A Logic for Order of Magnitude Reasoning with Negligibility, Non-Closeness and Distance*

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Abstract. This paper continues the research line on the multimodal logic of qualitative reasoning; specifically, it deals with the introduction of the notions non-closeness and distance. These concepts allow us to consider qualitative sum of medium and large numbers. We present a sound and complete axiomatization for this logic, together with some of its advantages by means of an example.

1 Introduction

Qualitative reasoning is an adequate tool for dealing with situations in which information is not sufficiently precise (e.g., exact numerical values are not available) or when numerical models are too complex. A form of qualitative reasoning is to manage numerical data in terms of orders of magnitude (see, for example, [10,11,13,15,17,20]). There are crucial problems in order of magnitude reasoning which remain to be solved: the difficulty to incorporate quantitative information when available, and the difficulty to control the inference process [10]. Two approaches to order of magnitude reasoning have been identified in [20]: Absolute Order of Magnitude, which is represented by a partition of the real line $\mathbb{R}$ where each element belongs to a qualitative class; and Relative Order of Magnitude, introducing a family of binary order of magnitude relations which establishes different comparison relations in $\mathbb{R}$ (e.g. *negligibility*, *closeness* and *distance*). In general, both models need to be combined in order to capture all the relevant information. This fact has led us to define a logic which bridges the absolute and relative order of magnitude models.

Previous works in logic to deal with qualitative reasoning, are presented in [2, 3, 16, 18, 22] for managing qualitative spatial reasoning, qualitative spatio-temporal representations, and the use of branching temporal logics to describe the possible solutions of ordinary differential equations when we have a lack of complete information about a system. However, an analogous development of order of magnitude reasoning from a logical standpoint has received little attention: to the best of our knowledge, the only logics dealing with order-of-magnitude reasoning have been developed in [6–8]. More recently, a relational

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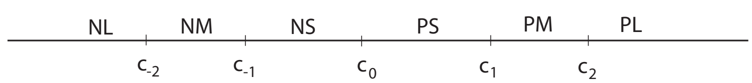
theorem prover has been developed in [9] for the logic of order-of-magnitude with negligibility introduced in [7].

The present paper generalizes the line of research presented in [8], where a notion of *negligibility* relation was considered, by introducing a logic to deal with two new relations: *non-closeness* and *distance* [12, 17, 21] defined in an arbitrarily chosen strict linearly ordered set. We present a sound and complete axiomatization for this logic, together with some of its advantages, which are shown by means of an example. The non-closeness relation is introduced following the ideas of the $Near$ relation in [17], that is, a real number x is $Near$ to y, when $y = x + Small$. If we work with real numbers, our definition says that x is *non-close* to y when either they have different *order of magnitude* or y is obtained by adding a *medium* or *large* number to x. The same idea is introduced to define distance: a real number is *distant* from another one when it is obtained by adding a *large* number. These definitions have the additional advantage that enables us to introducing the operation of qualitative sum of medium and large numbers.

We will consider a strict linearly ordered set $(\mathbb{S}, <)$ [3] divided into seven equivalence classes using five landmarks chosen depending on the context [14,19]. The system considered corresponds to the schematic representation shown below:



where $c_i \in \mathbb{S}$ for $i \in \{-2, -1, 0, 1, 2\}$ such that $c_j < c_{j+1}$ for all $j \in \{-2, -1, 0, 1\}$. In this work we consider the following set of qualitative classes:

$$\text{NL} = (-\infty, c_{-2}), \quad \text{NM} = [c_{-2}, c_{-1}) \quad \text{NS} = [c_{-1}, c_0), \quad \text{C}_0 = \{c_0\}$$

$$\text{PS} = (c_0, c_1], \quad \text{PM} = (c_1, c_2], \quad \text{PL} = (c_2, +\infty)$$

As it could be expected, the labels correspond to “negative large”, “negative medium”, “negative small”, “zero”, “positive small”, “positive medium” and “positive large”, respectively. By convention, the constants c_{-2}, c_2 are considered to belong to the medium-size classes, whereas c_{-1}, c_1 are considered to belong to the small-size classes.

The logic introduced in this paper is a special type of hybrid logic [1] because we just use a finite number of constants (i.e. nominals) which are used not only to represent points but also to represent distances. More differences arise from the specificity of our modal connectives and the fact that we do not have a nominal for each point, this fact would allow us to work with the set of real numbers.

The paper is organized as follows: In Section 2, the concepts of negligibility, non-closeness and distance are introduced; then, syntax and semantics of the proposed logic is introduced in Section 3 and some of its advantages on the basis

[3] for practical purposes, this set could be the real line.

of an example; the axiom system for our language is presented in Section 4. Finally, some conclusions and prospects of future work are presented.

2 Non-Closeness, Distance and Negligibility

As stated in the introduction, we will combine absolute and relative order of magnitude models. For this purpose, regarding the underlying representation model, it seems natural to consider an absolute order of magnitude model with a small number of landmarks, so that the size of the axiom system obtained is reasonable.

The concepts of order of magnitude, non-closeness, distance and negligibility we consider in this paper introduce the 'relative part' of the approach, which builds directly on the 'absolute part' just presented.

First of all, we define the following relation to give the intuitive meaning of constant distance.

Definition 1. *Let* $(\mathbb{S}, <)$ *a strict linearly ordered set which contains the constants* c_i *for* $i \in \{-2, -1, 0, 1, 2\}$ *as defined above. Given* $n \in \mathbb{N}$*, we define* $\overrightarrow{d_\alpha}$ *as a relation in* $\mathbb{S}$ *such that, for every* $x, y, z, x', y' \in \mathbb{S}$*:*

- $c_r \overrightarrow{d_\alpha} c_{r+1}$*, for* $r \in \{-1, 0\}$ *and* $c_s \overrightarrow{d_\alpha}^n c_{s+1}$ [4]*, for* $s \in \{-2, 1\}$*.*
- *If* $x \overrightarrow{d_\alpha} y$*, then* $x < y$
- *If* $x \overrightarrow{d_\alpha} y$ *and* $x \overrightarrow{d_\alpha} z$*, then* $y = z$*.*
- *If* $x \overrightarrow{d_\alpha} y$*,* $x' \overrightarrow{d_\alpha} y'$ *and* $x < x'$ *then* $y < y'$*.*

We denote by $\overleftarrow{d_\alpha}$ *the inverse of relation* $\overrightarrow{d_\alpha}$*.*

We assume in the previous definition that both constants c_{-1} and c_1 are at the same distance (called α) from c_0. Moreover, the distances from c_{-2} to c_{-1} and from c_1 to c_2 are assumed to be a multiple of α (that is, n times α). This choice arises from the idea of taking α as the basic pattern for measuring. As a consequence, the distance between two consecutive constants should be measurable in terms of α.

Definition 2 (Order of Magnitude). *Let* $(\mathbb{S}, <)$ *be defined as above. For every* $x, y \in \mathbb{S}$ *we say that* $x\text{OM}y$ *if and only if* $x, y \in \text{EQ}$*, where* EQ *denotes a qualitative class, that is, an element in the set* $\{\text{NL}, \text{NM}, \text{NS}, \text{C}_0, \text{PS}, \text{PM}, \text{PL}\}$*. Analogously, we define* $x\overline{\text{OM}}y$ *when* x, y *do not belong to the same class.*

Definition 3 (Non-Closeness and Distance). *Let* $(\mathbb{S}, <)$ *and* $n \in \mathbb{N}$ *be given as above. We define the relations* $\overrightarrow{\text{NC}}$ *and* $\overrightarrow{\text{D}}$ *in* $\mathbb{S}$ *as follows:*

$$\begin{array}{lll} x\overrightarrow{\text{NC}}y & \text{if and only if} & \text{either } x\overline{\text{OM}}y \text{ and } x < y \\ & & \text{or there exists } z \in \mathbb{S} \text{ such that } z < y \text{ and } x\overrightarrow{d_\alpha}z \\ x\overrightarrow{\text{D}}y & \text{if and only if} & \text{there exists } z \in \mathbb{S} \text{ such that } z < y \text{ and } x\overrightarrow{d_\alpha}^{n+1}z \end{array}$$

[4] $\overrightarrow{d_\alpha}^n$ is defined by $\overrightarrow{d_\alpha}^1 = \overrightarrow{d_\alpha}$ and $\overrightarrow{d_\alpha}^n = \overrightarrow{d_\alpha} \circ \overrightarrow{d_\alpha}^{n-1}$, for $n \in \mathbb{N}, n \geq 2$, being $\circ$ the usual composition of relations.

We denote by $\overleftarrow{\mathrm{NC}}$ *and* $\overleftarrow{\mathrm{D}}$ *the inverses of relations* $\overrightarrow{\mathrm{NC}}$ *and* $\overrightarrow{\mathrm{D}}$*, respectively.*

If we assume that $\mathbb{S}$ is a set of real numbers, the intuitive interpretation of non-closeness relation is that x is non-close to y if, and only if, either x and y have not the same order of magnitude, or y is obtained from x by adding a medium or large number. On the other hand, x is distant from y if and only if y is obtained from x by adding large number. On the other hand, we introduce the definition of non-closeness instead of closeness directly in order to have an easier way to prove the completeness of the axiom system given later. Nevertheless, as we will see in example below, this definition gives us enough expressive power.

In order to define the negligibility relation, it seems to be reasonable that if $x \neq c_0$ is *neglibible* with respect to y, then x is distant to y. With this aim, we give the following definition.

Definition 4 (Negligibility). *Let* $(\mathbb{S}, <)$ *be defined as above. If* $x, y \in \mathbb{S}$*, we say that* x *is* negligible *with respect to (wrt from now on)* y*, usually denoted* $x\overrightarrow{N}y$*, if and only if, we have one of the following cases:*

$$(i) \quad x = c_0 \qquad (ii) \quad x \in \mathrm{NS} \cup \mathrm{PS} \text{ and, either } c_{-1}\overleftarrow{\mathrm{D}}\,y \text{ or } c_1\overrightarrow{\mathrm{D}}\,y$$

We denote by $\overleftarrow{N}$ *the inverse of relation* $\overrightarrow{N}$*.*

Note that item (i) above corresponds to the intuitive idea that zero is negligible wrt any real number and item (ii) corresponds to the intuitive idea that a number *sufficiently small* is negligible wrt any number *sufficiently large*, independently of the sign of these numbers. This definition ensures that if $x \neq c_0$ and $x\overrightarrow{N}y$, then either $x\overleftarrow{\mathrm{D}}\,y$ or $x\overrightarrow{\mathrm{D}}\,y$.

3 Syntax and Semantics of the Language $\mathcal{L}(OM)^{\mathrm{NCD}}$

The language $\mathcal{L}(OM)^{\mathrm{NCD}}$ is an extension of $\mathcal{L}(OM)$ presented in [8]. To begin with, let us define informally the meaning of the modal connectives we will consider in our language. Their intuitive meanings of some of its connectives are given below (the rest are similar), where A is any formula:

- $\overrightarrow{\Box}A$ means A *is true for all point greater than the current one.*
- $\Box_{\overrightarrow{d_\alpha}}A$ is read A *is true for all point which is greater than the current one and its distance to this one is* α.
- $\Box_{\overrightarrow{\mathrm{N}}}A$ is read A *is true for all point with respect to which the current one is negligible.*
- $\Box_{\overrightarrow{\mathrm{NC}}}A$ is read A *is true for all point which is non-close and greater than the current one.*
- $\Box_{\overrightarrow{\mathrm{D}}}A$ is read A *is true for all point which is distant from and greater than the current one.*

The syntax of our logic is the usual modal propositional language on the modal connectives described above and a set of specific constants to denote the landmarks. Formally, the alphabet of our language is defined by using:

– A stock of atoms or propositional variables, $\mathcal{V}$.
– The classical connectives $\neg, \wedge, \vee, \rightarrow$ and the constant symbols $\top$ and $\bot$.
– The unary modal connectives $\overrightarrow{\Box}$, $\overleftarrow{\Box}$,$\Box_{\overrightarrow{\mathcal{R}}}$, $\Box_{\overleftarrow{\mathcal{R}}}$ being $\mathcal{R} \in \{d_\alpha, \textsc{nc}, \textsc{d}, N\}$.
– The finite set of specific constants defined by $\mathcal{C} = \{\overline{c}_{-2}, \overline{c}_{-1}, \overline{c}_0, \overline{c}_1, \overline{c}_2\}$.
– The auxiliary symbols (,).

Well-formed formulae of $\mathcal{L}(OM)^{\text{NCD}}$ are generated from $\mathcal{V} \cup \mathcal{C}$ by the construction rules of classical propositional logic plus the following rule which introduces the modal connectives:
If A is a formula, then so are $\overrightarrow{\Box}A$, $\overleftarrow{\Box}A$, $\Box_{\overrightarrow{\mathcal{R}}}A$ and $\Box_{\overleftarrow{\mathcal{R}}}A$ being $\mathcal{R} \in \{d_\alpha, \textsc{nc}, \textsc{d}, N\}$

As usual, the *mirror image* of A is the result of replacing in A the occurrences of $\overrightarrow{\Box}$, $\overleftarrow{\Box}$, $\Box_{\overrightarrow{\mathcal{R}}}$, $\Box_{\overleftarrow{\mathcal{R}}}$, $\overline{c}_j$, $\overline{c}_0$ by $\overleftarrow{\Box}$, $\overrightarrow{\Box}$, $\Box_{\overleftarrow{\mathcal{R}}}$, $\Box_{\overrightarrow{\mathcal{R}}}$, $\overline{c}_{-j}$ and $\overline{c}_0$, respectively, being $j \in \{-2, -1, 1, 2\}$.

Moreover, we use $\overrightarrow{\Diamond}$, $\overleftarrow{\Diamond}$, $\Diamond_{\overrightarrow{\mathcal{R}}}$, $\Diamond_{\overleftarrow{\mathcal{R}}}$ as abbreviations, respectively, of $\neg\overrightarrow{\Box}\neg$, $\neg\overleftarrow{\Box}\neg$, $\neg\Box_{\overrightarrow{\mathcal{R}}}\neg$ and $\neg\Box_{\overleftarrow{\mathcal{R}}}\neg$.

Definition 5. *A* qualitative frame *for $\mathcal{L}(OM)^{\text{NCD}}$ or, simply a frame, is a tuple $\Sigma = (\mathbb{S}, <, \overrightarrow{\mathcal{R}}, \overleftarrow{\mathcal{R}})$, being $(\mathbb{S}, <)$ a strict linearly ordered set which contains the constants c_i for $i \in \{-2, -1, 0, 1, 2\}$ as defined above, and $\mathcal{R} \in \{d_\alpha, \textsc{nc}, \textsc{d}, N\}$ are respectively the relations on $\mathbb{S}$ given in Definitions 1, 3 and 4.*

We can now give the definition of qualitative model. In its formulation, given R any relation in a set X and $x \in X$, we write $R(x)$ with the usual meaning:

$$R(x) = \{x' \in X \mid xRx'\}$$

Definition 6. *Let $\Sigma = (\mathbb{S}, <, \overrightarrow{\mathcal{R}}, \overleftarrow{\mathcal{R}})$ be a qualitative frame for $\mathcal{L}(OM)^{\text{NCD}}$, a* qualitative model *for Σ (or, simply Σ-model) is an ordered pair $\mathcal{M} = (\Sigma, h)$ where $h : \mathcal{V} \to 2^{\mathbb{S}}$ is a function called* interpretation. *Any interpretation can be uniquely extended to the set of all formulae in $\mathcal{L}(OM)^{\text{NCD}}$ (also denoted by h) by means of the usual conditions for the classical boolean connectives and for $\top$, $\bot$, and the following conditions, being $\mathcal{R} \in \{d_\alpha, \textsc{nc}, \textsc{d}, N\}$, and $i \in \{-2, -1, 0, 1, 2\}$ [5]:*

$$h(\overrightarrow{\Box}A) = \{x \in \mathbb{S} \mid (x, +\infty) \subseteq h(A)\} \qquad h(\overleftarrow{\Box}A) = \{x \in \mathbb{S} \mid (-\infty, x) \subseteq h(A)\}$$
$$h(\Box_{\overrightarrow{\mathcal{R}}}A) = \{x \in \mathbb{S} \mid \overrightarrow{\mathcal{R}}(x) \subseteq h(A)\} \qquad h(\Box_{\overleftarrow{\mathcal{R}}}A) = \{x \in \mathbb{S} \mid \overleftarrow{\mathcal{R}}(x) \subseteq h(A)\}$$
$$h(\overline{c}_i) = \{c_i\}$$

The concepts of truth and validity are defined in a standard way.

Notice that the connectives $\Box_{\overrightarrow{\textsc{nc}}}$, $\Box_{\overrightarrow{\textsc{d}}}$ allow us to manage the concepts of non-closeness and distance defined above which were not introduced in [8]. Thus, we extend the example presented in this previous paper with some uses of these new concepts.

[5] Note that these algebraic conditions for modal connectives are based on the intuitive meanings presented above.

Example 1. Let us suppose that we want to specify the behaviour of a device to automatically control the temperature, for example, in a museum, subject to have some specific conditions.

If we have to maintain the temperature close to some limit T, for practical purposes any value of the interval $[T-\epsilon, T+\epsilon]$ for small ϵ is admissible. Then the extreme points of this interval can be considered as the milestones c_{-1} and c_1, respectively.

Moreover, assume that if the temperature is out of this interval (for example, because the number of people within the museum is changing), it is necessary to put into operation some *heating* or *cooling* system. In addition, we have another interval $[T-\lambda, T+\lambda]$, such that if the temperature does not belong to this interval, we need to use an extra system of *cooling* or *heating*, because the default system is not enough. Now, the extreme points of this interval are the milestones c_{-2} and c_2, respectively.

We also assume that, when the normal system of *cooling* or *heating* is operating, a system to maintain the humidity is needed, and when the extra system is operating, we also need an extra system of humidification.

The qualitative classes $\textsc{nl}, \textsc{nm}, \textsc{ns} \cup \textsc{c}_0 \cup \textsc{ps}, \textsc{pm}$ and $\textsc{pl}$ can be interpreted by Very_Cold, Cold, Ok, Hot and Very_Hot, respectively. The following conditions specify the general behaviour of the system:

$$\textsc{Ok} \rightarrow \textit{off} \qquad \textsc{Very\_Cold} \rightarrow \textit{X-heating}$$

$$\textsc{Cold} \rightarrow \textit{heating} \qquad \textsc{Hot} \rightarrow \textit{cooling}$$

$$\textsc{Very\_Hot} \rightarrow \textit{X-cooling} \qquad (\textsc{Cold} \vee \textsc{Hot}) \rightarrow \textit{humidifier}$$

$$(\textsc{Very\_Cold} \vee \textsc{Very\_Hot}) \rightarrow \textit{X-humidifier}$$

The following formulae introduce relations among actions:

$$\textit{X-heating} \rightarrow (\neg \textit{heating} \wedge \neg \textit{off} \wedge \neg \textit{cooling} \wedge \neg \textit{X-cooling} \wedge \textit{X-humidifier})$$

$$\textit{heating} \rightarrow (\textit{humidifier} \wedge \neg \textit{X-cooling} \wedge \neg \textit{cooling} \wedge \neg \textit{off})$$

$$\textit{off} \rightarrow (\neg \textit{X-cooling} \wedge \neg \textit{cooling} \wedge \neg \textit{humidifier} \wedge \neg \textit{X-humidifier})$$

$$\textit{cooling} \rightarrow (\neg \textit{X-cooling} \wedge \textit{humidifier}) \qquad \textit{X-cooling} \rightarrow \textit{X-humidifier}$$

$$\textit{humidifier} \rightarrow (\textit{cooling} \vee \textit{heating}) \qquad \textit{X-humidifier} \rightarrow \neg \textit{humidifier}$$

where *off* means that the system is *off*, *cooling* means that we use the normal system of *cooling* and *X-cooling* means that we need to use an extra cooling system. Analogously, we have the meaning of *heating*, *X-heating*, *humidifier* and *X-humidifier*.

Some consequences of the previous specification that are obtained by using the proposed axiom system are the following:

1. The conditionals in the proper axioms turn out to be bi-conditionals, that is, we also have: $\textit{off} \rightarrow \textsc{Ok}$, $\textit{cooling} \rightarrow \textsc{Hot}$, etc.
2. $\textit{cooling} \rightarrow \overrightarrow{\Box}(\neg \textit{X-cooling} \rightarrow \textit{humidifier})$

3. $(\mathit{off} \land \neg \bar{c}_0) \rightarrow \Box_{\overrightarrow{\text{N}}}$ *X-humidifier*
4. $(\mathit{X\text{-}cooling} \lor \mathit{X\text{-}heating}) \rightarrow \Box_{\overleftarrow{\text{N}}}(\neg \mathit{humidifier} \land \neg \mathit{X\text{-}humidifier})$
5. $(\textsc{Ok} \land \overleftarrow{\Diamond}\,\bar{c}_0) \rightarrow (\Box_{\overrightarrow{\text{NC}}}(\mathit{humidifier} \lor \mathit{X\text{-}humidifier}) \land \Box_{\overrightarrow{\text{D}}}\mathit{X\text{-}humidifier})$
6. $\textsc{Hot} \rightarrow \Box_{\overrightarrow{\text{D}}}$ *X-humidifier*

We give now the intuitive meanings for the previous formulae.

- Formula 2 means that if the cooling system is running and the temperature increases, while the extra cooling system were not put in operation, the humidifier system is enough to maintain the desired conditions.
- Formula 3 says that if the system is off, but the temperature is not c_0, for every value wrt the current one is negligible, the extra humidifier system is needed.
- Formula 4 means that if the extra cooling or extra heating system are operating, the values which are negligible wrt that ones are not using neither humidifier nor humidifier systems.
- Formula 5 can be read in this way: if the temperature is Ok but greater than $\bar{c}_0$ and it is incremented by a medium or large positive value to obtain a non-close value, then we have to use the *humidifier* or *extra humidifier* system because the *cooling* or *heating* systems have been put into operation. Moreover, if this temperature is incremented by a positive large value to obtain a distant value, then we have to use the *extra humidifier* system.
- Formula 6 means that if the temperature is Hot and is incremented to obtain a distant value, then we have to use the *extra humidifier* system.

If we assume that the system is more efficient (in terms of energy saving) if the temperature is Ok and close to the milestone c_1, that is close but no greater, the following formula must be true:

$$\bar{c}_1 \rightarrow (\Box_{\text{NC}}\mathit{non\text{-}efficient} \land \Box_{\text{D}}\mathit{warning})\ ^{6}$$

This formula means that for every temperature non-close (smaller or greater) to c_1, the system is not running efficiently and if the temperature is distant to c_1, the system is wasting very much energy. Notice that, as c_1 is a milestone, every value greater than c_1 is not in the same order of magnitude and, as a consequence of Definition 3, it is non-close to c_1.

The following section is devoted to the axiomatization of this logic. For simplicity, from now on, we will assume that $n = 1$ in Definition 1, that is, the distance between every two consecutive constants is α. On the other hand, we will only consider modal connectives $\overrightarrow{\Box}, \overleftarrow{\Box}, \Box_{\overrightarrow{d_\alpha}}, \Box_{\overleftarrow{d_\alpha}}$, because the connectives $\Box_{\overrightarrow{\text{N}}}, \Box_{\overrightarrow{\text{NC}}}, \Box_{\overrightarrow{\text{D}}}$ (and its inverses) can be defined by using only the first ones. As

[6] We use $\Box_{\mathcal{R}}A$ as an abbreviation of $\Box_{\overleftarrow{\mathcal{R}}}A \land \Box_{\overrightarrow{\mathcal{R}}}A$, for $\mathcal{R} \in \{d_\alpha, \text{NC}, \text{D}, N\}$

an example, we give the definition of $\Box_{\overrightarrow{NC}}$:

$$\Box_{\overrightarrow{NC}} A \equiv \Box_{\overrightarrow{d_\alpha}} \overrightarrow{\Box} A \wedge (\bigvee_{j=0}^{2} \bar{c}_j \to \overrightarrow{\Box} A) \wedge$$
$$\wedge \bigwedge_{s=-2}^{0} (\overrightarrow{\Diamond} \bar{c}_s \to \overrightarrow{\Box}((\bar{c}_s \vee \overleftarrow{\Diamond} \bar{c}_s) \to A)) \wedge \bigwedge_{r=1}^{2} (\overrightarrow{\Diamond} \bar{c}_r \to \overrightarrow{\Box}(\overleftarrow{\Diamond} \bar{c}_r \to A))$$

4 Axiom system for $\mathcal{L}(OM)^{\mathrm{NCD}}$

We will denote OM^{NCD} the axiom system containing all the tautologies of classical propositional logic together with the following axiom schemata:

Axiom schemata for modal connectives:

K1 $\overrightarrow{\Box}(A \to B) \to (\overrightarrow{\Box} A \to \overrightarrow{\Box} B)$
K2 $A \to \overrightarrow{\Box} \overleftarrow{\Diamond} A$
K3 $\overrightarrow{\Box} A \to \overrightarrow{\Box} \overrightarrow{\Box} A$
K4 $\big(\overrightarrow{\Box}(A \vee B) \wedge \overrightarrow{\Box}(\overrightarrow{\Box} A \vee B) \wedge \overrightarrow{\Box}(A \vee \overrightarrow{\Box} B)\big) \to (\overrightarrow{\Box} A \vee \overrightarrow{\Box} B)$

Axiom schemata for constants:

C1 $\overleftarrow{\Diamond} \overline{c_i} \vee \overline{c_i} \vee \overrightarrow{\Diamond} \overline{c_i}$, where $i \in \{-2, -1, 0, 1, 2\}$
C2 $\overline{c_i} \to (\overleftarrow{\Box} \neg \overline{c_i} \wedge \overrightarrow{\Box} \neg \overline{c_i})$, being $i \in \{-2, -1, 0, 1, 2\}$

Axiom schemata for specific modal connectives:

d1 $\Box_{\overrightarrow{d_\alpha}}(A \to B) \to (\Box_{\overrightarrow{d_\alpha}} A \to \Box_{\overrightarrow{d_\alpha}} B)$
d2 $A \to \Box_{\overrightarrow{d_\alpha}} \Diamond_{\overleftarrow{d_\alpha}} A$.
d3 $\bar{c}_j \to \Diamond_{\overrightarrow{d_\alpha}} \bar{c}_{j+1}$, where $j \in \{-2, -1, 0, 1\}$ [7].
d4 $(\Diamond_{\overrightarrow{d_\alpha}} A \wedge \overrightarrow{\Diamond} \Diamond_{\overrightarrow{d_\alpha}} B) \to \overrightarrow{\Diamond}(A \wedge \overrightarrow{\Diamond} B)$
d5 $\Diamond_{\overrightarrow{d_\alpha}} A \to \Box_{\overrightarrow{d_\alpha}} A$
d6 $\overrightarrow{\Box} A \to \Box_{\overrightarrow{d_\alpha}} A$

We also consider as axioms the corresponding mirror images of K1–K4 and d1–d6.

Rules of Inference:

(MP) Modus Ponens for $\to$
(R$\overrightarrow{\Box}$) If $\vdash A$ then $\vdash \overrightarrow{\Box} A$
(R$\overleftarrow{\Box}$) If $\vdash A$ then $\vdash \overleftarrow{\Box} A$

[7] This is the unique axiom which is affected by our previous assumption that $n = 1$ in Definition 1.

Theorem 1 (Soundness and Completeness).

– *Every theorem of* OM^{NCD} *is a valid formula of* $\mathcal{L}(OM)^{\mathrm{NCD}}$.
– *Every valid formula of* $\mathcal{L}(OM)^{\mathrm{NCD}}$ *is a theorem of* OM^{NCD}.

The soundness of the axiom system is straightforward. Regarding completeness, a *step-by-step* proof (see, for example, [4] and [5]) can be given in the following terms: Given any consistent formula A , we have to prove that A is satisfiable. With this purpose, the step-by-step method defines a qualitative frame $\Sigma = (\mathbb{S}, <, \overrightarrow{\mathcal{R}}, \overleftarrow{\mathcal{R}})$ and a function f_Σ which assigns maximal consistent sets to any element of $\mathbb{S}$, such that $A \in f_\Sigma(x)$ for some $x \in \mathbb{S}$. The process to build such a frame is recursive, and follows the ideas of [7]: firstly, a pre-frame is generated which is later completed to an initial finite frame; later, successive extensions of this initial frame are defined until Σ is obtained. Although the method of proof is the same, the technical problems which arise from the use of this more complex language need special attention. Due to lack of space, the formal details are omitted.

5 Conclusions and Future Work

A multimodal logic for order of magnitude reasoning to deal with negligibility, non-closeness and distance has been introduced which enriches previous works in this line of research by introducing in some way qualitative sum of medium and large numbers. Some of the advantages of this logic have been studied on the basis of an example.

As a future work, our plans are to study the decidability and complexity of this logic. Last, but not least, we want to give a relational proof system based on dual tableaux for this extension in the line of [9].

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