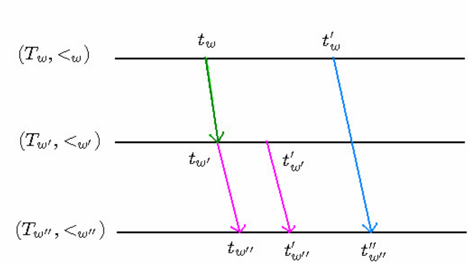
Let $P$ be a property of functional frames and $\mathbb{K}$ the class of all functional frames which have the property $P$. If $\mathbb{K} \subseteq \mathbb{J}$, we say that $P$ is definable in $\mathbb{J}$ if $\mathbb{K}$ is definable in $\mathbb{J}$. Thus, we say that $P$ is definable if $\mathbb{K}$ is definable.

### 3.1. *Transitivity*

We begin with the usual definition of transitivity for the induced relation $R_{\mathcal{F}}$.

**Definition 3.2** Let $\Sigma = (W, \mathcal{T}, \mathcal{F})$ be a functional frame. We will say that $\Sigma$ is **transitive** if the induced relation $R_{\mathcal{F}}$ is transitive, that is, for every pair $\stackrel{w\,w'}{\longrightarrow}, \stackrel{w'w''}{\longrightarrow} \in \mathcal{F}$, there exists $\stackrel{w\,w''}{\longrightarrow} \in \mathcal{F}$

**Example 3.3** The picture below is an example of transitive functional frame:




Notice that this definition does not depend on the particular coordinates. For this reason, although coordinate $t_w$ is not connected by an accessibility function with $t_{w''}$, transitivity holds because temporal flows $T_w$ and $T_{w''}$ are connected, in this case $t'_w$ with $t''_{w''}$. On the other hand, the frame of example 2.4 is not transitive because although $T_w$ is connected to $T_{w'}$ and $T_{w'}$ is connected to $T_{w''}$, $T_w$ is not connected to $T_{w''}$.

Now, we give an algebraic characterization of this property in order to obtain its definability.

Proposition 3.4 *Let $\Sigma = (W, \mathcal{T}, \mathcal{F})$ be a functional frame. $\Sigma$ is transitive iff for every coordinate $t_w$:*

$$(trans) \qquad \mathcal{F}(\mathcal{C}od(\mathcal{F}_{t_w})) \subseteq \mathcal{C}od(\mathcal{F}_w)$$

*Proof* Suppose that $\Sigma$ is transitive and let $t_{w''} \in \mathcal{F}(\mathcal{C}od(\mathcal{F}_{t_w}))$, that means that $t_{w''} = \stackrel{w'w''}{\longrightarrow}(t_{w'})$ with $t_{w'} \in \mathcal{C}od(\mathcal{F}_{t_w})$, that is, $t_{w'} \in T_{w'}$, being $t_w \in Dom(\stackrel{w\,w'}{\longrightarrow})$. As a consequence, $\stackrel{w\,w'}{\longrightarrow}, \stackrel{w'w''}{\longrightarrow} \in \mathcal{F}$, which implies, by the transitivity of $\Sigma$, that there exists $\stackrel{w\,w''}{\longrightarrow} \in \mathcal{F}_w$ and this leads to $t_{w''} \in \mathcal{C}od(\mathcal{F}_w)$, which proves $(trans)$.

Conversely, suppose $(trans)$ and $\stackrel{w\,w'}{\longrightarrow}, \stackrel{w'w''}{\longrightarrow} \in \mathcal{F}$. Now, from Definition 2.1 we have that $\stackrel{w\,w'}{\longrightarrow}, \stackrel{w'w''}{\longrightarrow}$ are non empty functions, that is, there exist coordinates $t_w$, $t_{w'}$, $t'_{w'}$ and $t'_{w''}$, such that $\stackrel{w\,w'}{\longrightarrow}(t_w) = t_{w'}$ and $\stackrel{w'w''}{\longrightarrow}(t'_{w'}) = t'_{w''}$, as a consequence, $t'_{w''} \in \mathcal{F}(\mathcal{C}od(\mathcal{F}_{t_w}))$. Now, by $(trans)$, we have that $t'_{w''} \in \mathcal{C}od(\mathcal{F}_w)$, this implies that there exists $\stackrel{w\,w''}{\longrightarrow} \in \mathcal{F}$, which means that $\Sigma$ is transitive and finishes our proof. $\square$

Now we can prove the definability of this property.

THEOREM 3.5 *The property of transitivity is definable by the formula* $\zeta\Box\zeta A \to \Box\zeta\Box A$

*Proof* Let $\Sigma = (W, \mathcal{T}, \mathcal{F})$ be transitive and $(\Sigma, h)$ a model on $\Sigma$, from the semantics of our connectives and Remark *1* above, we have:

$$t_w \in h(\zeta\Box\zeta A) \text{ iff } T_w \subseteq h(\Box\zeta A) \text{ iff } \mathcal{F}(T_w) \subseteq h(\zeta A) \text{ iff } \mathcal{C}od(\mathcal{F}_w) \subseteq h(A) \quad (1)$$
$$t_w \in h(\Box\zeta\Box A) \text{ iff } \mathcal{F}(\{t_w\}) \subseteq h(\zeta\Box A) \text{ iff } \mathcal{C}od(\mathcal{F}_{t_w}) \subseteq h(\Box A) \text{ iff } \mathcal{F}(\mathcal{C}od(\mathcal{F}_{t_w})) \subseteq h(A) \quad (2)$$

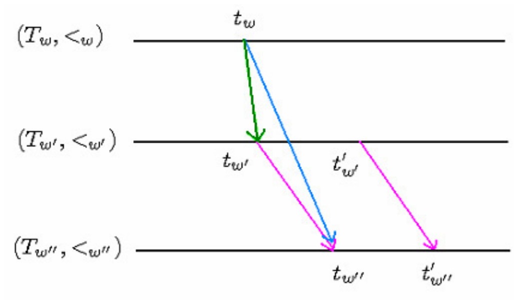
Now, for proving the validity of $\zeta\Box\zeta A \to \Box\zeta\Box A$, if $t_w \in h(\zeta\Box\zeta A)$, we have that $\mathcal{C}od(\mathcal{F}_w) \subseteq h(A)$ this means, by using Proposition 3.4, that $\mathcal{F}(\mathcal{C}od(\mathcal{F}_{t_w})) \subseteq h(A)$ and this implies $t_w \in h(\Box\zeta\Box A)$.

Reciprocally, if $\Sigma = (W, \mathcal{T}, \mathcal{F})$ is not transitive then, again by Proposition 3.4, there exists a coordinate $t_w$ such that $\mathcal{F}(\mathcal{C}od(\mathcal{F}_{t_w})) \nsubseteq \mathcal{C}od(\mathcal{F}_w)$. Now, if we define a functional model $(\Sigma, h)$ such that $h(p) = \mathcal{C}od(\mathcal{F}_w)$ being $p$ a propositional variable, we have, by using (1) and (2), that $t_w \in h(\zeta\Box\zeta p)$ but $t_w \notin h(\Box\zeta\Box p)$. This implies that $\zeta\Box\zeta p \to \Box\zeta\Box p$ is not true at $t_w$. $\Box$

As we have said above, we can give a stronger definition of transitivity, regarding the composition of accessibility functions.

**Definition 3.6** Let $\Sigma = (W, \mathcal{T}, \mathcal{F})$ be a functional frame such that $\overset{w\,w'}{\longrightarrow}(T_w) \subseteq Dom(\overset{w'w''}{\longrightarrow})$, for every $\overset{w\,w'}{\longrightarrow}, \overset{w'w''}{\longrightarrow} \in \mathcal{F}$. We say that $\Sigma$ is **strongly-transitive** if it is closed by composition, that is, if we have $\overset{w\,w'}{\longrightarrow}, \overset{w'w''}{\longrightarrow} \in \mathcal{F}$, then $(\overset{w'w''}{\longrightarrow} \circ \overset{w\,w'}{\longrightarrow}) \in \mathcal{F}$

**Example 3.7** The picture below is an example of strongly-transitive functional frame:




Let us observe that in this case $t_w$ connected to $t_{w'}$ and $t_{w'}$ connected to $t_{w''}$ implies $t_w$ connected to $t_{w''}$. This is the difference between strong-transitivity and transitivity. As a consequence, it is clear that if $\Sigma$ is strongly-transitive then it is transitive, but the converse is not true. We will have the same situation for every *strong* definition hereafter.

The following result gives an algebraic characterization of strong-transitivity, which enables us to prove its definability. For an easy reading, from now on, we omit the proofs because they are similar to the previous ones.

PROPOSITION 3.8 *Let* $\Sigma = (W, \mathcal{T}, \mathcal{F})$ *be a functional frame such that* $\overset{w\,w'}{\longrightarrow}(T_w) \subseteq Dom(\overset{w'w''}{\longrightarrow})$*, for all* $\overset{w\,w'}{\longrightarrow}, \overset{w'w''}{\longrightarrow} \in \mathcal{F}$. $\Sigma$ *is strongly-transitive iff for every coordinate* $t_w$*:*

$$(s\text{-}trans) \quad \mathcal{F}(\mathcal{F}(\{t_w\})) \subseteq \mathcal{F}(\{t_w\})$$

From the previous proposition, we can give the desired result of definability of the strong-transitivity.

THEOREM 3.9 *The property of strong-transitivity is definable in the class of functional frames* $\Sigma = (W, \mathcal{T}, \mathcal{F})$ *such that* $\overset{w\,w'}{\longrightarrow}(T_w) \subseteq Dom(\overset{w'w''}{\longrightarrow})$ *for all* $\overset{w\,w'}{\longrightarrow}, \overset{w'w''}{\longrightarrow} \in \mathcal{F}$*, by the formula* $\Box A \to \Box\Box A$.

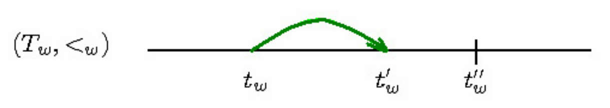
*Remark 1* Notice that the formula which defines strong-transitivity is the well known formula which defines transitivity in modal logic.

Following the method used in the previous section, in the rest of the paper we study different definitions for other properties as reflexivity, symmetry, euclideanity and seriality. Thus, we give algebraic characterizations for these definitions and we obtain their definability.

### 3.2. *Reflexivity*

**Definition 3.10** Let $\Sigma = (W, \mathcal{T}, \mathcal{F})$ be a functional frame. We say that $\Sigma$ is **reflexive** if the induced relation $R_{\mathcal{F}}$ is reflexive, that is, for every $w \in W$ there exists $\stackrel{w\,w}{\longrightarrow} \in \mathcal{F}$.

**Example 3.11** The picture below is an example of reflexive functional frame:




Let us observe that the existence of accessibility function $\stackrel{w\,w}{\longrightarrow}$ ensures the reflexivity, because coordinate $t_w$ is connected to $t'_w$. However, in this example, neither $t'_w$ nor $t''_w$ are connected to another coordinate.

Proposition 3.12 *Let $\Sigma = (W, \mathcal{T}, \mathcal{F})$ be a functional frame. $\Sigma$ is reflexive iff for every $w \in W$:*

$$(reflex) \qquad \mathcal{F}(T_w) \cap T_w \neq \varnothing$$

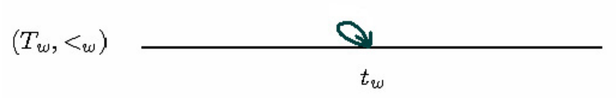
As a consequence, we have the following result of definability.

Theorem 3.13 *The property of reflexivity is definable by the formula $\zeta\Box A \to \tau A$*

We now present a more restrictive definition of reflexivity.

**Definition 3.14** Let $\Sigma = (W, \mathcal{T}, \mathcal{F})$ be a functional frame. We say that $\Sigma$ is **strongly-reflexive** if, for every $w \in W$ we have that $id_{T_w} \in \mathcal{F}$.

**Example 3.15** The picture below is an example of strongly-reflexive functional frame:




*Remark 2* We assume $Dom(id_{T_w}) = T_w$, for all $w \in W$, thus every coordinate $t_w$ is connected to $t_w$.

The following proposition is a consequence of the previous definition.

Proposition 3.16 *Let $\Sigma = (W, \mathcal{T}, \mathcal{F})$ be a functional frame. $\mathcal{F}$ is strongly-reflexive iff for every $w \in W$ and every coordinate $t_w$, we have:*

$$(s\text{-}reflex) \qquad t_w \in \mathcal{F}(\{t_w\})$$

Thus, we have the desired result of definability.

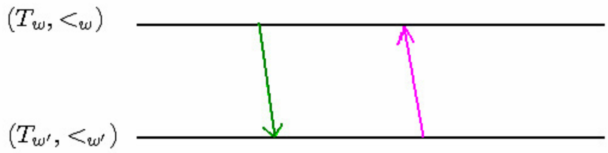
Theorem 3.17 *The property of strong-reflexivity is definable by the formula $\Box A \to A$*

### 3.3. *Symmetry*

As in the previous section, we can give two definitions.

**Definition 3.18** Let $\Sigma = (W, \mathcal{T}, \mathcal{F})$ be a functional frame. We say that $\Sigma$ is **symmetric** if the induced relation $R_{\mathcal{F}}$ is symmetric, that is, if $\stackrel{w\,w'}{\longrightarrow} \in \mathcal{F}$, there exists $\stackrel{w'w}{\longrightarrow} \in \mathcal{F}$

**Example 3.19** The picture below is an example of symmetric functional frame:




The previous definition says that if $T_w$ is connected to $T_{w'}$, then $T_{w'}$ must be connected to $T_w$.

Proposition 3.20 *Let $\Sigma = (W, \mathcal{T}, \mathcal{F})$ be a functional frame. $\Sigma$ is symmetric iff for every coordinate $t_w$ such that $\mathcal{F}_w \neq \varnothing$, we have that:*

$$(sym) \qquad t_w \in \bigcap_{\xrightarrow{w\,w'} \in \mathcal{F}_w} \mathcal{C}od(\mathcal{F}_{w'})$$

As a consequence, we have the definability of symmetry.

Theorem 3.21 *The property of symmetry is definable by the formula $A \rightarrow \zeta\Box\tau\Diamond\tau A$*

The strong definition of symmetry is related to the existence of inverse of the accessibility functions. To begin with, we give the definition of inverse in the context of our functional frames.

**Definition 3.22** Let $\Sigma = (W, \mathcal{T}, \mathcal{F})$ be a functional frame and $\xrightarrow{w\,w'} \in \mathcal{F}$ such that $\xrightarrow{w\,w'}$ is injective, the inverse of $\xrightarrow{w\,w'}$, denoted by $(\xrightarrow{w\,w'})^{-1}$, is a function $(\xrightarrow{w\,w'})^{-1} : T_{w'} \rightarrow T_w$ such that:

(i) $((\xrightarrow{w\,w'})^{-1} \circ \xrightarrow{w\,w'})(t_w) = t_w$ for every $t_w \in Dom(\xrightarrow{w\,w'})$
(ii) $(\xrightarrow{w\,w'} \circ (\xrightarrow{w\,w'})^{-1})(t_{w'}) = t_{w'}$ for every $t_{w'} \in \xrightarrow{w\,w'} (T_w)$

*Remark 3* Notice that in the previous definition, it is not required that $Dom(\xrightarrow{w\,w'}) = T_w$.

**Definition 3.23** Let $\Sigma = (W, \mathcal{T}, \mathcal{F})$ be a functional frame such that every accessibility function in $\mathcal{F}$ is injective. We will say that $\Sigma$ is **strongly-symmetric** if, for every $\xrightarrow{w\,w'} \in \mathcal{F}$, we have that $(\xrightarrow{w\,w'})^{-1} \in \mathcal{F}$.

**Example 3.24** The picture below is an example of strongly-symmetric functional frame:


In contrast to the previous example, strong-symmetry implies connection between coordinates.

Proposition 3.25 *Let $\Sigma = (W, \mathcal{T}, \mathcal{F})$ be a functional frame such that every accessibility function in $\mathcal{F}$ is injective. Then $\Sigma$ is symmetric iff for every $t_w \in Dom(\mathcal{F}_w)$, we have:*

$$(s\text{-}sym) \qquad t_w \in \bigcap_{t_{w'} \in \mathcal{F}(\{t_w\})} \mathcal{F}(\{t_{w'}\})$$

Now, we have the correspondening result of definability, in this case for a subclass of functional frames.

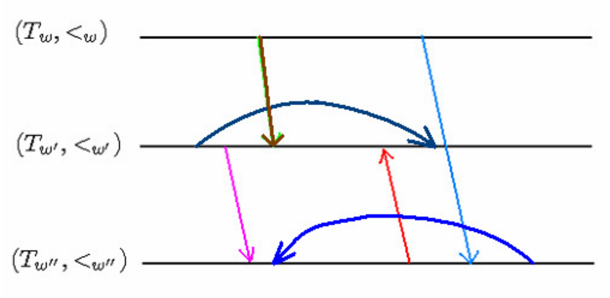
Theorem 3.26 *The property of strong-symmetry is definable in the class of functional frames with injective accessibility functions by the formula $A \rightarrow \Box\Diamond A$*

### 3.4. *Euclidean property*

As in previous sections, we begin with the intuitive definition of the property.

**Definition 3.27** Let $\Sigma = (W, \mathcal{T}, \mathcal{F})$ be a functional frame. We say that $\Sigma$ is **euclidean** if the induced relation $R_{\mathcal{F}}$ is euclidean, that is, if $\overset{w\,w'}{\longrightarrow}, \overset{w\,w''}{\longrightarrow} \in \mathcal{F}$ then $\overset{w'w''}{\longrightarrow} \in \mathcal{F}$.

**Example 3.28** The picture below is an example of euclidean functional frame:




Notice that as $T_w$ is connected to $T_{w'}$ and $T_w$ is connected to $T_{w''}$ then $T_{w'}$ must be connected to $T_{w''}$ and $T_{w''}$ must be connected to $T_{w'}$. Moreover, as $T_w$ is connected to $T_{w'}$, $T_{w'}$ must be connected to $T_{w'}$. Similarly to $T_{w''}$. However, this frame is not reflexive because $T_w$ is not connected to $T_w$.

Proposition 3.29 *Let $\Sigma = (W, \mathcal{T}, \mathcal{F})$ be a functional frame. $\mathcal{F}$ is euclidean iff for every coordinate $t_w$, we have that*

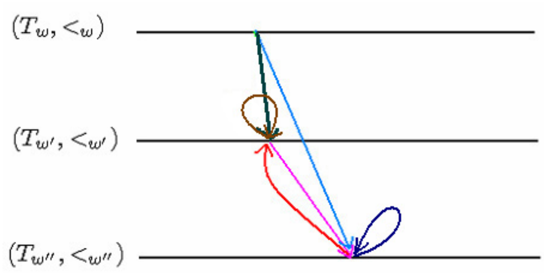
$$(eucl) \qquad \mathcal{F}(\{t_w\}) \subseteq \bigcap_{\overset{w\,w'}{\longrightarrow} \in \mathcal{F}_w} \mathcal{C}od(\mathcal{F}_{w'})$$

Theorem 3.30 *The euclidean property is definable by the formula $\Diamond A \to \zeta\Box\tau\Diamond\tau A$*

The following definition presents a strong version of the euclidean property. In contrast to the previous cases, we need to include the *weak* possibility.

**Definition 3.31** Let $\Sigma = (W, \mathcal{T}, \mathcal{F})$ a functional frame. We will say that $\Sigma$ is **strongly-euclidean** if it is euclidean and, moreover, for every coordinate $t_w$, such that $\overset{w\,w'}{\longrightarrow}(t_w) = t_{w'}$ and $\overset{w\,w''}{\longrightarrow}(t_w) = t_{w''}$, we have $\overset{w'w''}{\longrightarrow}(t_{w'}) = t_{w''}$.

**Example 3.32** The picture below is an example of strongly-euclidean functional frame:




Proposition 3.33 *Let $\Sigma = (W, \mathcal{T}, \mathcal{F})$ be a functional frame. Then, $\Sigma$ is strongly-euclidean iff it verifies $(eucl)$ and for every coordinate $t_w$, we have:*

$$(s\text{-}eucl) \qquad \mathcal{F}(\{t_w\}) \subseteq \bigcap_{t_{w'} \in \mathcal{F}(\{t_w\})} \mathcal{F}(\{t_{w'}\})$$

As a consequence, we obtain the definability of the property.

THEOREM 3.34 *The strong-euclidean property is definable by the set of formulae:*

$$\{\Diamond A \to \zeta\Box\tau\Diamond\tau A, \Diamond A \to \Box\Diamond A\}$$

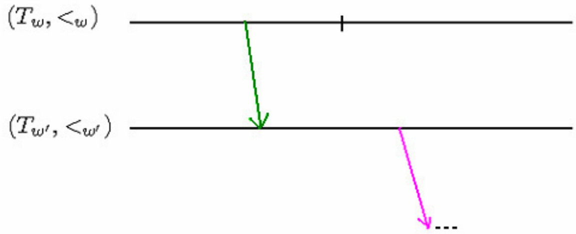
*Remark 4* Let us observe that the first formula defines the (weak) euclidean property.

### 3.5. *Serial property*

To begin with, we give the natural definition.

**Definition 3.35** Let $\Sigma = (W, \mathcal{T}, \mathcal{F})$ be a functional frame. We say that $\Sigma$ is **serial** if the induced relation $R_{\mathcal{F}}$ is serial, that is, for every $w \in W$, there exists $w' \in W$ such that $\stackrel{w\ w'}{\longrightarrow} \in \mathcal{F}$

**Example 3.36** The picture below is an example of serial functional frame:




Let us observe that, by definition, temporal flow $T_{w'}$ must be connected to another temporal flow and this one with another one, etc.

The following characterization of serial property is straightforward.

PROPOSITION 3.37 *Let $\Sigma = (W, \mathcal{T}, \mathcal{F})$ be a functional frame. $\mathcal{F}$ is serial iff we have that:*

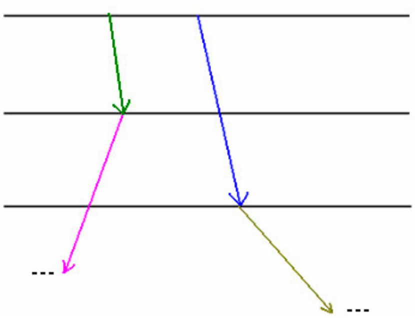
$$(serial) \qquad \textit{For every } w \in W, \textit{ we have that } \mathcal{F}_w \neq \varnothing$$

THEOREM 3.38 *The serial property is definable by the formula $\zeta\Box A \to \tau\Diamond A$*

Now, we give the strong definition.

**Definition 3.39** Let $\Sigma = (W, \mathcal{T}, \mathcal{F})$ be a functional frame. We say that $\Sigma$ is **strongly-serial** if, for every coordinate $t_w$, we have that $\mathcal{F}_w(\{t_w\}) \neq \varnothing$

**Example 3.40** The picture below is an example of strongly-serial functional frame:




As a consequence of the previous definition, we can give the definability of this property.

THEOREM 3.41 *The strong-serial property is definable by the formula $\Box A \to \Diamond A$*

## 4. Conclusions and future work

In this paper, we have generalized the definitions of transitivity, reflexivity, symmetry, euclideanity and seriality in the context of the functional approach for temporal×modal logic presented in [3]. Two different definitions for each property and their definability have been given. The study of the definability will allow us to give axiom systems for logics to deal with these properties. These logics can be applied in Computation for example to manage asynchronous systems in the context of multiagent and interactive systems.

As future work, it is planned to study the completeness of the logics with these properties. Moreover, we have planned to incorporate indexed connectives in order to specify which time flow is accessed, as in [4] with functional accessibility and in [5] were the accessibility is given by non deterministic operators.

Last, but not least, we will study the possibility of giving automated theorem provers for these logics.

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