Indexed Flows in Temporal $\times$ Modal Logic with Functional Semantics

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Abstract

*Two classical semantical approaches to studying logics which combine time and modality are the $\mathcal{T} \times \mathcal{W}$-frames and Kamp-frames (see Thomason, 84). In this paper we study a new kind of frame that extends the one introduced in [Burrieza and P. de Guzmán(2002)]. The motivation is twofold: theoretical, i.e., representing properties of the basic theory of functions (definability); and practical, their use in computational applications (considering time-flows as memory of computers connected in a net, each computer with its own clock). Specifically, we present a temporal $\times$ modal (labelled) logic, whose semantics are given by ind-*functional frames *in which* accessibility *functions are used in order to interconnect time-flows. This way, we can: (i) specify to what time-flow we want to go; (ii) carry out different comparisons among worlds with different time measures, and (iii) define properties of certain kinds of functions (in particular, of total, injective, surjective, constant, increasing and decreasing functions), without the need to resort to second-order theories. In addition, we define a minimal axiomatic system and give the* completeness theorem *(Henkin-style).*

1. Introduction

Two classical semantical approaches to study logics which combine time and modality are the $\mathcal{T} \times \mathcal{W}$-frames and Kamp-frames (see Thomason, 84). In this paper we study a new kind of frame that extends the one introduced in [Burrieza and P. de Guzmán(2002)]. The motivation is twofold: theoretical, i.e., representing properties of the basic theory of functions (definability); and practical, their use in computational applications (considering time-flows as memory of computers connected in a net, each computer with its own clock).

The semantic approach introduced in [Burrieza and P. de Guzmán(2002)], named *functional*, allows us to establish connections among time-flows in very different ways, which enables us to carry out different comparisons among worlds with different time measures. These connections are made by means of functions, called *accessibility* functions, and not by means of equivalence relations, as in $\mathcal{T} \times \mathcal{W}$-frames and Kamp-frames approaches. The theoretical study is interesting itself but also, in our opinion, and according to our experience when contacting users (who require applications for information and communication technologies), the functional approach considered in this paper is more adequate to the specifications used by them.

Specifically, we present a temporal $\times$ modal (labelled) logic, whose semantics are given by ind-*functional frames* in which *accessibility* functions are used in order to interconnect time-flows. This way, we can: (i) specify to what time-flow we want to go; (ii) carry out different comparisons among worlds with different time measures, and (iii) define basic properties of some kinds of functions (in particular, of total, injective, surjective, constant, increasing, and decreasing functions), without the need to resort to second-order theories. Thus, we require to label time-flows, so that our language will include indexed-modal connectives, that is, $< i >$-type connectives, so that an expression such as $<i> A$ can be intuitively read as follows: "A is true in flow i at the image of the reference instant (or where I am)".

In addition, we define a minimal axiomatic system and give the *completeness theorem* (Henkin-style).

The article is organized as follows: in Sect. 2 we introduce the family of temporal-(indexed-modal) languages and define its (algebraic-style) semantics. In Sect. 3 we present the results about definability of function properties. Finally, in Sect. 4 we give the proof of completeness of a minimal system for partial functions.

2. Languages $\mathcal{L}^{\mathcal{F}}_{(T\times W)}$-$\mathcal{I}$

Given a denumerable set of indices $\mathcal{I}$, the alphabet of the language $\mathcal{L}^{\mathcal{F}}_{(T\times W)}$-$\mathcal{I}$ is defined as follows: 1) a denumerable set, $\mathcal{V}$, of propositional variables; 2) the logic constants

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$\top$ and $\perp$, and the boolean connectives $\neg, \wedge, \vee$ and $\rightarrow$; 3) the temporal connectives G and H; 4) a family of unary modal connectives of the form $<i>$, for $i \in \mathcal{I}$.
The well-formed formulas (wffs) are generated by the construction rules of classical propositional logic, adding the following rule: If A is a wff, then GA, HA and $<i>A$ are wffs. We consider, as usual, the connectives F, P and $[i]$ to be defined connectives. The connectives G and H have their usual readings, but $<i>A$ has the following meaning: *A is true in flow i, at the image of the reference instant (from which I execute or speak)*. For its part, $[i]A$ has the following non-existential meaning: *if there exists an image of the reference instant in flow i, then A is true at such an image* and $[i]A$ is true at the reference instant when this has no i-image. So, if such an image exists in flow i, $<i>A$ has the same meaning as $[i]A$.

2.1. Semantics for $\mathcal{L}^{\mathcal{F}}_{(T\times W)}$-$\mathcal{I}$

Definition 1 *An ind-functional frame for $\mathcal{L}^{\mathcal{F}}_{(T\times W)}$-$\mathcal{I}$ is a tuple $\Sigma = (W, \mathcal{T}, \mathcal{F})$, where W is a nonempty set (set of labels for a set of time-flows), $\mathcal{T}$ is a nonempty set of strict linear orders, indexed by W, specifically:*

$\mathcal{T} = \{(T_w, <_w) \mid w \in W\}$, *where each T_w is non empty and, if $w \neq w'$, then $T_w \cap T_{w'} = \emptyset$.*

Finally, $\mathcal{F}$ is a set of non-empty functions, called **accessibility functions***, such that:*

1. *each function in $\mathcal{F}$ is a partial function from T_w to $T_{w'}$, for some $w \in W$ and some $w' \in W \cap \mathcal{I}$* [1]*;*
2. *for an arbitrary pair $(w, w') \in W \times (W \cap \mathcal{I})$, there is (in $\mathcal{F}$) at most one accessibility function from T_w to $T_{w'}$, denoted by $\xrightarrow{w\ w'}$.*

We will denote $\mathcal{F}_w = \{\xrightarrow{w\ w'} \in \mathcal{F} \mid w \in W\}$. Then $\mathcal{F} = \bigcup_{w\in W} \mathcal{F}_w$. The elements t_w of the disjoint union $Coor_\Sigma = \bigoplus_{w\in W} T_w$ are called **coordinates**.

Note that the definition of W and $\mathcal{T}$ depends only on the set (of labels) W, whereas $\mathcal{F}$ depends on W and $\mathcal{I}$.

Notation 1 *If $t_w \in Coor_\Sigma$ and $\mathcal{C} \subseteq Coor_\Sigma$:*

- $[t_w, \rightarrow) = \{t'_w \mid t_w \leq_w t'_w\}$; $(t_w, \rightarrow) = \{t'_w \mid t_w <_w t'_w\}$.
- $(\leftarrow, t_w] = \{t'_w \mid t'_w \leq_w t_w\}$; $(\leftarrow, t_w) = \{t'_w \mid t'_w <_w t_w\}$.
- $\mathcal{C}\uparrow = \bigcup_{t_w\in\mathcal{C}} (t_w, \rightarrow)$; $\mathcal{C}\overline{\underline{\uparrow}} = \bigcup_{t_w\in\mathcal{C}} [t_w, \rightarrow)$.
- $\mathcal{C}\downarrow = \bigcup_{t_w\in\mathcal{C}} (\leftarrow, t_w)$; $\mathcal{C}\overline{\underline{\downarrow}} = \bigcup_{t_w\in\mathcal{C}} (\leftarrow, t_w]$.

Definition 2 *An ind-functional model for $\mathcal{L}^{\mathcal{F}}_{(T\times W)}$-$\mathcal{I}$ is a tuple (Σ, h), where $\Sigma = (W, \mathcal{T}, \mathcal{F})$ is an ind-functional frame and h is a function $h : \mathcal{L}^{\mathcal{F}}_{(T\times W)}\text{-}\mathcal{I} \longrightarrow 2^{Coor_\Sigma}$, called an* **ind-functional interpretation***, satisfying the following conditions: The interpretation of the constants and of the boolean connectives is defined as usual,*

- $h(GA) = \{t_w \in Coor_\Sigma \mid (t_w, \rightarrow) \subseteq h(A)\}$
- $h(HA) = \{t_w \in Coor_\Sigma \mid (\leftarrow, t_w) \subseteq h(A)\}$
- $h(<i>A) = \{t_w \in Coor_\Sigma \mid \xrightarrow{w\ i} (\{t_w\}) \cap h(A) \neq \emptyset\}$.

[2]

Thus, the semantics of $[i]$ is the following:

$$h([i]A) = \{t_w \in Coor_\Sigma \mid \xrightarrow{w\ i} (\{t_w\}) \subseteq h(A)\}$$

The notions of satisfiability, validity and logical equivalence (denoted $\equiv$) are defined in a standard way. As a consequence of the semantics, $[i]A$ is valid in every ind-functional frame, $\Sigma = (W, \mathcal{T}, \mathcal{F})$ such that $\mathcal{I} \cap W = \emptyset$.

3 A Minimal Axiomatic System for $\mathcal{L}^{\mathcal{F}}_{(T\times W)}$-$\mathcal{I}$

In this section we introduce a minimal axiomatic system for $\mathcal{L}^{\mathcal{F}}_{(T\times W)}$-$\mathcal{I}$ to work with partial functions.

3.1 The system $\mathcal{S}^{\mathcal{F}}_{(T\times W)-\mathcal{I}}$-Parc

This system has the following axiom schemes:

1. Those of the minimal system of propositional linear temporal logic $\mathcal{K}_l$ and, for each $i \in \mathcal{I}$ the schema
$$[i](A \rightarrow B) \rightarrow ([i]A \rightarrow [i]B).$$
2. The following characteristic axiom schemes: for each $i \in \mathcal{I}$,
2.1 $<i>A \rightarrow [i]A$ (**Functionality**)
2.2 $(\lambda <i>A \wedge \lambda' <i>B) \rightarrow \lambda <i>(A \wedge (B \vee FB \vee PB))$ (**Confluence**)
where:
$$\begin{cases} \lambda = \gamma_1 <j_1> \gamma_2 \ldots <j_n> \gamma_{n+1}, & \gamma_i \in \{F, P, \epsilon\}, \\ \lambda' = \gamma'_1 <k_1> \gamma'_2 \ldots <k_m> \gamma'_{m+1}, & \gamma_i \in \{F, P, \epsilon\}, \end{cases}$$
being $n, m \geq 1$; $k_i, j_i \in \mathcal{I}$ and ϵ the empty chain.

The rules of inference are those of $\mathcal{K}_l$ and $\dfrac{A}{[i]A}$, for each $i \in \mathcal{I}$.

The concepts of *proof* and *theorem* are defined as usual.

Proposition 1 *The schema*

$$(<i>A \wedge <i>B) \rightarrow <i>(A \wedge B)$$

is a theorem of $\mathcal{S}^{\mathcal{F}}_{(T\times W)-\mathcal{I}}$-Parc.

[1] Note that we don't require that $\mathcal{I} \subseteq W$. Thus, the notion of validity can be given in a standard way.

[2] Consider that, in order that $<i>A$ be true in $t_w \in Coor_\Sigma$, it is necessary that $i \in \mathcal{I} \cap W$.

3.2 Definability of basic properties of functions

Definition 3 *Let $\mathcal{J}$ be a class of ind-functional frames and $\mathcal{K} \subseteq \mathcal{J}$. We say that $\mathcal{K}$ is $\mathcal{L}^{\mathcal{F}}_{(T\times W)}$-$\mathcal{I}$-* **definable in** $\mathcal{J}$ *if there exists a set Γ of formulas in $\mathcal{L}^{\mathcal{F}}_{(T\times W)}$-$\mathcal{I}$ such that for every frame $\Sigma \in \mathcal{J}$, we have that $\Sigma \in \mathcal{K}$ if and only if every formula of Γ is valid in Σ. If $\mathcal{J}$ is the class of all ind-functional frames, we say that $\mathcal{K}$ is $\mathcal{L}^{\mathcal{F}}_{(T\times W)}$-$\mathcal{I}$-***definable***. *Let P be a property of functions (injectivity, etc.) and $\mathcal{K}$ the class of all ind-functional frames whose functions have the property P. We say that P is $\mathcal{L}^{\mathcal{F}}_{(T\times W)}$-$\mathcal{I}$-definable if $\mathcal{K}$ is $\mathcal{L}^{\mathcal{F}}_{(T\times W)}$-$\mathcal{I}$-definable.*

Theorem 1 *The following classes of ind-functional frames are $\mathcal{L}^{\mathcal{F}}_{(T\times W)}$-$\mathcal{I}$-definable:*

$\mathcal{K}_1 = \{(W,\mathcal{T},\mathcal{F}) \mid \mathcal{F}$ *is a class of total functions* $\}$
$\mathcal{K}_2 = \{(W,\mathcal{T},\mathcal{F}) \mid \mathcal{F}$ *is a class of non total functions* $\}$
$\mathcal{K}_3 = \{(W,\mathcal{T},\mathcal{F}) \mid \mathcal{F}$ *is a class of total constant functions*$\}$
$\mathcal{K}_4 = \{(W,\mathcal{T},\mathcal{F}) \mid \mathcal{F}$ *is a class of total injective functions*$\}$
$\mathcal{K}_5 = \{(W,\mathcal{T},\mathcal{F}) \mid \mathcal{F}$ *is a class of surjective functions* $\}$
$\mathcal{K}_6 = \{(W,\mathcal{T},\mathcal{F}) \mid \mathcal{F}$ *is a class of total increasing functions*$\}$
$\mathcal{K}_7 = \{(W,\mathcal{T},\mathcal{F}) \mid \mathcal{F}$ *is a class of total strictly increasing functs.*$\}$
$\mathcal{K}_8 = \{(W,\mathcal{T},\mathcal{F}) \mid \mathcal{F}$ *is a class of total decreasing functions*$\}$
$\mathcal{K}_9 = \{(W,\mathcal{T},\mathcal{F}) \mid \mathcal{F}$ *is a class of total strictly decreasing functs.*$\}$

Proof:
We prove some of the items. For the rest, we give the set of formulas which defines the corresponding class.

1. It is sufficient to check that $\mathcal{K}_1$ is defined by the following set of formulas, denoted (Tot)-ind:

$$\{<i>(A \wedge GA \wedge HA) \rightarrow (G<i>A \wedge H<i>A) \mid i \in \mathcal{I}\}$$

In order to prove this result, we state that $\xrightarrow{w\ i}$ is a total function if and only if, for all $t_w \in T_w$:

$$\xrightarrow{w\ i}((t_w,\rightarrow)) \cup \xrightarrow{w\ i}((\leftarrow,t_w)) \overset{\dagger_1}{\subseteq} \xrightarrow{w\ i}(\{t_w\})\underline{\uparrow} \cup \xrightarrow{w\ i}(\{t_w\})\downarrow$$

Thus, we have that for all $i \in \mathcal{I}$

$$\begin{array}{ll} t_w \in h(<i>(A \wedge GA \wedge HA)) & iff \\ t_w \in h(<i>A) \cap h(<i>GA) \cap h(<i>HA) & iff \\ \xrightarrow{w\ i}(\{t_w\}) \cup (\xrightarrow{w\ i}(\{t_w\}),\rightarrow) \cup (\leftarrow, \xrightarrow{w\ i}(\{t_w\})) \subseteq h(A) & iff \\ \xrightarrow{w\ i}(\{t_w\})\underline{\uparrow} \cup \xrightarrow{w\ i}(\{t_w\})\downarrow \subseteq h(A)) & \end{array}$$

and, on the other hand, $t_w \in h(G<i>A)$ if and only if $\xrightarrow{w\ i}((t_w,\rightarrow)) \subseteq h(A)$. Analogously, $t_w \in h(H<i>A)$ if and only if $\xrightarrow{w\ i}((\leftarrow,t_w)) \subseteq h(A)$. Now, it is sufficient to consider $\dagger_1$ to finish the demonstration of validity.

Reciprocally, if $\Sigma = (W,\mathcal{T},\mathcal{F}) \notin \mathcal{K}_1$, then there is a non total function $\xrightarrow{w\ i} \in \mathcal{F}$. In this case, there exists a $t_w \in Coor_\Sigma$ such that $\dagger_1$ does not hold for t_w, that is, there exists a $t_i \in \xrightarrow{w\ i}((t_w,\rightarrow)) \cup \xrightarrow{w\ i}((\leftarrow,t_w))$ but $t_i \notin \xrightarrow{w\ i}(\{t_w\})\underline{\uparrow} \cup \xrightarrow{w\ i}(\{t_w\})\downarrow$. Now, since $i \in \mathcal{I}$, to refute the formula $<i>(p \wedge Gp \wedge Hp) \rightarrow (G<i>p \wedge H<i>p)$ in $t'_w \neq t_w$, such that $\xrightarrow{w\ i}(t'_w) = t_i$, it is enough to define an interpretation function h so that $h(p) = Coor_\Sigma$.

2. We shall prove that $\mathcal{K}_2$ is defined by the following set of formulas:

$$(Non\text{-}Tot)\text{-}ind \quad \{[i]\bot \vee F[i]\bot \vee P[i]\bot \mid i \in \mathcal{I}\}$$

Indeed, for every ind-functional frame Σ we have $\Sigma = (W,\mathcal{T},\mathcal{F}) \in \mathcal{K}_2$ if and only if for all $T_w \in \mathcal{T}$ and $\xrightarrow{w\ i} \in \mathcal{F}$, there is at least $t_w \in T_w$ such that t_w is not in the domain of $\xrightarrow{w\ i}$. From this, it should be clear that $t_w \in h([i]\bot)$ if and only if t_w does not belong to the domain of $\xrightarrow{w\ i}$. Then, considering the linearity of $<_w$, we have that $[i]\bot \vee F[i]\bot \vee P[i]\bot$ holds at every coordinate in T_w if and only if $\xrightarrow{w\ i}$ is a non-total function.

3. $\mathcal{K}_3$ is defined by $(Tot\text{-}Con)$-ind:
$\{<i>A \rightarrow (G<i>A \wedge H<i>A) \mid i \in \mathcal{I}\}$
4. $\mathcal{K}_4$ is defined by $(Tot\text{-}Inj)$-ind:
$\{<i>(GA \wedge HA) \rightarrow (G<i>A \wedge H<i>A)) \mid i \in \mathcal{I}\}$
5. $\mathcal{K}_5$ is defined by $(Surj)$-ind :
$\{(G[i]A \wedge H[i]A)) \rightarrow [i](GA \wedge HA) \mid i \in \mathcal{I}\}$
6. $\mathcal{K}_6$ is defined by $(Tot\text{-}Inc)$-ind: for all $i \in \mathcal{I}$,
$<i>(A \wedge GA) \rightarrow G<i>A))$ and
$<i>(A \wedge HA) \rightarrow H<i>A))$
7. $\mathcal{K}_7$ is defined by $(Tot\text{-}Str\text{-}Inc)$:
$\{<i>GA \rightarrow G<i>A, <i>HA \rightarrow H<i>A \mid i \in \mathcal{I}\}$
8. $\mathcal{K}_8$ is defined by $(Tot\text{-}Dec)$-ind: for all $i \in \mathcal{I}$,
$<i>(A \wedge GA) \rightarrow H<i>A)$ and
$<i>(A \wedge HA) \rightarrow G<i>A)$
9. $\mathcal{K}_9$ is defined by $(Tot\text{-}Str\text{-}Dec)$-$ind$:
$\{<i>GA \rightarrow H<i>A,\ <i>HA \rightarrow G<i>A \mid i \in \mathcal{I}\}$

4 Completeness of $\mathcal{S}^{\mathcal{F}}_{(T\times W)-\mathcal{I}}$-Parc

The proof of soundness is standard. In order to study the completeness, we assume the familiarity with the basic properties of maximally consistent sets in the propositional classical logic, and their standard definition, in the system $\mathcal{S}^{\mathcal{F}}_{(T\times W)-\mathcal{I}}$-Parc. We will denote by $\mathcal{MC}$ the family of maximally consistent sets (from now on, mc-sets).

We start with the following definitions, of interest for the rest of the development.

Definition 4 *Let $\Gamma_1, \Gamma_2 \in \mathcal{MC}$ and $i \in \mathcal{I}$. Then we define:*

(a) $\Gamma_1 \prec_T \Gamma_2$ *if and only if* $\{A \mid GA \in \Gamma_1\} \subseteq \Gamma_2$

(b) $\Gamma_1 \prec_i \Gamma_2$ *if and only if* $\emptyset \neq \{A \mid <i>A \in \Gamma_1\} \subseteq \Gamma_2$.

Definition 5 *In $\mathcal{MC}$ we define the following equivalence relation, denoted $\sim_T$. If $\Gamma_1, \Gamma_2 \in \mathcal{MC}$, then:*
$\Gamma_1 \sim_T \Gamma_2$ *iff* $[\Gamma_1 \prec_T \Gamma_2,$ *or* $\Gamma_2 \prec_T \Gamma_1,$ *or* $\Gamma_1 = \Gamma_2]$

Definition 6 *In $\mathcal{MC}$ we define the relation $\prec_i^{\sim T}$ as follows. If $\Gamma_1, \Gamma_2 \in \mathcal{MC}$ and $i \in \mathcal{I}$, then $\Gamma_1 \prec_i^{\sim T} \Gamma_2$ if and only if one of the two following conditions is satisfied:*

- $\Gamma_1 \sim_T \Gamma_2$.
- *There are some Γ_3 and Γ_4 such that $\Gamma_1 \sim_T \Gamma_3$, $\Gamma_3 \prec_i \Gamma_4$ and $\Gamma_4 \sim_T \Gamma_2$.*

Definition 7 *We define the relation $\searrow$ in $\mathcal{MC}$ as follows: if $\Gamma, \Gamma' \in \mathcal{MC}$, then: $\Gamma \searrow \Gamma'$ if and only if there are $n \geq 0$, $i_1, \ldots, i_n \in \mathcal{I}$ and $\Gamma_0, \ldots, \Gamma_n \in \mathcal{MC}$ such that*

$$\Gamma = \Gamma_0 \prec_{i_1}^{\sim T} \Gamma_1 \prec_{i_2}^{\sim T} \Gamma_2 \prec_{i_3}^{\sim T} \ldots \prec_{i_n}^{\sim T} \Gamma_n = \Gamma'$$

The following lemma is standard in modal and tense logic.

Lemma 1

1. *Any consistent set of formulas in $\mathcal{S}^{\mathcal{F}}_{(T\times W)-\mathcal{I}}$-Parc can be extended to an mc-set in $\mathcal{S}^{\mathcal{F}}_{(T\times W)-\mathcal{I}}$-Parc (Lindenbaum's lemma)*
2. *Let $\Gamma_1 \in \mathcal{MC}$ and $i \in \mathcal{I}$:*
   (a) *If $FA \in \Gamma_1$, there exists $\Gamma_2 \in \mathcal{MC}$ such that $\Gamma_1 \prec_T \Gamma_2$ and $A \in \Gamma_2$.*
   (b) *If $PA \in \Gamma_1$, there exists $\Gamma_2 \in \mathcal{MC}$ such that $\Gamma_2 \prec_T \Gamma_1$ and $A \in \Gamma_2$.*
   (c) *If $<i> A \in \Gamma_1$, there exists $\Gamma_2 \in \mathcal{MC}$ such that $\Gamma_1 \prec_i \Gamma_2$ and $A \in \Gamma_2$.*
3. *Let $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{MC}$. If $\Gamma_1 \prec_T \Gamma_2$ and $\Gamma_2 \prec_T \Gamma_3$, then $\Gamma_1 \prec_T \Gamma_3$.*
4. *Let $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{MC}$. Then:*
   (a) *If $\Gamma_1 \prec_T \Gamma_2$ and $\Gamma_1 \prec_T \Gamma_3$, then $\Gamma_2 \sim_T \Gamma_3$.*
   (b) *If $\Gamma_2 \prec_T \Gamma_1$ and $\Gamma_3 \prec_T \Gamma_1$, then $\Gamma_2 \sim_T \Gamma_3$.*

The following lemma is specific to our system.

Lemma 2 *Let $\Gamma_1, \Gamma_2 \in \mathcal{MC}$ and $i \in \mathcal{I}$. Then we have:*

1. *$\Gamma_1 \sim_T \Gamma_2$ iff there exists $\gamma \in \{F, P, \epsilon\}$ such that $\{\gamma A \mid A \in \Gamma_2\} \subseteq \Gamma_1$.*
2. *$\Gamma_1 \prec_i \Gamma_2$ iff one of the following condition is satisfied:*
   $i)$ $\{A \mid [i]A \in \Gamma_1\} \subseteq \Gamma_2$;
   $ii)$ $\{<i> A \mid A \in \Gamma_2\} \subseteq \Gamma_1$
3. *$\Gamma_1 \prec_i^{\sim T} \Gamma_2$ iff one of the following conditions is satisfied:*
   (i) *there exists $\gamma \in \{F, P, \epsilon\}$ such that*
   $$\{\gamma A \mid A \in \Gamma_2\} \subseteq \Gamma_1,$$
   (ii) *there are $\gamma_1, \gamma_2 \in \{F, P, \epsilon\}$ such that*
   $$\emptyset \neq \{\gamma_1 <i> \gamma_2 A \mid A \in \Gamma_2\} \subseteq \Gamma_1$$

Corollary 1 *Let $\Gamma_1, \Gamma_2 \in \mathcal{MC}$. Then $\Gamma_1 \searrow \Gamma_2$ if and only if one of the following conditions is satisfied:*

a) there exists $\gamma \in \{F, P, \epsilon\}$ such that
$$\{\gamma A \mid A \in \Gamma_2\} \subseteq \Gamma_1.$$

b) there are $\gamma_1, \ldots \gamma_{n+1} \in \{F, P, \epsilon\}$ and $i_1, \ldots, i_n \in \mathcal{I}$, with $n \geq 1$, such that
$$\{\gamma_1 <i_1> \gamma_2 \ldots <i_n> \gamma_{n+1} A \mid A \in \Gamma_2\} \subseteq \Gamma_1.$$

Theorem 2 (Diamond theorem) *Let $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{MC}$ such that:*

1. *$\Gamma_1 \searrow \Gamma_2$ and $\Gamma_1 \searrow \Gamma_3$.*
2. *there are $i \in \mathcal{I}$ and $\Omega_1 \in \mathcal{MC}$ such that*
   2.1) $\Gamma_2 \prec_i \Omega_1$ *2.2)* $\{A \mid <i> A \in \Gamma_3\} \neq \emptyset$

Then, there exists $\Gamma_4 \in \mathcal{MC}$ such that
$$\Gamma_2 \searrow \Gamma_4 \text{ and } \Gamma_3 \searrow \Gamma_4.$$
More concretely, there exists $\Omega_2 \in \mathcal{MC}$ such that
$$\Gamma_3 \prec_i \Omega_2 \text{ and } \Omega_2 \sim_T \Omega_1.$$

Proof:
In order to prove that there exists $\Omega_2 \in \mathcal{MC}$ with the desired properties, it suffices to prove that one of the following conditions is satisfied:

$a)$ $\{A \mid <i> A \in \Gamma_3\} \subseteq \Omega_1$.
$b)$ $\{A \mid <i> A \in \Gamma_3\} \cup \{A \mid GA \in \Omega_1\}$ is consistent.
$c)$ $\{A \mid <i> A \in \Gamma_3\} \cup \{A \mid HA \in \Omega_1\}$ is consistent.

Indeed, if condition $a)$ is satisfied, then it is enough to take $\Omega_2 = \Omega_1$. On the other hand, if condition $b)$ is satisfied, then Lindenbaum's lemma guarantees that there exists at least one mc extension, Ω_2, such that $\Gamma_3 \prec_i \Omega_2$ and $\Omega_1 \prec_T \Omega_2$. Analogously, if condition $c)$ is satisfied, then Lindenbaum's lemma again guarantees that there exists at least one *mc* extension, Ω_2, such that $\Gamma_3 \prec_i \Omega_2$ and $\Omega_2 \prec_T \Omega_1$.

We assume that none of the conditions a)–c) holds: Since condition $a)$ does not hold, we have that there exists $A \notin \Omega_1$ such that $<i> A \in \Gamma_3$. Also, since condition b) does not hold, there are $B_1, \ldots B_{r_1}$ and $C_1, \ldots C_{r_2}$ such that:

- $<i> B_1, \ldots <i> B_{r_1} \in \Gamma_3$
- $GC_1, \ldots GC_{r_2} \in \Omega_1$
- $\vdash \neg(B \wedge C)$, where $B = B_1 \wedge \ldots \wedge B_{r_1}$ and $C = C_1 \wedge \ldots \wedge C_{r_2}$

Similarly, since condition $c)$ does not hold, there are $D_1, \ldots D_{r_3}$ and $E_1, \ldots E_{r_4}$ such that:

- $<i> D_1, \ldots <i> D_{r_3} \in \Gamma_3$
- $HE_1, \ldots HE_{r_4} \in \Omega_1$

- $\vdash \neg(D \wedge E)$, where $D = D_1 \wedge \ldots \wedge D_{r_3}$ and $E = E_1 \wedge \ldots \wedge E_{r_4}$

Now, it is clear that $\neg A \wedge GC \wedge HE \in \Omega_1$ and, by proposition 1, we can obtain $<\!i\!>(A \wedge \neg C \wedge \neg E) \in \Gamma_3$.

Since $\Gamma_1 \searrow \Gamma_3$, by corollary 1, we have that one of the following conditions holds:

a') there exists $\gamma \in \{F, P, \epsilon\}$ such that $\{\gamma A \,|\, A \in \Gamma_2\} \subseteq \Gamma_1$
b') there are $\gamma_1, \ldots \gamma_{n+1} \in \{F, P, \epsilon\}$, $i_1, \ldots, i_n \in \mathcal{I}$, $n \geq 1$, such that $\{\gamma_1 <\!i_1\!> \gamma_2 \ldots <\!i_n\!> \gamma_{n+1} A \,|\, A \in \Gamma_2\} \subseteq \Gamma_1$.

Therefore, there exists λ such that either $\lambda = \gamma$ or $\lambda = \gamma_1 <\!i_1\!> \gamma_2 \ldots <\!i_n\!> \gamma_{n+1}$ and:

$$\lambda <\!i\!> (A \wedge \neg C \wedge \neg E) \in \Gamma_1 \quad (\dagger_1)$$

On the other hand, since $\Gamma_2 \prec_i \Omega_1$, we have

$$<\!i\!> (\neg A \wedge GC \wedge HE) \in \Gamma_2$$

and, since $\Gamma_1 \searrow \Gamma_2$, by the same reasoning, once again by corollary 1, there exists λ' such that either $\lambda' = \gamma'$ or $\lambda' = \gamma'_1 <\!i'_1\!> \gamma'_2 \ldots <\!i'_m\!> \gamma'_{m+1}$ and

$$\lambda' <\!i\!> (\neg A \wedge GC \wedge HE) \in \Gamma_1 \quad (\dagger_2)$$

Now, by $(\dagger_1)$, $(\dagger_2)$ and the axiom of confluence, we have: $\lambda <\!i\!> (\alpha \wedge (\beta \vee F\beta \vee P\beta)) \in \Gamma_1$, where $\alpha = A \wedge \neg C \wedge \neg E$ and $\beta = \neg A \wedge GC \wedge HE$. But $\alpha \wedge (\beta \vee F\beta \vee P\beta) \equiv \bot$. Thus, the desired contradiction is obtained.

Definition 8 *Let Σ be an ind-functional frame. A* **trace** *of Σ is a function $\Phi_\Sigma : \mathcal{C}oor_\Sigma \longrightarrow 2^{\mathcal{L}^{\mathcal{F}}_{(T\times W)}\text{-}\mathcal{I}}$ such that, for all $t_w \in \mathcal{C}oor_\Sigma$, the set $\Phi_\Sigma(t_w)$ is an mc-set.*

Definition 9 *A trace Φ_Σ is called:*

- **temporally coherent** *if, for all $t_w, t'_w \in \mathcal{C}oor_\Sigma$: if $t'_w \in (t_w, \rightarrow)$, then $\Phi_\Sigma(t_w) \prec_T \Phi_\Sigma(t'_w)$*
- ind-**modally coherent** *if, for all $t_i, t_w \in \mathcal{C}oor_\Sigma$ with $i \in \mathcal{I} \cap W$: if $t_i = \stackrel{w\ i}{\longrightarrow} (t_w)$, then $\Phi_\Sigma(t_w) \prec_i \Phi_\Sigma(t_i)$*
- **coherent** *if it is temporally coherent and ind-modally coherent*
- **prophetic** *if it is temporally coherent and, moreover, for all $A \in \mathcal{L}^{\mathcal{F}}_{(T\times W)}\text{-}\mathcal{I}$ and $t_w \in \mathcal{C}oor_\Sigma$:*
(1) *if $FA \in \Phi_\Sigma(t_w)$, there exists $t'_w \in (t_w, \rightarrow)$ such that $A \in \Phi_\Sigma(t'_w)$*
- **historic** *if it is temporally coherent and, moreover, for all formula $A \in \mathcal{L}^{\mathcal{F}}_{(T\times W)}\text{-}\mathcal{I}$ and $t_w \in \mathcal{C}oor_\Sigma$:*
(2) *if $PA \in \Phi_\Sigma(t_w)$, there exists $t'_w \in (\leftarrow, t_w)$ such that $A \in \Phi_\Sigma(t'_w)$*
- **ind-possibilistic** *if it is ind-modally coherent and, moreover, for all formula $A \in \mathcal{L}^{\mathcal{F}}_{(T\times W)}\text{-}\mathcal{I}$, $t_w \in \mathcal{C}oor_\Sigma$ and $i \in \mathcal{I} \cap W$:*
(3) *if $<\!i\!> A \in \Phi_\Sigma(t_w)$, there exists $t_i = \stackrel{w\ i}{\longrightarrow} (t_w)$ such that $A \in \Phi_\Sigma(t_i)$*

The conditional (1) *(resp.,* (2) *or* (3)*) is called a* **prophetic** *(resp.,* **historic** *or* **ind-possibilistic***)* **conditional for** Φ_Σ **with respect to** FA *(resp., PA or $<\!i\!> A$)* **and** t_w.

An ind-functional frame, Φ_Σ, is called **full** *if it is prophetic, historic and ind-possibilistic.*

Definition 10 *Let W_Ξ be a denumerable infinite set such that $\mathcal{I} \subset W_\Xi$ and $T_\Xi = \bigcup_{w\in W_\Xi} T_w$ where, for all $w \in W_\Xi$, T_w is a denumerable infinite set. We will consider the class, Ξ, of the ind-functional frames $(W', \mathcal{T}', \mathcal{F}')$ such that:*

- W' is a nonempty finite subset of W_Ξ.
- $\mathcal{T}'_\Xi = \{(T'_w, <'_w) \mid w \in W'\}$, where T'_w is a nonempty finite subset of T_w.

If $\Sigma_1 = (W_1, \mathcal{T}_1, \mathcal{F}_1), \Sigma_2 = (W_2, \mathcal{T}_2, \mathcal{F}_2) \in \Xi$, we say that Σ_2 is **an extension of** Σ_1 *if the following conditions are satisfied:*

- *$W_1 \subseteq W_2$;*
- *either $\mathcal{T}_1 \subset \mathcal{T}_2$, or for each $(T_w, <_w) \in \mathcal{T}_1$, the set $\mathcal{T}_2$ contains an extension of $(T_w, <_w)$.*
- *either $\mathcal{F}_1 \subset \mathcal{F}_2$ or, for each $\stackrel{w\ i}{\longrightarrow} \in \mathcal{F}_1$, the set $\mathcal{F}_2$ contains a function which extends $\stackrel{w\ i}{\longrightarrow}$.*

Definition 11 *Let Ξ be as in definition 10, and let $\Phi_{\Sigma'}$ be a trace of an ind-functional frame $\Sigma' = (W', \mathcal{T}', \mathcal{F}') \in \Xi$.*

I) *Given a prophetic conditional:*

(1) *if $FA \in \Phi_{\Sigma'}(t_w)$, there exists a $t'_w \in (t_w, \rightarrow)$ such that $A \in \Phi_{\Sigma'}(t'_w)$.*

We say that (1) *is* **inactive** *if its antecedent is not fulfilled, that is, if one of the following conditions is satisfied:*

(i) $t_w \notin \mathcal{C}oor_{\Sigma'}$,

(ii) $t_w \in \mathcal{C}oor_{\Sigma'}$, *but $FA \notin \Phi_{\Sigma'}(t_w)$.*

We say that (1) *is* **active** *if its antecedent is fulfilled but its consequent is not, that is, $t_w \in \mathcal{C}oor_{\Sigma'}$ and $FA \in \Phi_{\Sigma'}(t_w)$, but there is no $t'_w \in (t_w, \rightarrow)$ such that $A \in \Phi_{\Sigma'}(t'_w)$. We say that* (1) *is* **exhausted** *if its consequent is fulfilled, that is, there exists $t'_w \in (t_w, \rightarrow)$ such that $A \in \Phi_{\Sigma'}(t'_w)$.*

II) *Given a historic conditional:*

(2) *if $PA \in \Phi_{\Sigma'}(t_w)$, there exists $t'_w \in (\leftarrow, t_w)$ such that $A \in \Phi_{\Sigma'}(t'_w)$.*

We say that (2) *is* **inactive** *if its antecedent is not fulfilled, that is, if one of the following conditions is satisfied:*

(i) $t_w \notin \mathcal{C}oor_{\Sigma'}$, (ii) $t_w \in \mathcal{C}oor_{\Sigma'}$ *but $PA \notin \Phi_{\Sigma'}(t_w)$.*

We say that (2) *is* **active** *if its antecedent is fulfilled but its consequent is not, that is, $t_w \in \mathcal{C}oor_{\Sigma'}$ and*

$PA \in \Phi_{\Sigma'}(t_w)$, but there is no $t'_w \in (\leftarrow, t_w)$ such that $A \in \Phi_{\Sigma'}(t'_w)$. We say that (2) *is* **exhausted** *if its consequent is fulfilled, that is, there exists a $t'_w \in (\leftarrow, t_w)$ such that $A \in \Phi_{\Sigma'}(t'_w)$.*

III) *Given an ind-possibilistic conditional:*

(3) *if $<i> A \in \Phi_{\Sigma'}(t_w)$, then there exists*

$$t_i = \stackrel{w\ i}{\longrightarrow} (t_w) \text{ such that } A \in \Phi_{\Sigma'}(t_i).$$

We say that the conditional (3) *is* **inactive** *if its antecedent is not fulfilled, that is, if one of the following conditions is satisfied:*

(i) $t_w \notin \mathcal{C}oor_{\Sigma'}$,

(ii) $t_w \in \mathcal{C}oor_{\Sigma'}$ *but* $<i> A \notin \Phi_{\Sigma'}(t_w)$.

We say that the conditional (3) *is* **active** *if its antecedent is fulfilled but its consequent is not, that is, $t_w \in \mathcal{C}oord_{\Sigma'}$ and $<i> A \in \Phi_{\Sigma'}(t_w)$, but there is no $t_i = \stackrel{w\ i}{\longrightarrow} (t_w)$ such that $A \in \Phi_{\Sigma'}(t_i)$. We say that* (3) *is* **exhausted** *if its consequent is fulfilled, that is, there exists a $t_i = \stackrel{w\ i}{\longrightarrow} (t_w)$ such that $A \in \Phi_{\Sigma'}(t_i)$.*

Lemma 3 (trace lemma) *Let Φ_Σ be an full trace of an ind-functional frame $\Sigma = (W, \mathcal{T}, \mathcal{F})$ and h an ind-functional interpretation assigning each propositional variable, p, the set $h(p) = \{t_w \in Coor_\Sigma \mid p \in \Phi_\Sigma(t_w)\}$. Then, for any formula A, we have*

$$h(A) = \{t_w \in Coor_\Sigma \mid A \in \Phi_\Sigma(t_w)\}$$

In order to prove the completeness theorem, for each consistent formula, A, we will construct (using the class Ξ in definition 10) an ind-functional frame $\Sigma = (W, \mathcal{T}, \mathcal{F})$ and a full trace, Φ_Σ, such that $A \in \Phi_\Sigma(t_w)$ for some $t_w \in \mathcal{C}oor_\Sigma$.

To this end, we define:

- an enumeration of W_Ξ: $W_\Xi = \{w_n \mid n \in \mathbb{N}\}$.
- an enumeration of $T_\Xi = \bigcup_{w \in W_\Xi} T_w$:
  $T_\Xi = \bigcup_{n \in \mathbb{N}} T_{w_n}$; $\quad T_{w_n} = \{t_{(n,m)} \mid m \in \mathbb{N}\}$
- an enumeration of $\mathcal{L}^{\mathcal{F}}_{(T \times W)}$-$\mathcal{I}$: $A_0, A_1, \ldots A_n, \ldots$.

Therefore, we can also assign a code number for each prophetic conditional (historic conditional, ind-possibilistic conditional, etc.) in the usual way.

Now, given a consistent formula A, the construction of Σ and Φ_Σ goes step by step as follows:

We begin with a finite ind-functional frame $\Sigma_0 = (W_0, \mathcal{T}_0, \mathcal{F}_0) \in \Xi$, with $W_0 = \{w_0\}$, where $w_0 \in W_\Xi - \mathcal{I}$; $\mathcal{T}_0 = \{(\{t_{(0,0)}\}, \emptyset)\}$, $\mathcal{F}_0 = \emptyset$ and a trace Φ_{Σ_0}, such that $\Phi_{\Sigma_0}(t_{(0,0)}) = \Gamma_0$, where Γ_0 is an mc-set containing A.

Assume that $\Sigma_n = (W_n, \mathcal{T}_n, \mathcal{F}_n)$ and Φ_{Σ_n} are defined. Then Σ_{n+1} and $\Phi_{\Sigma_{n+1}}$ are defined as follows:

- If all conditionals are not active, then $\Sigma_{n+1} = \Sigma_n$, $\Phi_{\Sigma_{n+1}} = \Phi_{\Sigma_n}$ and the construction is finished.
- Otherwise, i.e., if there is some conditional (α) for Φ_{Σ_n} which is active, then we choose the conditional (α') with the lowest code number and the *exhausting lemma* below ensures that there exists a finite extension $\Sigma_{n+1} = (W_{n+1}, \mathcal{T}_{n+1}, \mathcal{F}_{n+1})$ of Σ_n and a finite extension $\Phi_{\Sigma_{n+1}}$ of Φ_{Σ_n}, such that the conditional (α') for $\Phi_{\Sigma_{n+1}}$ is exhausted.

The result is a sequence of finite ind-functional frames

$$(W_0, \mathcal{T}_0, \mathcal{F}_0), (W_1, \mathcal{T}_1, \mathcal{F}_1), ..., (W_n, \mathcal{T}_n, \mathcal{F}_n), ...,$$

whose union is the ind-functional frame Σ, and a sequence of corresponding traces, $\Phi_{\Sigma_0}, \Phi_{\Sigma_1}, \ldots, \Phi_{\Sigma_n}, \ldots$, whose union is Φ_Σ.

Each finite ind-functional frame of the above sequence satisfies the condition of linearity of temporal orders and each trace of it is coherent, but in general, it fails to be prophetic, historic or ind-possibilistic.

However, as we shall show, the trace Φ_Σ has all these properties. Finally, we will demonstrate that the trace lemma ensures that A is satisfied in Σ.

Lemma 4 (Exhausting lemma) *Let Ξ be as in definition 10, Φ_{Σ_n} a coherent trace of an ind-functional frame $\Sigma_n = (W_n, \mathcal{T}_n, \mathcal{F}_n) \in \Xi$ and let (α) be a prophetic (historic or ind-possibilistic) conditional for Φ_{Σ_n} which is active. Then there exists a coherent trace $\Phi_{\Sigma_{n+1}}$, extension of Φ_{Σ_n}, such that (α) is a conditional for $\Phi_{\Sigma_{n+1}}$ which is exhausted.*

Proof:
The proof for prophetic or historic conditionals is standard in temporal logic. Consider the special case of ind-possibilistic conditionals.

Let Φ_{Σ_n} be a coherent trace of an ind-functional frame $\Sigma_n = (W_n, \mathcal{T}_n, \mathcal{F}_n) \in \Xi$, let $i \in \mathcal{I}$ and assume that the following ind-possibilistic conditional for Φ_{Σ_n} is active:
(2) if $<i> A \in \Phi_{\Sigma_n}(t_w)$, then there exists

$$t_i = \stackrel{w\ i}{\longrightarrow} (t_w) \text{ such that } A \in \Phi_{\Sigma_n}(t_i).$$

Thus, we have that $<i> A \in \Phi_{\Sigma_n}(t_w)$, but there is no $t_i = \stackrel{w\ i}{\longrightarrow} (t_w)$ such that $A \in \Phi_{\Sigma_n}(t_i)$. Now, we have to consider two cases:

I) $i \notin W_n$. Then, by item 2 (c) in lemma 1, there exists an mc-set, Γ, such that $\Phi_{\Sigma_n}(t_w) \prec_i \Gamma$ and $A \in \Gamma$; then we need a new flow of time labelled with i, T_i, which requires to extend W_n and, also, to introduce a new coordinate t_i associated with Γ so that $t_i = \stackrel{w\ i}{\longrightarrow} (t_w)$. So, we proceed as follows:

- $W_{n+1} = W_n \cup \{i\}$

- $\mathcal{T}_{n+1} = \mathcal{T}_n \cup \{(T_i, <_i)\}$, where
  $(T_i, <_i) = (\{t_i\}, \emptyset)$
- $\mathcal{F}_{n+1} = \mathcal{F} \cup \{\xrightarrow{w\ i}\}$, where $\xrightarrow{w\ i} = \{(t_w, t_i)\}$
- $\Phi_{\Sigma_{n+1}} = \Phi_{\Sigma_n} \cup \{(t_i, \Gamma)\}$

It is immediate that linearity is preserved and that $\Phi_{\Sigma_{n+1}}$ is coherent.

II) $i \in W_n$. In this case, we must consider the following situations:

(II.1) $\xrightarrow{w\ i}$ is defined in $\mathcal{F}_n$.

(II.2) $\xrightarrow{w\ i}$ is not defined in $\mathcal{F}_n$.

(II.1): Let t_i be the minimum [3] of $\xrightarrow{w\ i} (T_w)$ and t'_w such that $t_i = \xrightarrow{w\ i} (t'_w)$. Thus, we have that

$$\Phi_{\Sigma_n}(t'_w) \prec_i \Phi_{\Sigma_n}(t_i) \qquad (\dagger_1)$$

and, since $\Phi_{\Sigma_n}(t_w) \sim_T \Phi_{\Sigma_n}(t'_w)$, we obtain $\Phi_{\Sigma_n}(t_w) \searrow \Phi_{\Sigma_n}(t'_w)$. On the other hand, it is evident that $\Phi_{\Sigma_n}(t_w) \searrow \Phi_{\Sigma_n}(t_w)$. Thus, by diamond theorem, from $(\dagger_1)$ we obtain that there exists an *mc*-set, Γ, such that

$$\Phi_{\Sigma_n}(t_w) \prec_i \Gamma \quad \text{and} \quad \Gamma \sim_T \Phi_{\Sigma_n}(t_i) \qquad (\dagger_2)$$

$(\dagger_2)$ means that one of the following three conditions is satisfied:

(i) $\Gamma = \Phi_{\Sigma_n}(t_i)$;

(ii) $\Phi_{\Sigma_n}(t_i) \prec_T \Gamma$;

(iii) $\Gamma \prec_T \Phi_{\Sigma_n}(t_i)$.

In all of these cases we have $W_{n+1} = W_n$. Now,

– if item (i) holds, we have:

$\mathcal{T}_{n+1} = \mathcal{T}_n$;

$\mathcal{F}_{n+1} = (\mathcal{F}_n - \{\xrightarrow{w\ i}\}) \cup \{\xrightarrow{w\ i}{}'\}$, where

$\xrightarrow{w\ i}{}' = \xrightarrow{w\ i} \cup \{(t_w, t_i)\}$

$\Phi_{\Sigma_{n+1}} = \Phi_{\Sigma_n}$

It is evident that $\Phi_{\Sigma_{n+1}}$ is coherent.

– if item (ii) holds, we have to consider the number of successors of t_i in T_i:

(ii.a) if the number of successors of t_i in T_i is zero, then a new coordinate t'_i is chosen, to be associated with Γ and we have:

$\mathcal{T}_{n+1} = (\mathcal{T}_n - \{(T_i, <_i)\}) \cup \{(T'_i, <'_i)\}$, where

- $T'_i = T_i \cup \{t'_i\}$
- $<'_i = <_i \cup \{(t_i, t'_i)\} \cup \{(t^*_i, t'_i) \mid t^*_i <_i t_i\}$

$\mathcal{F}_{n+1} = (\mathcal{F}_n - \{\xrightarrow{w\ i}\}) \cup \{\xrightarrow{w\ i}{}'\}$, where:

$\xrightarrow{w\ i}{}' = \xrightarrow{w\ i} \cup \{(t_w, t'_i)\}$ (*)

$\Phi_{\Sigma_{n+1}} = \Phi_{\Sigma_n} \cup \{(t'_i, \Gamma)\}$ (**)

Clearly, linearity is preserved and lemma 1 completes the proof of the coherence of $\Phi_{\Sigma_{n+1}}$.

(ii.b) On the other hand, if the number of successors of t_i in T_i is $s > 0$, then we consider the immediate successor of t_i, say t^1_i. Now, since $\Phi_{\Sigma_n}(t_i) \prec_T \Phi_{\Sigma_n}(t^1_i)$, by item 4 (a) in lemma 1, we obtain one of the three following conditions:

(ii.b.1) $\Gamma = \Phi_{\Sigma_n}(t^1_i)$

(ii.b.2) $\Gamma \prec_T \Phi_{\Sigma_n}(t^1_i)$

(ii.b.3) $\Phi_{\Sigma_n}(t^1_i) \prec_T \Gamma$

For case (ii.b.1) the reasoning is the same as in (i). The case (ii.b.2) gives rise to

$$\Phi_{\Sigma_n}(t_i) \prec_T \Gamma \prec_T \Phi_{\Sigma_n}(t^1_i)$$

Then we select a new coordinate, t'_i, to associate it to Γ. Thus:

$\mathcal{T}_{n+1} = (\mathcal{T}_n - \{(T_i, <_i)\}) \cup \{(T'_i, <'_i)\}$, where

- $T'_i = T_i \cup \{t'_i\}$
- $<'_i = <_i \cup \{(t_i, t'_i), (t'_i, t^1_i)\} \cup \{(t^*_i, t'_i) \mid t^*_i <_i t_i\} \cup \{(t'_i, t^*_i) \mid t^1_i <_i t^*_i\}$

$\mathcal{F}_{n+1}$ and $\Phi_{\Sigma_{n+1}}$ are defined as in (*) and (**) respectively.

Linearity in the new frame and coherence of the new trace are again preserved.

In case (ii.b.3) we consider the immediate successor of t^1_i, namely t^2_i, and we proceed in a similar way.

By iterating this operation at most s times, we fix the image of t_w associating an mc-set to it, preserving coherence and linearity.

Finally, case (iii) is analogous to case (ii).

Consider case (II.2), that is, $\xrightarrow{w\ i}$ is not defined in $\mathcal{F}_n$. It means (by construction of Σ_n) that then there will be some time-flow, $T_{w'}$, with $w' \neq w$ and $\xrightarrow{w'\ i} \in \mathcal{F}_n$. Let t_i be the minimum of $\xrightarrow{w'\ i} (T_{w'})$ and $t_{w'}$ such that $\xrightarrow{w'\ i} (t_{w'}) = t_i$.

Thus $\Phi_{\Sigma_n}(t_{w'}) \prec_i \Phi_{\Sigma_n}(t_i)$. Now, (again by construction of Σ_n) we have three subcases:

(II.2.1) $\Phi_{\Sigma_n}(t_w) \searrow \Phi_{\Sigma_n}(t_{w'})$

(II.2.2) $\Phi_{\Sigma_n}(t_{w'}) \searrow \Phi_{\Sigma_n}(t_w)$

(II.2.3) there exists a flow $T_{w''}$, with $w'' \neq w$ and $w'' \neq w'$, and there exists $t_{w''} \in T_{w''}$ such that:

$$\Phi_{\Sigma_n}(t_{w''}) \searrow \Phi_{\Sigma_n}(t_w) \text{ and } \Phi_{\Sigma_n}(t_{w''}) \searrow \Phi_{\Sigma_n}(t_{w'})$$

[3] We could also consider the maximum.

In case (II.2.1), given that $\Phi_{\Sigma_n}(t_w) \searrow \Phi_{\Sigma_n}(t_w)$, by diamond theorem, there exists an *mc*-set Γ such that $\Phi_{\Sigma_n}(t_w) \prec_i \Gamma$ and $\Gamma \sim_T \Phi_{\Sigma_n}(t_i)$. Thus, we can observe, once again, one of the three situations referred to in paragraph (II.1):

(i) $\Gamma = \Phi_{\Sigma_n}(t_i)$;

(ii) $\Phi_{\Sigma_n}(t_i) \prec_T \Gamma$;

(iii) $\Gamma \prec_T \Phi_{\Sigma_n}(t_i)$.

and we can reason analogously. [4]

In case (II.2.2), i.e., if $\Phi_{\Sigma_n}(t_{w'}) \searrow \Phi_{\Sigma_n}(t_w)$, since $\Phi_{\Sigma_n}(t_{w'}) \searrow \Phi_{\Sigma_n}(t_{w'})$, the diamond theorem ensures that there exists an *mc*-set Γ such that $\Phi_{\Sigma_n}(t_w) \prec_i \Gamma$ and $\Gamma \sim_T \Phi_{\Sigma_n}(t_i)$. Thus, we have the same situation as in the previous cases (i), (ii) and (iii) and we can carry out the same reasoning.

Finally, in case (II.2.3), i.e., if there exists a time-flow $T_{w''}$, with $w'' \neq w$ and $w'' \neq w'$, and there exists $t_{w''} \in T_{w''}$ such that $\Phi_{\Sigma_n}(t_{w''}) \searrow \Phi_{\Sigma_n}(t_w)$ and $\Phi_{\Sigma_n}(t_{w''}) \searrow \Phi_{\Sigma_n}(t_{w'})$, once again by diamond theorem, there exists an *mc*-set Γ such that

$$\Phi_{\Sigma_n}(t_w) \prec_i \Gamma \quad \text{and} \quad \Gamma \sim_T \Phi_{\Sigma_n}(t_i)$$

and we can repeat the same reasoning.

Now we can formulate the following theorem.

Theorem 3 (Completeness theorem for $\mathcal{S}^{\mathcal{F}}_{(T\times W)-\mathcal{I}}$-Parc.) *If a formula $A \in \mathcal{L}^{\mathcal{F}}_{(T\times W)}$-$\mathcal{I}$ is valid in the class of every ind-functional frame, then A is a theorem of $\mathcal{S}^{\mathcal{F}}_{(T\times W)-\mathcal{I}}$-Parc.*

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[4] It suffices to consider that, in this case, $\xrightarrow{w\ i} \notin \mathcal{F}_n$ and, therefore, in the extensions of $\mathcal{F}_n$ for (i), (ii) and (iii), we will obtain that $\mathcal{F}_{n+1} = \mathcal{F}_n \cup \{\xrightarrow{w\ i}\}$, being concretely $\xrightarrow{w\ i} = \{(t_w, t_i)\}$ in subcase (i). In subcases (ii) and (iii), we will obtain, respectively, that $\xrightarrow{w\ i} (t_w)$ is situated on the right or on the left of t_i.

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