Order of magnitude qualitative reasoning with bidirectional negligibility*

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Abstract. In this paper, we enrich the logic of order of magnitude qualitative reasoning by means of a new notion of negligibility which has very useful properties with respect to operations of real numbers. A complete axiom system is presented for the proposed logic, and the new negligibility relation is compared with previous ones and its advantages are presented on the basis of an example.

1 Introduction

Qualitative reasoning is an emergent area of AI. It is an adequate tool for dealing with situations in which information is not sufficiently precise (e.g., numerical values are not available).

A form of qualitative reasoning is to manage numerical data in terms of orders of magnitude (OM) (see, for example, [7–9,11,13,15]). There are two approaches for OM that can be combined: Absolute Order of Magnitude (AOM), which is represented by a partition of the real line $\mathbb{R}$ and each element of $\mathbb{R}$ belongs to a qualitative class and Relative Order of Magnitude (ROM), introducing a family of binary order of magnitude relations which establishes different comparison relations in $\mathbb{R}$ (e.g., *comparability*, *negligibility* and *closeness* [13]).

Several logics have been defined to deal with qualitative reasoning, such as the Region Connection Calculus [2, 14] for managing qualitative spatial reasoning; or the multimodal logics used in [3, 17] to deal with qualitative spatio-temporal representations, and the use of branching temporal logics to describe the possible solutions of ordinary differential equations when we have a lack of complete information about a system [12]: however, an analogous development of order-of-magnitude reasoning from a logical approach standpoint has received little attention: to the best of our knowledge, the only logics dealing with order-of-magnitude reasoning have been developed in [5, 6]. The present paper continues the line of research of a logical approach to order-of-magnitude reasoning.

Regarding the underlying representation model, it seems natural to consider an absolute order of magnitude model with a small number of landmarks, so that the size of the proof system obtained is reasonable. Following usual practice, we

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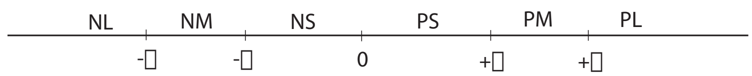
will divide the real line into seven equivalence classes using five landmarks chosen depending on the context [10, 16].

The system considered corresponds to the schematic representation shown in the picture below



were α, β are two positive real numbers such that $\alpha <_{\mathbb{R}} \beta$, and $\leq_{\mathbb{R}}$ the usual order in $\mathbb{R}$. Moreover, in this work we consider that:

$$\begin{array}{llll} \mathrm{NL} = (-\infty, -\beta), & \mathrm{NM} = [-\beta, -\alpha), & \mathrm{NS} = [-\alpha, 0), & [0] = \{0\}, \\ & \mathrm{PS} = (0, \alpha], & \mathrm{PM} = (\alpha, \beta], & \mathrm{PL} = (\beta, +\infty) \end{array}$$

The labels correspond to "negative large", "negative medium", "negative small", "zero", "positive small", "positive medium" and "positive large", respectively.

The main result of this paper is the introduction of a new bidirectional negligibility relation in the logic for qualitative reasoning with orders of magnitude introduced in [6]. This new negligibility relation is "more quantitative" than the original one and, thus, is closer to the one presented in [11] but less complex. As a consequence, our approach (between purely qualitative and purely quantitative) allows us to have the expressive power of the logic and a lot of useful properties of the negligibility relation.

The notion of negligibility that will be used in the rest of the paper is given in the following definition:

Definition 1. *Given $\alpha, \beta, x, y \in \mathbb{R}$, such that $0 <_{\mathbb{R}} \alpha <_{\mathbb{R}} \beta$, we say that x is* **negligible** *with respect to (wrt from now on) y, in symbols $x \, N_{\mathbb{R}} \, y$, if and only if, we have one of the following possibilities:*

$$(i) \quad x = 0 \qquad (ii) \quad x \in \mathrm{NS} \cup \mathrm{PS} \text{ and } y \in \mathrm{NL} \cup \mathrm{PL}$$

Note that item (i) above corresponds to the intuitive idea that 0 is negligible wrt any real number and item (ii) corresponds to the intuitive idea that a number *sufficiently small* is negligible wrt any number *sufficiently large*, independently of the sign of these numbers. Thus, we have that $N_{\mathbb{R}}$ is not a restriction of $<_{\mathbb{R}}$. In this way, we say that $N_{\mathbb{R}}$ is *bidirectional*.

The paper is organized as follows: In Section 2, the syntax and semantics of the proposed logic is introduced; in Section 3, the axiom system for our language is presented; in Section 4, a comparison with alternative definitions of negligibility is done, and an example is presented on which some features of the proposed relation are shown. Finally, some conclusions and future work are presented.

2 Syntax and Semantics of the Language $\mathcal{L}(MQ)^{N_{\mathbb{R}}}$

In a similar way to [6], we define the connectives $\overrightarrow{\Box}$, $\overleftarrow{\Box}$, $\Box_N$ and $\overline{\Box}_N$ to deal with the relations $<$ and N, respectively. The intuitive meanings of each modal connective is as follows:

- $\overrightarrow{\Box}\, A$ means A *is true for all element greater than the current one.*
- $\overleftarrow{\Box}\, A$ means A *is true for all number less than the current one.*
- $\Box_N A$ is read A *is true for all number from which the current one is negligible.*
- $\overline{\Box}_N A$ is read A *is true for all number which is negligible from the current one.*

Now, we define the language $\mathcal{L}(MQ)^{N_{\mathbb{R}}}$ of our logic. The alphabet of $\mathcal{L}(MQ)^{N_{\mathbb{R}}}$ is defined by using:

– A stock of atoms or propositional variables, $\mathcal{V}$.
– The classical connectives $\neg, \wedge, \vee, \rightarrow$ and the constants $\top$ and $\bot$.
– The unary modal connectives $\overrightarrow{\Box}$, $\overleftarrow{\Box}$, $\Box_N$ and $\overline{\Box}_N$.
– The finite set of specific constants $\mathcal{C}$ defined by $\mathcal{C} = \{\beta^-, \alpha^-, \overline{0}, \alpha^+, \beta^+\}$.
– The auxiliary symbols $($, $)$.

Formulae of $\mathcal{L}(MQ)^{N_{\mathbb{R}}}$ are generated from $\mathcal{V} \cup \mathcal{C}$ by the construction rules of classical propositional logic adding the following rule:

If A is a formula, then so are $\overrightarrow{\Box}\, A$, $\overleftarrow{\Box}\, A$, $\Box_N A$ and $\overline{\Box}_N A$. The *mirror image* of A is the result of replacing in A the occurrence of $\overrightarrow{\Box}, \overleftarrow{\Box}$, $\Box_N$, $\overline{\Box}_N$, $\beta^-, \alpha^-, \overline{0}, \alpha^+, \beta^+$ by $\overleftarrow{\Box}, \overrightarrow{\Box}$, $\overline{\Box}_N$, $\Box_N$, $\beta^+, \alpha^+, \overline{0}, \alpha^-, \beta^-$ respectively. As usual in modal logic, we use $\overrightarrow{\Diamond}$, $\overleftarrow{\Diamond}$, $\Diamond_N$, and $\overline{\Diamond}_N$ as abbreviations respectively of $\neg \overrightarrow{\Box} \neg$, $\neg \overleftarrow{\Box} \neg$, $\neg \Box_N \neg$ and $\neg \overline{\Box}_N \neg$.

Definition 2. *A* **qualitative frame** *for* $\mathcal{L}(MQ)^{N_{\mathbb{R}}}$ *is a tuple* $\Sigma = (\mathbb{S}, C, <, N)$, *where:*

1. $\mathbb{S}$ *is a nonempty set of real numbers.*
2. $C \subseteq \mathbb{S}$ *where* $C = \{-\beta, -\alpha, 0, \alpha, \beta\}$.
3. $<$ *and* N *are, respectively, the restriction to* $\mathbb{S}$ *of the relations* $<_{\mathbb{R}}$ *and* $N_{\mathbb{R}}$ *above.*

If condition 2 it not fulfilled, we say that $(\mathbb{S}, <, N)$ *is a* **pre-frame**.

Notation: We will sometimes assume the following notations, in order to simplify the presentation of results:

- $c_1 = -\beta, c_2 = -\alpha, c_3 = 0, c_4 = +\alpha, c_5 = +\beta$
- $\overline{c}_1 = \beta^-, \overline{c}_2 = \alpha^-, \overline{c}_3 = \overline{0}, \overline{c}_4 = \alpha^+ \overline{c}_5 = \beta^+$
- If $X \subseteq \mathbb{R}$, R is a relation in X and $a \in X$:

$$R(a) = \{a' \in \mathbb{R} \mid aRa'\}$$
$$R^{-1}(a) = \{a' \in \mathbb{R} \mid a'Ra\}$$

Definition 3. *Let* $\Sigma = (\mathbb{S}, C, <, N)$ *be a qualitative frame for* $\mathcal{L}(MQ)^{N_{\mathbb{R}}}$*, a* **qualitative model** *for* Σ *(or, simply* Σ*-model) is an ordered pair* $\mathcal{M} = (\Sigma, h)$ *where* $h : \mathcal{V} \to 2^{\mathbb{S}}$ *is a function called* **interpretation**. *Any interpretation can be uniquely extended to the set of all formulae in* $\mathcal{L}(MQ)^{N_{\mathbb{R}}}$ *(also denoted by* h*) by means of the usual conditions for the classical boolean connectives and for* $\top$*,* $\bot$*, and the following conditions* [3]*:*

- $h(\overrightarrow{\Box} A) = \{x \in \mathbb{S} \mid (x, +\infty) \subseteq h(A)\}$.
- $h(\overleftarrow{\Box} A) = \{x \in \mathbb{S} \mid (-\infty, x) \subseteq h(A)\}$.
- $h(\Box_N A) = \{x \in \mathbb{S} \mid N(x) \subseteq h(A)\}$.
- $h(\overline{\Box}_N A) = \{x \in \mathbb{S} \mid N^{-1}(x) \subseteq h(A)\}$.
- $h(\overline{c}_i) = \{c_i\}$ *for all* $i \in \{1, 2, 3, 4, 5\}$.

The concepts of truth and validity are defined in a standard way.

3 Axiom system for $\mathcal{L}(MQ)^{N_{\mathbb{R}}}$

We will denote $MQ^{N_{\mathbb{R}}}$ the axiom system with all the tautologies of classical propositional logic together with the following axiom schemata:

Axiom schemata for modal connectives:

K1 $\overrightarrow{\Box} (A \to B) \to (\overrightarrow{\Box} A \to \overrightarrow{\Box} B)$

K2 $A \to \overrightarrow{\Box} \overleftarrow{\Diamond} A$

K3 $\overrightarrow{\Box} A \to \overrightarrow{\Box} \overrightarrow{\Box} A$

K4 $\big(\overrightarrow{\Box} (A \vee B) \wedge \overrightarrow{\Box} (\overrightarrow{\Box} A \vee B) \wedge \overrightarrow{\Box} (A \vee \overrightarrow{\Box} B)\big) \to (\overrightarrow{\Box} A \vee \overrightarrow{\Box} B)$

Axiom schemata for constants:

C1 $\overleftarrow{\Diamond} \overline{c}_i \vee \overline{c}_i \vee \overrightarrow{\Diamond} \overline{c}_i$, where $i \in \{1, 2, 3, 4, 5\}$

C2 $\overline{c}_i \to (\overleftarrow{\Box} \neg \overline{c}_i \wedge \overrightarrow{\Box} \neg \overline{c}_i)$, being $i \in \{1, 2, 3, 4, 5\}$

C3 $\overline{c}_i \to \overrightarrow{\Diamond} \overline{c}_{i+1}$, where $i \in \{1, 2, 3, 4\}$.

Axiom schemata for negligibility connectives:

N1 $\Box_N (A \to B) \to (\Box_N A \to \Box_N B)$

N2 $A \to \Box_N \overline{\Diamond}_N A$

N3 $(\overleftarrow{\Box} A \wedge A \wedge \overrightarrow{\Box} A) \to \Box_N A$

N4 $(\overrightarrow{\Diamond} \alpha^- \vee \overleftarrow{\Diamond} \alpha^+) \to \Box_N \bot$

N5 $\overline{0} \to (\Box_N A \to (\overleftarrow{\Box} A \wedge A \wedge \overrightarrow{\Box} A))$

N6 $(\neg \overline{0} \wedge (\alpha^- \vee (\overleftarrow{\Diamond} \alpha^- \wedge \overrightarrow{\Diamond} \alpha^+) \vee \alpha^+)) \to \Box_N (\overrightarrow{\Diamond} \beta^- \vee \overleftarrow{\Diamond} \beta^+)$

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[3] These algebraic conditions for modal connectives are based on the intuitive meanings presented above.

N7 $\big(\neg\bar{0} \wedge (\alpha^- \vee (\overleftarrow{\Diamond}\, \alpha^- \wedge \overrightarrow{\Diamond}\, \alpha^+) \vee \alpha^+)\big) \rightarrow$

$$\big(\Box_N A \rightarrow (\overleftarrow{\Box}\, (\overrightarrow{\Diamond}\, \beta^- \rightarrow A) \wedge \overrightarrow{\Box}\, (\overleftarrow{\Diamond}\, \beta^+ \rightarrow A)\big)$$

We also consider as axioms the corresponding mirror images of axioms K1-K4, and axioms N1-N3.

Rules of Inference:

(MP) Modus Ponens for $\rightarrow$
(R$\overrightarrow{\Box}$) If $\vdash A$ then $\vdash \overrightarrow{\Box}\, A$
(R$\overleftarrow{\Box}$) If $\vdash A$ then $\vdash \overleftarrow{\Box}\, A$

Informal reading of specific axioms is the following:

- Axioms C1–C3 formalize, respectively, the existence, uniqueness and ordering of the constants.
- Axioms N1-N2 are standard in modal logic.
- Axiom N3 means that the bidirectional relation N is a restriction of $< \cup >$.
- Axiom N4 expresses that neither large nor medium elements are negligible with respect to any number.
- Axiom N5 means that 0 is negligible wrt every number.
- Axiom N6 means that all elements wrt which small elements are negligible are large. Axiom N7 means that any small number is negligible wrt any large number. Summarising, axioms N6 and N7 mean that $x \neq 0$ is negligible wrt y if and only if x is small and y is large (as expressed in Definition 1).

Theorem 1 (Soundness and Completeness).

- *Every theorem of* $MQ^{N_{\mathbb{R}}}$ *is a valid formula of* $\mathcal{L}(MQ)^{N_{\mathbb{R}}}$.
- *Every valid formula of* $\mathcal{L}(MQ)^{N_{\mathbb{R}}}$ *is a theorem of* $MQ^{N_{\mathbb{R}}}$.

The soundness of the axiom system is straightforward.
Regarding completeness, a *step-by-step* proof (see, for example, [1] and [4]) can be given in the following terms:

Given any consistent formula A , we have to prove that A is satisfiable. With this purpose, the step-by-step method defines a qualitative frame $\Sigma = (\mathbb{S}, C, <, N)$ and a function f_Σ which assigns maximal consistent sets to any element of $\mathbb{S}$, such that $A \in f_\Sigma(x)$ for some $x \in \mathbb{S}$. The process to build such a frame is recursive, and follows the ideas of [6]: firstly, a pre-frame is generated which is later completed to an initial finite frame; later, successive extensions of this initial frame are defined until Σ is obtained.

Although the method of proof is the same, the technical problems which arise from the use of this more complex language need special attention. Due to lack of space, the formal details are omitted.

4 Comparison of $N_{\mathbb{R}}$ with other definitions of negligibility

In the definition of our negligibility relation $N_{\mathbb{R}}$, we pursue a balance between the applicability to different types of problems and the complexity of the logic that we want to construct. For this reason, this section is devoted to compare the properties of $N_{\mathbb{R}}$ with other definitions of negligibility, in particular, the ones presented in [6, 11].

The negligibility relation presented in [6], denoted by $\prec$ is defined as a restriction of $<_{\mathbb{R}}$ satisfying the following properties: (i) If $x \prec y <_{\mathbb{R}} z$, then $x \prec z$; (ii) If $x <_{\mathbb{R}} y \prec z$, then $x \prec z$; (iii) If $x \prec y$, then either $x \notin INF$ or $y \notin INF$, being $\alpha \in \mathbb{R}$ and considering the real line divided in three equivalence classes using two landmarks α and $-\alpha$ as follows:

$$\begin{aligned} OBS^- &= \{x \in \mathbb{R} \mid x \leq_{\mathbb{R}} -\alpha\}; \\ INF &= \{x \in \mathbb{R} \mid -\alpha \leq_{\mathbb{R}} x \leq_{\mathbb{R}} \alpha\}; \\ OBS^+ &= \{x \in \mathbb{R} \mid \alpha \leq_{\mathbb{R}} x\} \end{aligned}$$

The main differences between $N_{\mathbb{R}}$ and $\prec$ are that $\prec$ is a restriction of $<_{\mathbb{R}}$ and $N_{\mathbb{R}}$ is not. Indeed $N_{\mathbb{R}}$ is bidirectional and this fact allows us to compare positive and negative numbers. Another difference is that $N_{\mathbb{R}}$ uses the standard division of the real line in seven classes while $\prec$ only uses three. Thus, $N_{\mathbb{R}}$ verifies a version of property (iii) in the definition of $\prec$, taking into account that we have a different number of intervals and substituting INF by $NS \cup PS$. $N_{\mathbb{R}}$ satisfies neither (i) nor (ii) because, as commented above, $N_{\mathbb{R}}$ is not a restriction of $<_{\mathbb{R}}$. However, if we work just with positive numbers, then the two properties are fulfilled. $N_{\mathbb{R}}$ verifies one of the properties of quasi-density studied in [6], i.e. if $x\, N_{\mathbb{R}}\, y$, then there exists $z \in \mathbb{R}$ such that $x\, N_{\mathbb{R}}\, z <_{\mathbb{R}} y$, for all $x \in \mathbb{R}$ but does not satisfy the other one, i.e. if $x\, N_{\mathbb{R}}\, y$, then there exists $z \in \mathbb{R}$ such that $x <_{\mathbb{R}} z\, N_{\mathbb{R}}\, y$, because it fails when $x = 0$ or $x = \alpha$. However, none of these properties is obtained from the definition of $\prec$.

On the other hand, the negligibility relation (also bidirectional) presented in [11], denoted by *Ne*, is defined as follows: Given $\alpha, \beta \in \mathbb{R}$ such that $0 <_{\mathbb{R}} \alpha <_{\mathbb{R}} \beta$, then $x\ Ne\ y$ if and only if, $|\frac{x}{y}| <_{\mathbb{R}} \frac{\alpha}{\beta}$. The main difference wrt our relation is that *Ne* is not supported by a logic whereas $N_{\mathbb{R}}$ is; in addition, $N_{\mathbb{R}}$ maintains the majority of the properties of *Ne* as we can see below.

Proposition 1. *Consider $x, y, z, t \in \mathbb{R}$, then we have:*

1. $0\, N_{\mathbb{R}}\, x$
2. *If $y \neq 0$ and $x\, N_{\mathbb{R}}\, y$, then $|x| < |y|$.*
3. *If $x \neq 0$ and $x\, N_{\mathbb{R}}\, y$, then it is not the case that $y\, N_{\mathbb{R}}\, x$*
4. *If $x\, N_{\mathbb{R}}\, y$ and $y\, N_{\mathbb{R}}\, z$, then $x\, N_{\mathbb{R}}\, z$*
5. *If $x\, N_{\mathbb{R}}\, y$, then $(\pm x)\, N_{\mathbb{R}}\, (\pm y)$*
6. *If $x\, N_{\mathbb{R}}\, y$, $|z| \leq |x|$ and $|y| \leq |t|$, then $z\, N_{\mathbb{R}}\, t$*
7. *If $x\, N_{\mathbb{R}}\, y$, then $sign(x + y) = sign(y)$*
8. *If $sign(x+y) = +$ and $sign(x) = -$, then $sign(y) = +$ and it is not the case that $y\, N_{\mathbb{R}}\, x$*

9. *If* $x\, N_{\mathbb{R}}\, y$ *and* $sign(y) = sign(z)$, *then* $x\, N_{\mathbb{R}}\, (y+z)$
10. *If* $y \neq 0$ *and* $(x-y)\, N_{\mathbb{R}}\, y$, *then it is not the case that* $y\, N_{\mathbb{R}}\, x$
11. *If* $\beta \geq 2\alpha$, $y \in PL$ *and* $x\, N_{\mathbb{R}}\, y$, *then* $x + y \in PM \cup PL$.
12. *If* $\beta \geq 2\alpha$, $y \in NL$ *and* $x\, N_{\mathbb{R}}\, y$, *then* $x + y \in NM \cup NL$.
13. *If* $x\, N_{\mathbb{R}}\, y$, $|z| \leq 1 \leq |t|$, *then* $zx\, N_{\mathbb{R}}\, ty$

Summarising, we can say that $N_{\mathbb{R}}$ presents different features than $\prec$ which make it closer to *Ne* and, consequently, $N_{\mathbb{R}}$ is somewhere in between the other two relations, sharing with them some qualitative and some quantitative interesting properties, apart from the fact that both $\prec$ and $N_{\mathbb{R}}$ are founded on a logic.

We finish this section with the next example which is devoted to specify a small use of our logic. We want to emphasize the use of the bidirectionallity and properties of our negligibility relation. Moreover, the use of the axiom system allows us to obtain some interesting properties from the specification of the example.

Example 1. Let us suppose that we want to specify the behaviour of a device to automatically control the temperature, for example, in a museum, where we need to have some specific conditions. If we have to maintain the temperature close to some limit T, for practical purposes any value of the interval $[T-\epsilon, T+\epsilon]$ for small ϵ is admissible. Then the extreme points of this interval can be considered as the milestones $-\alpha$ and α, respectively. Moreover, assume that if the temperature is out of this interval (for example, because the number of people within the museum is changing), it is necessary to put into operation some *heating* or *cooling* system. In addition, we have another interval $[T-\lambda, T+\lambda]$, such that if the temperature does not belong to this interval, we need to use an extra system of *cooling* or *heating*, because the normal system is not enough. Now, the extreme points of this interval are the milestones $-\beta$ and β, respectively. We also assume that when the normal system of *cooling* or *heating* is operating, a system to maintain the humidity is needed, and when the extra system is operating, we also need an extra system of humidification.

The intervals $\mathrm{NL}, \mathrm{NM}, \mathrm{NS} \cup [0] \cup \mathrm{PS}, \mathrm{PM}$ and PL can be interpreted by VERY_COLD, COLD, OK, HOT and VERY_HOT, respectively. The proper axioms that specify the general behaviour of the system are stated below:

OK $\rightarrow$ *off*	VERY_COLD $\rightarrow$ *X-heating*
COLD $\rightarrow$ *heating*	HOT $\rightarrow$ *cooling*
VERY_HOT $\rightarrow$ *X-cooling*	(COLD $\vee$ HOT) $\rightarrow$ *humidifier*

(VERY_COLD $\vee$ VERY_HOT) $\rightarrow$ *X-humidifier*

The following formulae introduce relations among actions:

$$\textit{X-heating} \to (\neg \textit{heating} \wedge \neg \textit{off} \wedge \neg \textit{cooling} \wedge \neg \textit{X-cooling} \wedge \textit{X-humidifier})$$
$$\textit{heating} \to (\textit{humidifier} \wedge \neg \textit{extra-cooling} \wedge \neg \textit{cooling} \wedge \neg \textit{off})$$
$$\textit{off} \to (\neg \textit{X-cooling} \wedge \neg \textit{cooling} \wedge \neg \textit{humidifier} \wedge \neg \textit{X-humidifier})$$
$$\textit{cooling} \to (\neg \textit{X-cooling} \wedge \textit{humidifier})$$
$$\textit{X-cooling} \to \textit{X-humidifier}$$
$$\textit{humidifier} \to (\textit{cooling} \vee \textit{heating})$$
$$\textit{X-humidifier} \to \neg \textit{humidifier}$$

where *off* means that the system is *off*, *cooling* means that we use the normal system of *cooling* and *X-cooling* means that we need to use an extra cooling system. Analogously, we have the meaning of *heating*, *X-heating*, *humidifier* and *X-humidifier*.

Some consequences of the previous specification that are obtained by using the proposed axiom system are the following:

1. The conditionals in the proper axioms turn out to be bi-conditionals, that is, we also have: $\textit{off} \to \textsc{Ok}$, $\textit{cooling} \to \textsc{Hot}$, etc.
2. $\neg \textit{off} \to \Box_N \bot$
3. $(\textit{cooling} \vee \textit{X-cooling}) \to (\textit{humidifier} \vee \textit{X-humidifier})$
4. $\textit{cooling} \to \overrightarrow{\Box} (\neg \textit{X-cooling} \to \textit{humidifier})$
5. $(\textit{off} \wedge \neg \overline{0}) \to \Box_N \textit{X-humidifier}$
6. $\textit{humidifier} \to \overline{\Box}_N (\neg \overline{0} \to \bot)$
7. $\textit{X-humidifier} \to \overline{\Box}_N \textit{off}$
8. $(\textit{X-cooling} \vee \textit{X-heating}) \to \overline{\Box}_N (\neg \textit{humidifier} \wedge \neg \textit{X-humidifier})$

If we assume the intuitive hypothesis $\beta \geq 2\alpha$, the properties 11 and 12 of Proposition 1 can be applied in this example as follows, where p represents *the temperature is incremented (positively or negatively) in the actual value*:

$$(\textsc{Ok} \wedge \neg \overline{0} \wedge \Diamond_N p) \to (\textit{humidifier} \vee \textit{X-humidifier})$$

This formula can be read in this way: if the temperature is Ok but nonzero and is incremented in a value from which the actual value is negligible, then we have to use the *humidifier* or extra *humidifier* system because the *cooling* or *heating* systems have been put into operation.

5 Conclusions and future work

A sound and complete system of qualitative order-of-magnitude reasoning has been introduced with a new notion of negligibility. This new negligibility relation is between those presented in [11] and [6], thus sharing convenient properties from both the qualitative and the quantitative sides. As a result, we have been able to construct a logic with a reasonable number of axioms, which is useful to work in situations where we need to use different properties of the negligibility relation.

As future work, our plan is to investigate a logic with modal operators which represent, in some way, the sum and product of real numbers. Thus, it would be very interesting for the applications (see, for example [13]) to study the behaviour of the negligibility relation $N_{\mathbb{R}}$ wrt the sum and product of real numbers as we have commented in Section 4. It is easy to see that if $x \, N_{\mathbb{R}} \, y$ and $z \in \mathbb{R}$, neither $(x+z) \, N_{\mathbb{R}} \, (y+z)$ nor $xz \, N_{\mathbb{R}} \, yz$ are true in general; thus, we are interested in finding new relations which guarantee such kind of properties under certain conditions. Specifically, we will investigate the definition of new relations to obtain the definability of those properties of sum and product introduced in Proposition 1, in order to extend the current logic with the corresponding new modal connectives.

Last but not least, we are planning to develop theorem provers for our system based either on tableaux or on resolution.

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