A PDL Approach for Qualitative Velocity

A. BURRIEZA
Dept. Philosophy. University of Málaga
29071 Málaga. Spain
burrieza@uma.es

E. MUÑOZ-VELASCO
Dept. Applied Mathematics. University of Málaga
29071 Málaga. Spain
emilio@ctima.uma.es

M. OJEDA-ACIEGO
Dept. Applied Mathematics. University of Málaga
29071 Málaga. Spain
aciego@ctima.uma.es

We introduce the syntax, semantics, and an axiom system for a PDL-based extension of the logic for order of magnitude qualitative reasoning, developed in order to deal with the concept of qualitative velocity, which together with qualitative distance and orientation, are important notions in order to represent spatial reasoning for moving objects, such as robots. The main advantages of using a PDL-based approach are, on the one hand, all the well-known advantages of using logic in AI, and, on the other hand, the possibility of constructing complex relations from simpler ones, the flexibility for using different levels of granularity, its possible extension by adding other spatial components, and the use of a language close to programming languages.

Keywords: qualitative reasoning, order-of-magnitude reasoning, propositional dynamic logic, hybrid logic

1. Introduction

Qualitative reasoning, QR, tries to simulate the way humans think in almost all situations. For example, we do not need to know the exact value of velocity, distance, position of our car in order to park it when we are so lucky to find a parking spot in the city center. A form of QR is order-of-magnitude reasoning, where the quantitative information is substituted by a finite number of qualitative classes, for example: *zero*, *small*, *medium*, and *large*; moreover, some relations between the qualitative classes, such as *negligibility, closeness* ..., may be defined [1,2]. The level of granularity, that is, the number of qualitative classes used, depends on the problem in question. Recent applications of order-of-magnitude reasoning can be seen, for example, in [3,4].

The use of logic in QR, as in other areas of AI, improves the capability of formal representation of real world problems and provides insights into their most suitable solving methods. Some ideas arisen from logic, such as theorem-proving and model-construction techniques are used in AI [5]. As an example, in [6] three uses of logic in AI are considered: as a tool of analysis, as a basis for knowledge representation, and as a programming language. There are several applications of logics for QR, [7,8], many of them in spatial reasoning. As examples of logics for order of magnitude reasoning, see [9,10]; a theorem prover for one of these logics can be seen in [11], which has been implemented in [12].

Propositional Dynamic Logic, PDL, provides the possibility of constructing complex relations from simpler ones and the use of a language very close to programming languages. Some recent applications of PDL in AI can be seen in [13,14,15]. In addition, PDL is a decidable logic [16] whereas, for example, first-order logic is not. In our approach, we extend PDL with a finite number of constants in order to represent the qualitative classes, for this reason, it is a special kind of Hybrid Logic [17,18].

The concepts of qualitative velocity [19,20,21], together with qualitative distance and orientation, are very important in order to represent spatial reasoning for moving objects, such as robots. Several papers have been published in this line (see, for example [22,23,24,25]) which try to make progress in the development of qualitative kinematics models, as given in [26,27,28]. The problem of the relative movement of one physical object with respect to another can been treated by the Region Connection Calculus [29] and the Qualitative Trajectory Calculus [30]. However, as far as we know, there is no work which proposes a logic framework to manage qualitative velocity. From a programming point of view, ReadyLog [31,32] has been successfully applied for controlling soccer robots in RoboCup competitions [33]. ReadyLog is an extension of GoLog [34], based on the situation calculus [35], a first-order logic for reasoning about actions and their effects in a dynamically changing world. However, as said above, in our approach with PDL, we integrate the qualitative classes in our language and, moreover, we have the possibility of constructing complex programs from simpler ones, by using composition, union, etc. As in [19], we consider qualitative velocity of an object B with respect to another object A, represented by two components: module and orientation, each one given by a qualitative class. If we consider the velocity of B with respect to A, and the velocity of C with respect to B, then the *composition* of these two velocities consists of obtaining the velocity of C with respect to A. For example, if (Q, l) represents a velocity of B with respect to A, which is *quick* towards the *left*, and (N, r) is a velocity of C with respect to B, which is *normal* towards the *right*, then its composition is the velocity of C with respect to A, that could be either (Q, l) or (N, l), that is, a *quick* or *normal* velocity towards the left. The results of these compositions could depend on the specific problem we are dealing with. In the following section, we make some *common-sense* assumptions about these compositions (denoted by **P1-P6**), as a way of example. From now on, for representing a velocity, we will give its module and direction by omitting, for simplicity, the reference to the objects A, B, C, etc. whenever this information is clear from the context.

In this paper, we present the syntax, semantics and an axiom system for an extension of PDL for order of magnitude qualitative reasoning to deal with the concept of qualitative velocity. We study the decidability of the satisfiability problem and the soundness and completeness of the proposed logic. The main advantages of this approach are, on the one hand, all the advantages of using logic in AI commented above, and, on the other hand, the possibility of constructing complex relations from simpler ones, the flexibility for using different levels of granularity, its possible extension by adding other spatial components, and the use of a language close to programming languages. We consider also an extension of our approach to fuzzy qualitative reasoning [24,36,37], by introducing qualitative classes as four-tuple fuzzy numbers, thus we could use fuzzy arithmetic operations in order to get the results of the composition of velocities.

The paper is organized as follows. In Section 2, the syntax and semantics of the proposed logic is introduced, and explained on the basis of some examples. In Section 3, we give an example of axiom system for our logic and we discuss completeness and the decidability of the problem of satisfiability. In Section 4, an extension of our approach by considering fuzzy qualitative velocities is presented; finally, some conclusions and future works are discussed in Section 5.

2. Syntax and Semantics

To begin with, let us consider the following example. The sets $L_1 = \{$zero (Z), slow (S), normal (N), quick (Q)$\}$, and $L_2 = \{$front-left (fl), straight-front (sf), front-right (fr), left (l), none (n), right (r), back-left (bl), straight-back (sb), back-right (br)$\}$, represent, respectively, the qualitative velocities (its module) and the orientations. Therefore, a pair (S,bl) may represent a *slow* velocity towards the *back-left* orientation.

In order to introduce the language of our logic, we consider a set of formulas Φ and a set of programs Π, which are defined recursively on disjoint sets Φ_0 and Π_0, respectively. Φ_0 is called the set of *atomic formulas* which can be thought of as abstractions of properties of states. Similarly, Π_0 is called the set of *atomic programs* which are intended to represent basic instructions.

Formulas:

- $\Phi_0 = \mathbb{V} \cup \mathbb{C}$, where $\mathbb{V}$ is a denumerable set consisting of propositional variables and $\mathbb{C} = L_1 \times \cdots \times L_k$, where $L_1, \dots, L_k$ are intended to represent finite sets of labels.
- If φ and ψ are formulas and a is a program, then $\varphi \to \psi$ (propositional implication), $\bot$ (propositional falsity) and $[a]\varphi$ (program necessity) are also formulas. As usual, $\vee$ and $\wedge$ represent logical disjunction and conjunction, respectively; whereas $\langle a \rangle$ represents program possibility.

Notice that, as the elements of $\mathbb{C}$ have k components, we could consider different spatial components, such as position, distance, cardinal directions, etc. In the previous example and from now, we will consider for simplicity only two components of velocity: module and orientation (i.e. $\mathbb{C} = L_1 \times L_2$), but this approach can be extended.

Programs:

- $\Pi_0 = \{\otimes_\star \mid \star \in \mathbb{C}\}$, a set of specific programs.
- If a and b are programs and φ is a formula, then $(a; b)$ ("do a followed by b"), $a \cup b$ ("do either a or b, nondeterministically"), a^* ("repeat a a nondeterministically chosen finite number of times") and $\varphi?$ ("proceed if φ is true, else fail") are also programs.

Example 1. We can consider the program $(\mathsf{S}, \mathsf{bl})?$ in order to represent the intuitive meaning of being a slow velocity towards the back-left orientation; $(\mathsf{S}, \mathsf{bl})? \cup (\mathsf{Q}, \mathsf{r})?$ means, being either a slow velocity towards the back-left orientation, or a quick velocity towards the right orientation. The intuitive meaning of program $\otimes_{(\mathrm{S,bl})}$ is to compose a slow velocity towards the back-left orientation with the current state, $\otimes^*_{(\mathrm{S,bl})}$ represents the composition a finite number of times of a slow velocity towards the back-left orientation, and $\otimes_{(\mathrm{S,bl})}; \otimes_{(\mathrm{N,sb})}$ means to compose a slow velocity towards the back-left orientation *followed* by the composition of a normal velocity towards the straight back orientation.

We now define the *semantics* of our logic. A *model* $\mathcal{M}$ is a tuple $(W_1 \times \cdots \times W_k, m)$ where each W_i is a non-empty set divided into $|L_i|$ qualitative classes,[a] being $|L_i|$ the number of elements of the set of labels L_i defined above. By abuse of notation, we will use the same symbols to represent the qualitative classes and its corresponding formulas. On the other hand, m is a meaning function such that $m(p) \subseteq W$, for every propositional variable, $m(\star) = \star$, for every $\star \in \mathbb{C}$ and $m(a) \subseteq W \times W$, for all atomic program a. Moreover, if φ and ψ are formulas and a, b are programs, then we have the following:

- $m(\varphi \to \psi) = (W \smallsetminus m(\varphi)) \cup m(\psi)$
- $m(\bot) = \varnothing$
- $m([a]\varphi) = \{w \in W : \text{for all } v \in W, \text{ if } (w, v) \in m(a) \text{ then } v \in m(\varphi)\}$
- $m(a \cup b) = m(a) \cup m(b)$
- $m(a; b) = m(a); m(b)$
- $m(a^*) = m(a)^*$ (reflexive and transitive closure of relation $m(a)$).
- $m(\varphi?) = \{(w, w) : w \in m(\varphi)\}$

Given a model $\mathcal{M} = (W, m)$, a formula φ is *true* in $u \in W$ whenever we have that $u \in m(\varphi)$. We say that φ is *satisfiable* if there exists $u \in W$ such as φ is true

[a]In order to simplify the notation, from now on, we will use W instead of $W_1 \times \cdots \times W_k$.

in u. Moreover, φ is *valid in a model* $\mathcal{M} = (W, m)$ if φ is true in all $u \in W$, that is, if $m(\varphi) = W$. Finally, φ is *valid* if φ is valid in all models.

The informal meaning of some formulas is given below. The states $u \in W$ referred to, are to be understood as objects affected by some form of qualitative velocity. With this idea in mind, when we say that u is a slow velocity towards some orientation, we are abusing notation to refer to an object which is moving at a "slow velocity towards some orientation" with respect to another object. Let φ be any propositional formula, then:

- $\langle(\mathsf{S}, \mathsf{bl})?\rangle\, \varphi$ is true in u iff u is a slow velocity towards the back-left orientation, and φ is true in u.
- $\langle(\mathsf{S}, \mathsf{bl})? \cup (\mathsf{Q}, \mathsf{r})?\rangle\, \varphi$ is true in u iff u is either a slow velocity towards the back-left orientation or a quick velocity towards the right orientation, and φ is true in u.
- $[\otimes_{(\mathsf{Q},\mathsf{r})}]\, \varphi$ is true in u iff for every velocity u' obtained by composing u with a quick velocity towards the right orientation, φ is true in u'.
- $[\otimes^*_{(\mathsf{Q},\mathsf{r})}]\, \varphi$ is true in u iff for every velocity u' obtained by the repetition of the composition of u with a quick velocity towards the right orientation a nondeterministically chosen finite number of times, φ is true in u'.
- $[\otimes_{(\mathsf{S},\mathsf{bl})}; \otimes_{(\mathsf{N},\mathsf{sb})}]\, \varphi$ is true in u iff for every velocity u' obtained by composing u with a slow velocity towards the back left orientation, followed by a normal velocity towards the straight back orientation, φ is true in u'.
- $(\mathsf{Q}, \mathsf{r}) \rightarrow [\otimes_{(\mathsf{S},\mathsf{bl})}]\big((\mathsf{N}, \mathsf{br}) \vee (\mathsf{Q}, \mathsf{r})\big)$ means that, if the current velocity is quick towards the right orientation then, for every value obtained by composing it with a slow velocity towards the back-left orientation, we have either a normal velocity towards the back-right orientation or a quick velocity towards the right orientation.
- $[((\mathsf{Z}, \mathsf{n})?; \otimes_{(\mathsf{S},\mathsf{r})})^*; \neg(\mathsf{Z}, \mathsf{n})?]\neg(\mathsf{Z}, \mathsf{n})$ says that *while* the velocity is zero with no orientation, it has to be composed with a small velocity towards the right.
- $(\mathsf{Q}, \mathsf{sb}) \rightarrow [(\otimes_{(\mathsf{Q},\mathsf{sf})}; (\neg(\mathsf{N}, \mathsf{sf})?; \otimes_{(\mathsf{Q},\mathsf{sf})})^*; (\mathsf{N}, \mathsf{sf})?](\mathsf{N}, \mathsf{sf})$ if the velocity is quick towards the straight back, *repeat* the composition with a quick velocity towards the straight front *until* the velocity is normal towards the straight front.

Observe that, in the last two formulas, we use the advantages of PDL for expressing programming commands such as *while ... do* and *repeat ... until*.

We can construct the desired logic depending on the granularity and the specific properties required. For simplicity in the presentation, let us consider a simple case in which the set of qualitative velocities is $L_1 = \{\mathsf{z}, \mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3\}$, where $\mathsf{z}, \mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3$ represent zero, slow, normal and quick, respectively; and the set of qualitative orientations is $L_2 = \{\mathsf{n}, \mathsf{o}_1, \mathsf{o}_2, \mathsf{o}_3, \mathsf{o}_4\}$, representing, respectively, none, front, right, back, and left orientations.

Remark 1. In the presentation of the following properties, we have to take into account the specific framework we are working in. We are considering just four qualitative classes for the module of the velocities, and five qualitative classes for the orientation of the velocity. This means that we have to somehow paraphrase the physical properties in terms of the qualitative framework stated in the previous paragraph, i.e., the properties are given for qualitative classes and can be changed according to the assumptions of the problem under study.

Assume that the following properties are required for the composition of *qualitative* velocities.

P1 the composition of a zero velocity with any other velocity and orientation, gives as a result the latter velocity and orientation, that is, $\otimes_{(\mathsf{z},\mathsf{n})}$ plays the role of the neutral element for the composition of velocities.

P2 the composition of two velocities with the same module but with opposite orientations, is the zero velocity.

P3 the composition of two velocities with the same module but with perpendicular orientations, will be a velocity with the same module, towards any of both orientations.

P4 the composition of two different velocities with perpendicular orientation, will be the maximum of both velocities, with the orientation of the maximum velocity.

P5 the composition of two different velocities with the same orientation will be, either the maximum of both velocities, or even a velocity greater than this one, with the same orientation.

P6 the composition of two different velocities with opposite orientation, will be either the maximum of both velocities or even a velocity smaller than this one, with the orientation of the maximum velocity.

The previous properties can be expressed semantically as follows. Given a model (W, m), for every $\mathsf{v}, \mathsf{v}', \mathsf{v_r}, \mathsf{v_s}, \mathsf{v_p}, \mathsf{v_q} \in L_1$, $\mathsf{o}, \mathsf{o}', \mathsf{o_j}, \mathsf{o_{j+1}}, \mathsf{o_{j+2}} \in L_2$, we have:

(1) $m(\otimes_{(\mathsf{v},\mathsf{o})}; \otimes_{(\mathsf{z},\mathsf{n})}) = m(\otimes_{(\mathsf{v},\mathsf{o})})$

(2) $m(\otimes_{(\mathsf{v},\mathsf{o_j})}; \otimes_{(\mathsf{v},\mathsf{o_{j+2}})}) = m(\otimes_{(\mathsf{z},\mathsf{n})})$, $\mathsf{j} = 1, 2$.

(3) $m(\otimes_{(\mathsf{v},\mathsf{o_{j+1}})})(m(\mathsf{v}, \mathsf{o_j})) \subseteq m(\mathsf{v}, \mathsf{o_j}) \cup m(\mathsf{v}, \mathsf{o_{j+1}})$, $\mathsf{j} = 1, 2, 3$.

(4) $m(\otimes_{(\mathsf{v_s},\mathsf{o_{j+1}})})(m(\mathsf{v_r}, \mathsf{o_j})) \subseteq m(\mathsf{v_s}, \mathsf{o_{j+1}})$, $\mathsf{j} = 1, 2, 3$ and $\mathsf{r} < \mathsf{s}$.

(5) $m(\otimes_{(\mathsf{v_s},\mathsf{o})})(m(\mathsf{v_r}, \mathsf{o})) \subseteq m(\mathsf{v_s}, \mathsf{o}) \cup m(\mathsf{v_3}, \mathsf{o})$, being $\mathsf{r} < \mathsf{s}$, and $\mathsf{s} = 2, 3$.

(6) $m(\otimes_{(\mathsf{v_s},\mathsf{o_{j+2}})})(m(\mathsf{v_r}, \mathsf{o_j})) \subseteq m(\mathsf{v_s}, \mathsf{o_{j+2}}) \cup m(\mathsf{v_q}, \mathsf{o_{j+2}})$, being $\mathsf{j} = 1, 2$; $\mathsf{r} < \mathsf{s}$, $\mathsf{s} = 2, 3$; and $\mathsf{q} = \mathsf{s} - 1$.

As an example of 6, if $(\mathsf{v_r}, \mathsf{o_j})$ =(slow, front), and $(\mathsf{v_s}, \mathsf{o_{j+2}})$=(quick, back), then its composition will be either (quick, back), or (normal, back), being normal the qualitative class immediately smaller than quick.

Example 2. Let us consider the case study of ball interception of simulated soccer agents, presented in [21]. Suppose that the ball is located at a point B and is moving with a velocity $(\mathsf{v_b}, \mathsf{o_b})$ and the robot is at point R and it is not moving at this instant

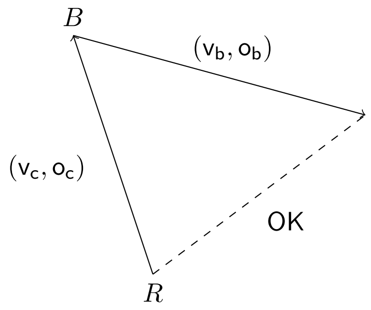

Fig. 1. Catching a ball when the robot is not moving

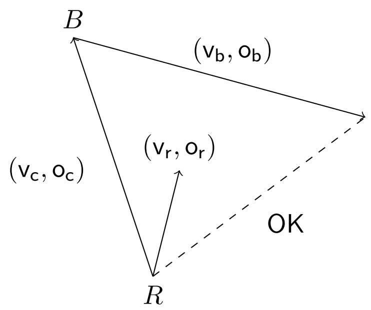

Fig. 2. Catching a ball when the robot is moving

(see Figure 1). Suppose also that the robot can calculate the qualitative velocity needed to catch the ball at the current instant and position and this velocity is $(\mathsf{v_c}, \mathsf{o_c})$. A simple vectorial argument leads us to the fact that the composition of both velocities has to be the velocity needed to catch the ball. This condition can be expressed in our language by the formula $(\mathsf{z}, \mathsf{n}) \to [\otimes_{(\mathsf{v_c}, \mathsf{o_c})}; \otimes_{(\mathsf{v_b}, \mathsf{o_b})}]\mathsf{OK}$, where (z, n) means that the robot is not moving at this instant, and OK means that the velocity of the robot is the correct one in order to catch the ball. This situation can be checked as the robot is moving towards the ball and has to be corrected if it is not OK. In the case that the robot is also moving with velocity $(\mathsf{v_r}, \mathsf{o_r})$ (see Figure 2), the desired velocity in order to capture the ball is the composition of this velocity with $(\mathsf{v_r}, -\mathsf{o_r})$, $(\mathsf{v_c}, \mathsf{o_c})$ and $(\mathsf{v_b}, \mathsf{o_b})$, where $-\mathsf{o_r}$ means the opposite orientation to $\mathsf{o_r}$. As a consequence, the formula $(\mathsf{v_r}, \mathsf{o_r}) \to [\otimes_{(\mathsf{v_r}, -\mathsf{o_r})}; \otimes_{(\mathsf{v_c}, \mathsf{o_c})}; \otimes_{(\mathsf{v_b}, \mathsf{o_b})}]\mathsf{OK}$ represents the fact that the robot is moving at the desired velocity and orientation. If the previous formula is not true, the velocity (either its module, or its orientation, or both) has to be corrected. Using again the expressiveness of PDL, we can represent the correction of the velocity (if needed) as follows: Let us suppose that the previous formula is not true. This means that the velocity has to be corrected. For simplicity, let us suppose that the current velocity is $(\mathsf{v}, \mathsf{o}_{j-1})$, instead of $(\mathsf{v}, \mathsf{o}_j)$, that is, the robot is moving towards a orientation which is *on the left* to the desired one. In this case, the correction can be expressed by the following formula:

$$(\mathsf{v}, \mathsf{o}_{j-1}) \to [(\otimes_{(\mathsf{s},\mathsf{o}_{j+1})}; (\neg(\mathsf{v}_r, \mathsf{o}_r)?; (\otimes_{(\mathsf{s},\mathsf{o}_{r+1})})^*; (\mathsf{v}, \mathsf{o}_j)?](\mathsf{v}, \mathsf{o}_j)$$

which means that if the orientation of velocity is to the left of the desired one, *repeat* the composition with a slow velocity towards the right of the desired direction *until* the velocity is the correct one.

From a syntactical point of view, the conditions reflecting the required properties have to be included as axioms of our system. This situation is considered in the following section.

3. The axiom system $\mathcal{QV}$

We introduce an axiom system, called $\mathcal{QV}$, in order to deal with the required properties **P1-P6** presented in the previous section. Let us consider the following specific axioms.

Specific axiom schemata:

For every $\mathsf{v}, \mathsf{v}', \mathsf{v}_r, \mathsf{v}_s, \mathsf{v}_p, \mathsf{v}_q \in L_1$, $\mathsf{o}, \mathsf{o}', \mathsf{o}_j, \mathsf{o}_{j+1}, \mathsf{o}_{j+2} \in L_2$:

S1 $[\otimes_{(\mathsf{v},\mathsf{o})}; \otimes_{(\mathsf{z},\mathsf{n})}]\varphi \leftrightarrow [\otimes_{(\mathsf{v},\mathsf{o})}]\varphi$
S2 $[\otimes_{(\mathsf{v},\mathsf{o}_j)}; \otimes_{(\mathsf{v},\mathsf{o}_{j+2})}]\varphi \leftrightarrow [\otimes_{(\mathsf{z},\mathsf{n})}]\varphi$, $j = 1, 2$.
S3 $(\mathsf{v}, \mathsf{o}_j) \to [\otimes_{(\mathsf{v},\mathsf{o}_{j+1})}]\big((\mathsf{v}, \mathsf{o}_j) \vee (\mathsf{v}, \mathsf{o}_{j+1})\big)$, $j = 1, 2, 3$.
S4 $(\mathsf{v}_r, \mathsf{o}_j) \to [\otimes_{(\mathsf{v}_s,\mathsf{o}_{j+1})}](\mathsf{v}_s, \mathsf{o}_{j+1})$, $j = 1, 2, 3$ and $r < s$.
S5 $(\mathsf{v}_r, \mathsf{o}) \to [\otimes_{(\mathsf{v}_s,\mathsf{o})}](\mathsf{v}_s, \mathsf{o}) \vee (\mathsf{v}_3, \mathsf{o})$, being $r < s$, and $s = 2, 3$.
S6 $(\mathsf{v}_r, \mathsf{o}_j) \to [\otimes_{(\mathsf{v}_s,\mathsf{o}_{j+2})}]\big((\mathsf{v}_s, \mathsf{o}_{j+2}) \vee (\mathsf{v}_q, \mathsf{o}_{j+2})\big)$, being $j = 1, 2$; $r < s$, $s = 2, 3$; and $q = s - 1$.

QE $\bigvee_{(\mathsf{v},\mathsf{o}) \in L_1 \times L_2} (\mathsf{v}, \mathsf{o})$
QU $\star \to \neg\#$ for every $\star \in L_1 \times L_2$ and $\# \in L_1 \times L_2 - \{\star\}$

The previous axioms have the following intuitive meaning:

- **S1-S6** reflect the properties **P1-P6** assumed above.
- **QE** and **QU** mean the existence and uniqueness of the qualitative classes, respectively.

The rest of axioms are those specific to PDL.

Axiom schemata for PDL:

A1 All instances of tautologies of the propositional calculus.
A2 $[a](\varphi \to \psi) \to ([a]\varphi \to [a]\psi)$
A3 $[a](\varphi \wedge \psi) \to ([a]\varphi \wedge [a]\psi)$
A4 $[a \cup b]\varphi \to ([a]\varphi \vee [b]\varphi)$

A5 $[a;b]\varphi \rightarrow [a][b]\varphi$
A6 $[\varphi?]\psi \rightarrow (\varphi \rightarrow \psi)$
A7 $(\varphi \wedge [a][a^*]\varphi) \rightarrow [a^*]\varphi$
A8 $(\varphi \wedge [a^*](\varphi \rightarrow [a]\varphi)) \rightarrow [a^*]\varphi$ (induction axiom)

Inference Rules:
(MP) $\varphi, \varphi \rightarrow \psi \vdash \psi$ (*Modus Ponens*) **(G)** $\varphi \vdash [a]\varphi$ (generalization)

In order to prove the soundness of our system, we give the following result.

Lemma 1. *All the axioms of $\mathcal{QV}$ are valid formulas and all the inference rules preserve validity.*

Proof. As a way of example, we prove the validity of axiom **S4** of $\mathcal{QV}$, that is, $(\mathsf{v_r}, \mathsf{o_j}) \rightarrow [\otimes_{(\mathsf{v_s},\mathsf{o_{j+1}})}](\mathsf{v_s}, \mathsf{o_{j+1}})$, $\mathsf{j} = 1, 2, 3$ and $\mathsf{r} < \mathsf{s}$. We must prove that, for every model $\mathcal{M} = (W, m)$, we have that $m((\mathsf{v_r}, \mathsf{o_j})) \subseteq m([\otimes_{(\mathsf{v_s},\mathsf{o_{j+1}})}](\mathsf{v_s}, \mathsf{o_{j+1}}))$ holds. If $w \in m((\mathsf{v_r}, \mathsf{o_j}))$, then, by definition, $w \in (\mathsf{v_r}, \mathsf{o_j})$. Let $w' \in W$ be such that $(w, w') \in m(\otimes_{(\mathsf{v_s},\mathsf{o_{j+1}})})$, this means that $w' \in m(\otimes_{(\mathsf{v_s},\mathsf{o_{j+1}})})(m((\mathsf{v_r}, \mathsf{o_j})))$ [b]. Now, we use the semantic condition for property **P4**, given above:

$$(4) \quad m(\otimes_{(\mathsf{v_s},\mathsf{o_{j+1}})})(m(\mathsf{v_r}, \mathsf{o_j})) \subseteq m(\mathsf{v_s}, \mathsf{o_{j+1}}), \ \mathsf{j} = 1, 2, 3 \text{ and } \mathsf{r} < \mathsf{s}$$

Thus, we obtain that $w' \in m(\mathsf{v_s}, \mathsf{o_{j+1}})$, that is, $w \in m([\otimes_{(\mathsf{v_s},\mathsf{o_{j+1}})}](\mathsf{v_s}, \mathsf{o_{j+1}}))$. □

As a consequence, we have the soundness of our system as follows.

Theorem 1. *For every formula φ, if φ is a theorem then φ is a valid formula.*

The decidability of the satisfiability problem and the completeness proof for this logic can be obtained following the line used in [10].

For decidability, the *small model property* has to be proved. This property says that if a formula φ is satisfiable, then it is satisfied in a model with no more than $2^{|\varphi|}$ elements, where $|\varphi|$ is the number of symbols of φ. This result can be obtained by the technique of *filtrations* used in modal logic. However, while in modal logic it is used the concept of subformula, in PDL we have to rely on the Fisher-Lander Closure. First of all, we define the following two functions by simultaneous induction, being Φ the set of formulas, Π the set of programs of our logic and $\varphi, \psi \in \Phi$, $a, b \in \Pi$:

$$FL : \Phi \rightarrow 2^{\Phi}; \qquad FL^{\Box} : \{[a]\varphi \mid a \in \Pi, \varphi \in \Phi\} \rightarrow 2^{\Phi}$$

(a) $FL(p) = \{p\}$, for every propositional variable p.
(b) $FL(\star) = \star$, for all $\star \in \mathbb{C}$.
(c) $FL(\varphi \rightarrow \psi) = \{\varphi \rightarrow \psi\} \cup FL(\varphi) \cup FL(\psi)$

[b] Recall that if R is a relation defined in a set W, and $X \subseteq W$, then:

$$R(X) = \{w \in W \mid \text{there exists } x \in X \text{ such that } (x, w) \in R\}$$

(d) $FL(\bot) = \{\bot\}$
(e) $FL([a]\varphi) = FL^{\Box}([a]\varphi) \cup FL(\varphi)$
(f) $FL^{\Box}([a]\varphi) = \{[a]\varphi\}$, being a an atomic program.
(g) $FL^{\Box}([a \cup b]\varphi) = \{[a \cup b]\varphi\} \cup FL^{\Box}([a]\varphi) \cup FL^{\Box}([b]\varphi)$
(h) $FL^{\Box}([a; b]\varphi) = \{[a; b]\varphi\} \cup FL^{\Box}([a][b]\varphi) \cup FL^{\Box}([b]\varphi)$
(i) $FL^{\Box}([a^*]\varphi) = \{[a^*]\varphi\} \cup FL^{\Box}([a][a^*]\varphi)$
(j) $FL^{\Box}([\psi?]\varphi) = \{[\psi?]\varphi\} \cup FL(\psi)$

$FL(\varphi)$ is called the *Fisher-Lander closure* of formula φ.

The following result provides upper bounds for the number of elements of $FL(\varphi)$, denoted also by $|FL(\varphi)|$, in terms of $|\varphi|$. It is proved by simultaneous induction following the ideas presented in [16], taking into account our specific definition of $FL(\star) = \star$, for all $\star \in \mathbb{C}$, in the basis case of this induction.

Lemma 2.

(a) For any formula φ, $|FL(\varphi)| \leq |\varphi|$.
(b) For any formula $[a]\varphi$, $|FL^{\Box}([a]\varphi)| \leq |a|$, being $|a|$ the number of symbols of program a.

We now define the concept of filtration. First of all, given a formula φ and a model (W, m), we define the following equivalence relation on W:

$$u \equiv v \overset{\text{def}}{\Longleftrightarrow} \forall\, \psi \in FL(\varphi)[u \in m(\psi) \text{ iff } v \in m(\psi)]$$

The filtration structure $(\overline{W}, \overline{m})$ of (W, m) by $FL(\varphi)$ is defined on the quotient set $\overline{W} = W/\equiv$, and the qualitative classes in $\overline{W}$ are defined by $\overline{\star} = \{\overline{u} \mid u \in \star\}$, for every $\star \in \mathbb{C}$. Furthermore, the map $\overline{m}$ is defined as follows:

(1) $\overline{m}(p) = \{\overline{u} \mid u \in m(p)\}$, for every propositional variable p.
(2) $\overline{m}(\star) = m(\star) = \star$, for all $\star \in \mathbb{C}$.
(3) $\overline{m}(a) = \{(\overline{u}, \overline{v}) \mid \exists u' \in \overline{u} \text{ and } \exists v' \in \overline{v} \text{ such that } (u', v') \in m(a)\}$, for every atomic program a.

$\overline{m}$ is extended by recursion to compound propositions and programs as described previously in the definition of model.

The following two lemmas are crucial in this section and can be proved again following the ideas presented in [16]. To do this, we have to take into account two facts: first, our definition of Fisher-Lander closure includes the qualitative classes; second, the properties (1)–(6) required to our models for atomic programs, such as $m(\otimes_{(\mathsf{v},\mathsf{o})}; \otimes_{(\mathsf{z},\mathsf{n})}) = m(\otimes_{(\mathsf{v},\mathsf{o})})$, are maintained in the filtration structure, as a direct consequence of our previous definitions.

Lemma 3. $(\overline{W}, \overline{m})$ *is a finite model.*

Now, the Filtration Lemma is as follows.

Lemma 4. *Let (W, m) be a model and $(\overline{W}, \overline{m})$ defined from a formula φ as above. Consider $u, v \in W$, then:*

(1) For all $\psi \in FL(\varphi)$, $u \in m(\psi)$ iff $\overline{u} \in \overline{m}(\psi)$.
(2) For all $[a]\psi \in FL(\varphi)$,
   (a) if $(u, v) \in m(a)$ then $(\overline{u}, \overline{v}) \in \overline{m}(a)$;
   (b) if $(\overline{u}, \overline{v}) \in \overline{m}(a)$ and $u \in m([a]\psi)$, then $v \in m(\psi)$.

As a consequence or the previous lemmas, we can give the following result, called the Small Model Theorem.

Theorem 2. *Let φ a satisfiable formula, then φ is satisfied in a model with no more than $2^{|\varphi|}$ states.*

Proof. If φ is satisfiable, then there exists a model (W, m) and $u \in W$ such that $u \in m(\varphi)$. Let us consider $FL(\varphi)$ the Fisher-Lander closure of φ and the filtration model $(\overline{W}, \overline{m})$ of (W, m) by $FL(\varphi)$ defined previously. From Lemma 3, $(\overline{W}, \overline{m})$ is a finite model and by Lemma 4 (Filtration Lemma), we have that $\overline{u} \in \overline{m}(\varphi)$. As a consequence, φ is satisfied in a finite model. Moreover, $\overline{W}$ has no more elements than the truth assignments to formulas in $FL(\varphi)$, which by Lemma 2 is at most $2^{|\varphi|}$. □

In order to get the completeness of our system, that is, every valid formula is a theorem, we construct a *nonstandard model* from maximal consistent sets of formulas and we use a filtration for nonstandard models to collapse it to a finite *standard* model.

A *nonstandard model* is any structure $\mathcal{N} = (N, m_{\mathcal{N}})$ such as it is a model in the sense of Section 2 in every respect, except that, for every program a, $m_{\mathcal{N}}(a^*)$ needs not be the reflexive and transitive closure of $m_{\mathcal{N}}(a)$, but only a reflexive and transitive relation which contains $m_{\mathcal{N}}(a)$. Given a nonstandard model $(N, m_{\mathcal{N}})$ and a formula φ, we can construct the filtration model $(\overline{N}, \overline{m}_{\mathcal{N}})$ as above, and the Filtration Lemma (Lemma 4) also holds in this case.

As stated before, to obtain completeness, we define a nonstandard model $(N, m_{\mathcal{N}})$ as follows: N contains all the maximal consistent sets of formulas of our logic and $m_{\mathcal{N}}$ is defined, for every formula φ and every program a, by:

$$m_{\mathcal{N}}(\varphi) = \{u \mid \varphi \in u\}; \quad m_{\mathcal{N}}(a) = \{(u, v) \mid \text{for all } \varphi, \text{ if } [a]\varphi \in u \text{ then } \varphi \in v\}$$

Using the previous definition, all the properties for nonstandard models are satisfied, even the ones for our specific atomic programs, as we can see in the following result.

Lemma 5. *$(N, m_{\mathcal{N}})$ verifies all the required properties for non-standard models.*

Proof. As a way of example, let us prove property (4) of models, that is:

$$m_{\mathcal{N}}(\otimes_{(\mathsf{v_s}, \mathsf{o_{j+1}})})(m_{\mathcal{N}}(\mathsf{v_r}, \mathsf{o_j})) \subseteq m_{\mathcal{N}}(\mathsf{v_s}, \mathsf{o_{j+1}}), \mathsf{j} = 1, 2, 3 \text{ and } \mathsf{r} < \mathsf{s}$$

If $u \in m_{\mathcal{N}}(\otimes_{(\mathsf{v_s},\mathsf{o_{j+1}})})(m_{\mathcal{N}}(\mathsf{v_r},\mathsf{o_j}))$, this means that there exists $u' \in m_{\mathcal{N}}(\mathsf{v_r},\mathsf{o_j})$, such that $(u',u) \in m_{\mathcal{N}}(\otimes_{(\mathsf{v_s},\mathsf{o_{j+1}})})$, that is, $(\mathsf{v_r},\mathsf{o_j}) \in u'$. By the above definition of $m_{\mathcal{N}}(\otimes_{(\mathsf{v_s},\mathsf{o_{j+1}})})$, for all formula φ, $[\otimes_{(\mathsf{v_s},\mathsf{o_{j+1}})}]\varphi \in u'$ implies $\varphi \in u$. Now, $(\mathsf{v_r},\mathsf{o_j}) \in u'$ implies, by axiom **S4**, that $[\otimes_{(\mathsf{v_s},\mathsf{o_{j+1}})}](\mathsf{v_s},\mathsf{o_{j+1}}) \in u'$. As a consequence, $(\mathsf{v_s},\mathsf{o_{j+1}}) \in u$, which proves that $u \in m_{\mathcal{N}}(\mathsf{v_s},\mathsf{o_{j+1}})$. □

Now, we can give the following completeness result.

Theorem 3. *For every formula φ, if φ is valid then φ is a theorem.*

Proof. We need to prove that if φ is consistent, then it is satisfied. If φ is consistent, it is contained in a maximal consistent set u, which is a state of the nonstandard model constructed above. By the Filtration Lemma for nonstandard models, φ is satisfied in the state $\overline{u}$ of the filtration model $(\overline{N},\overline{m}_{\mathcal{N}})$. □

4. Towards fuzzy qualitative reasoning

In this section, we introduce some ideas about how our approach could be extended to fuzzy qualitative reasoning, in the line of [37,38]. As velocity and orientation are given by qualitative classes, we consider fuzzy qualitative spaces in order to represent them in *fuzzy qualitative polar coordinates*, as given in [38], and we translate them to *fuzzy qualitative Cartesian coordinates*. Then, we apply fuzzy arithmetic operations [37] in order to obtain the composition of the velocity. First, we consider fuzzy numbers in order to represent the qualitative classes of velocity and orientation. We use the membership distribution of a normal fuzzy number given by the 4-tuple $[a,b,\tau,\beta]$, where $a \leq b$ and $a \times b \geq 0$. In our case, the values of a and b would represent the milestones which determine each qualitative class. For example, if we consider for the module of the velocity the qualitative classes $\mathsf{z},\mathsf{v_1},\mathsf{v_2},\mathsf{v_3}$, (zero, slow, normal and quick) and its values are normalised to the numeric range $[0,1]$, then they could be represented as follows:

$$\mathsf{z} = [0,0,0,0]; \mathsf{v_1} = [0,0.2,0,0.2]; \mathsf{v_2} = [0.4,0.7,0.1,0.2]; \mathsf{v_3} = [0.9,1,0.1,0]$$

Similarly, we could give a representation for the qualitative classes which represent the orientation. Now, we follow the ideas about fuzzy qualitative trigonometry presented in [38]. For simplicity we consider now seven qualitative classes n, fl, l, bl, fr, r, and br to represent, respectively, the none, front-left, left, back-left, front-right, right and back-right orientations. These classes could be represented by fuzzy 4-tuples as follows.

$$\mathsf{n} = [0,0,0,0];$$

$$\mathsf{fl} = [0,\frac{\pi}{4},0.1,0.1]; \mathsf{l} = [\frac{\pi}{4}+0.1,\frac{3\pi}{4}+0.1,0.1,0.1]; \mathsf{bl} = [\frac{3\pi}{4}+0.2,\pi,0.1,0];$$

$$\mathsf{fr} = [-\frac{\pi}{4},0,0.1,0.1]; \mathsf{r} = [-\frac{3\pi}{4}-0.1,-\frac{\pi}{4}-0.1,0.1,0.1]; \mathsf{br} = [-\pi,-\frac{3\pi}{4}-0.2,0,0.1]$$

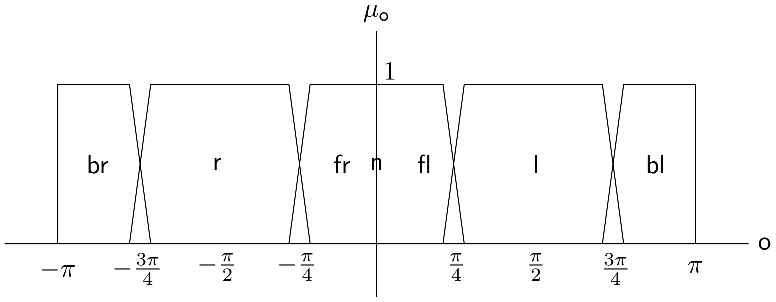

Fig. 3. The fuzzy qualitative classes for orientation

As stated above, the composition of velocities can be obtained from fuzzy qualitative Cartesian coordinates by means of fuzzy arithmetic operations. We have each velocity given by its fuzzy qualitative polar coordinates $[[r_1, r_2, r_3, r_4][\theta_1, \theta_1, \theta_1, \theta_1]]$, where $[r_1, r_2, r_3, r_4]$ represents the fuzzy qualitative velocity (its module) and $[\theta_1, \theta_1, \theta_1, \theta_1]$ the fuzzy qualitative orientation. We could obtain first their fuzzy Cartesian coordinates (for more details, see [38]), and then use the fuzzy definition of sum (that is, $[a, b, \tau, \beta] + [c, d, \gamma, \delta] = [a + c, b + d, \tau + \gamma, \beta + \delta]$) and other fuzzy arithmetic operations, to obtain the desired composition of these velocities. In the previous step, we have to take into account that these velocities could not be given with respect to the same reference system, because, as stated in the introduction, we are considering composition of a velocity of B with respect to A and a velocity of C with respect to B, in order to obtain the velocity of C with respect to A. Thus, we could obtain the required properties for the composition of velocities (similar to our properties **P1-P6**), but in this case these properties are obtained from fuzzy qualitative operations. These properties should have to be expressed as axioms for the new logic constructed for this approach, in a similar way as done in the previous sections. We believe that this approach would give more expressiveness and could be more appropriate when the granularity increases and could be extended following the ideas presented in [24] .

5. Conclusions and future work

A PDL-based extension of the logics for order of magnitude qualitative reasoning has been presented in order to deal with qualitative velocity. We have exploited the expressiveness of PDL not only for dealing with the qualitative composition of velocities with any orientation, but also for using programming commands, such as *while ... do* and *repeat ... until*. An axiom system for a simple case has been defined, which may be extended depending on the application required. The formulas which syntactically express the needed properties have been included in the axiom system. The completeness and decidability of the satisfiability problem of the given logic has been discussed. Moreover, an extension of this logic approach by using fuzzy qualitative reasoning is proposed.

As future work, we consider to exploit the advantages of the fuzzy qualitative approach and the study of completeness and decidability for the proposed extension. Moreover, we will extend this approach to 3D, as in [20], and we will consider other spatial components, such as position, distance, cardinal direction, etc. Last, but not least, we have planned the design of a theorem prover for this logic, in the line of [39].

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