A Multimodal Logic for Closeness

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Abstract. We introduce a multimodal logic for order of magnitude reasoning which includes the notions of closeness and negligibility, we provide an axiom system which is sound and complete.

1 Introduction

There are some multimodal logics for order of magnitude reasoning dealing with the relations of negligibility and comparability, see for instance [2,4,8]; however, as far as we know, the only published reference on the notion of closeness in a logic-based context is [6], where the notions of closeness and distance are treated using Propositional Dynamic Logic, and their definitions are based on the concept of qualitative sum; specifically, in [6] two values are assumed to be close if one of them can be obtained from the other by adding a small number, and small numbers are defined as those belonging to a fixed interval.

In this work, we consider a new logic-based alternative to the notion of closeness in the context of multimodal logics. Our notion of closeness stems from the idea that two values are considered to be *close* if they are inside a prescribed area or *proximity interval*. This idea applies to the situations described in the previous paragraph, although it may differ from other intuitions based on distances since it leads to an equivalence relation, particularly, transitivity holds. Neither reflexivity nor symmetry of closeness generate any discussion among the different authors, but transitivity does. The original notion of closeness given by Raiman in [9] allows a certain form of transitivity which he had to tame by using a number of arbitrary limitations to avoid an unrestricted application of chaining. This arbitrariness was criticized in [1], in which a fuzzy set-based approach for handling relative orders of magnitude was introduced. It is remarkable to note that the criticism was made against the arbitrary limitations on chaining the relation, or the impossibility of considering suitable modified versions of transitivity, but not on transitivity per se.

The limitations stated above do not apply to our approach, which can be seen as founded on the notion of granularity as given in [7], which was already suggested in [11]. The main difficulties in accepting closeness as a transitive relation arise in a distance-based interpretation because, then, its unrestricted use would collapse the relation since all the elements would be close. As stated above, our notion will be based not on distance but on membership to a certain element of a given set of proximity intervals, since our driving force is to define an abstract framework for dealing with natural or artificial barriers.

On the other hand, the negligibility notion provided in this paper is a slight generalization of the one given in [5] where, following the line of other classical approaches, for instance [10], the class of 0 is considered to be just a singleton. This choice makes little sense in a qualitative approach, since considering the class of 0 to be just a singleton would require to have measures with infinite precision. Instead, we consider the qualitative class INF of *infinitesimals* which, of course, will be all close to each other. Note that these infinitesimals will be interpreted as numbers indistinguishable from 0 in the sense that their difference cannot be measured, not in the sense of hyperreal numbers.

In this work, we introduce a multimodal logic for order of magnitude reasoning which manages the notions of closeness and negligibility, then an axiom system is introduced which is sound and complete.

2 Preliminary definitions

We will consider a subset of real numbers $(\mathbb{S}, <)$ divided into the following qualitative classes:

$$\begin{array}{lll} \mathrm{NL} = (-\infty, -\gamma) & & \mathrm{PS} = (+\alpha, +\beta] \\ \mathrm{NM} = [-\gamma, -\beta) & \mathrm{INF} = [-\alpha, +\alpha] & \mathrm{PM} = (+\beta, +\gamma] \\ \mathrm{NS} = [-\beta, -\alpha) & & \mathrm{PL} = (+\gamma, +\infty) \end{array}$$

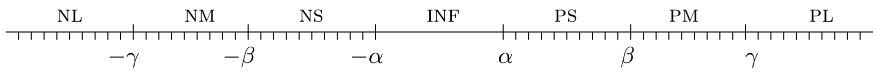

Fig. 1.1. Proximity intervals.

Note that all the intervals are considered relative to $\mathbb{S}$.

The labels correspond to "negative large" (NL), "negative medium"(NM), "negative small"(NS), "infinitesimals"(INF), "positive small" (PS), "positive medium" (PM) and "positive large" (PL). It is worth to note that this classification is slightly more general than the standard one [10], since the qualitative class containing the element 0, i.e. INF, needs not be a singleton; this allows for considering values very close to zero as null values in practice, which is more in line with a qualitative approach where accurate measurements are not always possible.

We will consider each qualitative class to be divided into disjoint intervals called *proximity intervals*, as shown in Figure 1.1. The qualitative class INF is itself one proximity interval.

Definition 1. *Let* $(\mathbb{S}, <)$ *be the set of numbers introduced above.*

- *An* r-proximity structure *is a finite set* $\mathcal{I}(\mathbb{S}) = \{I_1, I_2, \ldots, I_r\}$ *of intervals in* $\mathbb{S}$*, such that:*
  1. *For all* $I_i, I_j \in \mathcal{I}(\mathbb{S})$*, if* $i \neq j$*, then* $I_i \cap I_j = \varnothing$.
  2. $I_1 \cup I_2 \cup \cdots \cup I_r = \mathbb{S}$.
  3. *For all* $x, y \in \mathbb{S}$ *and* $I_i \in \mathcal{I}(\mathbb{S})$*, if* $x, y \in I_i$*, then* x, y *belong to the same qualitative class.*
  4. INF $\in \mathcal{I}(\mathbb{S})$.
- *Given a proximity structure* $\mathcal{I}(\mathbb{S})$*, the binary relation of closeness* $\mathfrak{c}$ *is defined, for all* $x, y \in \mathbb{S}$*, as follows:* $x \,\mathfrak{c}\, y$ *if and only if there exists* $I_i \in \mathcal{I}(\mathbb{S})$ *such that* $x, y \in I_i$.

Notice that, by definition, the number of proximity intervals is finite, regardless of the cardinality of the set $\mathbb{S}$. This choice is justified by the applications (the number of values we can consider is always finite) and the nature of the measuring devices that after reaching a certain limit, they do not distinguish among nearly equal amounts; for instance, consider the limits to represent numbers in a pocket calculator, thermometer, speedometer, etc.

The informal notion of negligibility we will use in this paper is the following: x is said to be *negligible* with respect to y if and only if either (i) x is infinitesimal and y is not, or (ii) x is small (but not infinitesimal) and y is *sufficiently large*. Formally:

Definition 2. *The binary relation of negligibility* $\mathfrak{n}$ *is defined on* $(\mathbb{S}, <)$ *as* $x \,\mathfrak{n}\, y$ *if and only if one of the following situations holds:*

(i) $x \in$ INF *and* $y \notin$ INF*,*
(ii) $x \in$ NS $\cup$ PS *and* $y \in$ NL $\cup$ PL.

3 A logic for closeness

In this section, we will use as special modal connectives $\overrightarrow{\Box}$ and $\overleftarrow{\Box}$ to deal with the usual ordering $<$, so $\overrightarrow{\Box}A$ and $\overleftarrow{\Box}A$ have the informal readings: *A is true for all numbers greater than the current one* and *A is true for all number less than the current one*, respectively. Two other modal operators will be used, $\boxed{c}$ for closeness, where the informal reading of $\boxed{c}A$ is: *A is true for all number close to the current one*, and $\boxed{n}$ for negligibility, where $\boxed{n}A$ means *A is true for all number with respect to the current one is negligible.*

The alphabet of the language $\mathcal{L}(MQ)^{\mathcal{P}}$ is defined by using a stock of atoms or propositional variables, $\mathcal{V}$, the classical connectives $\neg, \wedge, \vee$ and $\rightarrow$; the constants for milestones $\alpha^-, \alpha^+, \beta^-, \beta^+, \gamma^-, \gamma^+$; a finite set $\mathcal{C}$ of constants for proximity intervals, $\mathcal{C} = \{c_1, \ldots, c_r\}$ [1]; the unary modal connectives $\overrightarrow{\Box}$, $\overleftarrow{\Box}$, $\boxed{n}$, $\boxed{c}$, and the parentheses '(' and ')'. We define the formulas of $\mathcal{L}(MQ)^{\mathcal{P}}$ as follows:

$$A = p \mid \xi \mid c_i \mid \neg A \mid (A \wedge A) \mid (A \vee A) \mid (A \rightarrow A) \mid \overrightarrow{\Box}A \mid \overleftarrow{\Box}A \mid \boxed{n}A \mid \boxed{c}A$$

where $p \in \mathcal{V}$, $\xi \in \{\alpha^+, \alpha^-, \beta^+, \beta^-, \gamma^+, \gamma^-\}$ and $c_i \in \mathcal{C}$. In order to refer to any constant for positive milestones as α^+ we will use ξ^+ and for negative ones as β^- we will use ξ^-.

[1] There are at least as many elements in $\mathcal{C}$ as qualitative classes.

The *mirror image* of a formula A is the result of replacing in A each occurrence of $\overrightarrow{\Box}$, $\overleftarrow{\Box}$, α^+, β^+ and γ^+ respectively by $\overleftarrow{\Box}$, $\overrightarrow{\Box}$, α^-, β^- and γ^- and reciprocally. We will use the symbols $\overrightarrow{\Diamond}$, $\overleftarrow{\Diamond}$, $\langle\mathsf{c}\rangle$, $\langle\mathsf{n}\rangle$ as abbreviations, respectively, of $\neg\overrightarrow{\Box}\neg$, $\neg\overleftarrow{\Box}\neg$, $\neg\boxed{\mathsf{c}}\neg$ and $\neg\boxed{\mathsf{n}}\neg$. Moreover, we will introduce $\mathsf{nl}, \ldots \mathsf{pl}$ as abbreviations for qualitative classes, for instance, ps for $(\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\Diamond}\beta^+) \vee \beta^+$. By means of qc we denote any element of the set $\{\mathsf{nl}, \mathsf{nm}, \mathsf{ns}, \mathsf{inf}, \mathsf{ps}, \mathsf{pm}, \mathsf{pl}\}$.

The cardinality r of the set $\mathcal{C}$ of constants for proximity intervals will play an important role since it, somehow, encodes the granularity of the underlying logic. This implies that, actually, *we are introducing a family of logics which depend parametrically on* r.

Definition 3. *A* multimodal qualitative frame *for* $\mathcal{L}(MQ)^{\mathcal{P}}$ *(a* frame*, for short) is a tuple* $\Sigma = (\mathbb{S}, \mathcal{D}, <, \mathcal{I}(\mathbb{S}), \mathcal{P})$*, where:*

1. $(\mathbb{S}, <)$ *is an ordered subset of real numbers.*
2. $\mathcal{D} = \{+\alpha, -\alpha, +\beta, -\beta, +\gamma, -\gamma\}$ *is a set of designated points in* $\mathbb{S}$ *satisfying* $-\gamma < -\beta < -\alpha < +\alpha < +\beta < +\gamma$.
3. $\mathcal{I}(\mathbb{S})$ *is an r-proximity structure.*
4. $\mathcal{P}$ *is a bijection (called* proximity function*),* $\mathcal{P}\colon \mathcal{C} \longrightarrow \mathcal{I}(\mathbb{S})$*, that assigns to each proximity constant c a proximity interval.*

Definition 4. *Let* Σ *be a frame for* $\mathcal{L}(MQ)^{\mathcal{P}}$*, a* multimodal qualitative model on Σ *(a* MQ-model*, for short) is an ordered pair* $\mathcal{M} = (\Sigma, h)$*, where* h *is a* meaning function *(or,* interpretation*)* $h\colon \mathcal{V} \longrightarrow 2^{\mathbb{S}}$*. Any interpretation can be uniquely extended to the set of all formulas in* $\mathcal{L}(MQ)^{\mathcal{P}}$ *(also denoted by* h*) by means of the usual conditions for the classical Boolean connectives and the following conditions:*

$$
\begin{aligned}
h(\overrightarrow{\Box}A) &= \{x \in \mathbb{S} \mid y \in h(A) \textit{ for all } y \textit{ such that } x < y\}\\
h(\overleftarrow{\Box}A) &= \{x \in \mathbb{S} \mid y \in h(A) \textit{ for all } y \textit{ such that } y < x\}\\
h(\boxed{\mathsf{c}}A) &= \{x \in \mathbb{S} \mid y \in h(A) \textit{ for all } y \textit{ such that } x \,\mathsf{c}\, y\}\\
h(\boxed{\mathsf{n}}A) &= \{x \in \mathbb{S} \mid y \in h(A) \textit{ for all } y \textit{ such that } x \,\mathsf{n}\, y\}\\
h(\alpha^+) &= \{+\alpha\} \qquad h(\beta^+) = \{+\beta\} \qquad h(\gamma^+) = \{+\gamma\}\\
h(\alpha^-) &= \{-\alpha\} \qquad h(\beta^-) = \{-\beta\} \qquad h(\gamma^-) = \{-\gamma\}\\
h(c_i) &= \{x \in \mathbb{S} \mid x \in \mathcal{P}(c_i)\}
\end{aligned}
$$

The definitions of truth*,* satisfiability *and* validity *are the usual ones.*

Now, we consider the axiom system $MQ^{\mathcal{P}}$ for the language $\mathcal{L}(MQ)^{\mathcal{P}}$, consisting of all the tautologies of classical propositional logic together with the following axiom schemata and rules of inference:

For white connectives

K1 $\overrightarrow{\Box}(A \to B) \to (\overrightarrow{\Box}A \to \overrightarrow{\Box}B)$
K2 $A \to \overrightarrow{\Box}\overleftarrow{\Diamond}A$
K3 $\overrightarrow{\Box}A \to \overrightarrow{\Box}\overrightarrow{\Box}A$
K4 $\big(\overrightarrow{\Box}(A \vee B) \wedge \overrightarrow{\Box}(\overrightarrow{\Box}A \vee B) \wedge \overrightarrow{\Box}(A \vee \overrightarrow{\Box}B)\big) \to \big(\overrightarrow{\Box}A \vee \overrightarrow{\Box}B\big)$

For constants $\xi \in \{\alpha^+, \beta^+, \gamma^+, \alpha^-, \beta^-, \gamma^-\}$

c1 $\overleftarrow{\Diamond}\xi \vee \xi \vee \overrightarrow{\Diamond}\xi$
c2 $\xi \to (\overleftarrow{\Box}\neg\xi \wedge \overrightarrow{\Box}\neg\xi)$
c3 $\gamma^- \to \overrightarrow{\Diamond}\beta^-$
c4 $\beta^- \to \overrightarrow{\Diamond}\alpha^-$
c5 $\alpha^- \to \overrightarrow{\Diamond}\alpha^+$
c6 $\alpha^+ \to \overrightarrow{\Diamond}\beta^+$
c7 $\beta^+ \to \overrightarrow{\Diamond}\gamma^+$

For proximity constants (for all $i, j \in \{1, \ldots, n\}$)

p1 $\bigvee_{i=1}^{n} c_i$
p2 $c_i \to \neg c_j$ (for $i \neq j$)
p3 $(\overleftarrow{\Diamond}c_i \wedge \overrightarrow{\Diamond}c_i) \to c_i$
p4 $\overleftarrow{\Diamond}c_i \vee c_i \vee \overrightarrow{\Diamond}c_i$

Mixed axioms (for all $i \in \{1, \ldots, n\}$)

m1 $(c_i \wedge \mathsf{qc}) \rightarrow \big(\overleftarrow{\Box}(c_i \rightarrow \mathsf{qc}) \wedge \overrightarrow{\Box}(c_i \rightarrow \mathsf{qc})\big)$
m2 $(c_i \wedge \mathsf{inf}) \rightarrow \big(\overleftarrow{\Box}(\mathsf{inf} \rightarrow c_i) \wedge \overrightarrow{\Box}(\mathsf{inf} \rightarrow c_i)\big)$
m3 $\boxed{\scriptstyle c}A \leftrightarrow \Big(A \wedge \bigvee_{i=1}^{r} \Big(c_i \wedge \overleftarrow{\Box}(c_i \rightarrow A) \wedge \overrightarrow{\Box}(c_i \rightarrow A)\Big)\Big)$
m4 $\boxed{\scriptstyle n}A \leftrightarrow \Bigg(\Big(\mathsf{inf} \rightarrow \big(\overleftarrow{\Box}(\neg\mathsf{inf} \rightarrow A) \wedge \overrightarrow{\Box}(\neg\mathsf{inf} \rightarrow A)\big)\Big) \wedge$
$\Big((\mathsf{ns} \vee \mathsf{ps}) \rightarrow \big(\overleftarrow{\Box}(\mathsf{nl} \rightarrow A) \wedge \overrightarrow{\Box}(\mathsf{pl} \rightarrow A)\big)\Big)\Bigg)$

The mirror images of **K1, K2** and **K4** are also considered as axioms.

The intuitive meaning of the previous axioms is the following: **K1-K4** (and their mirror images) constitute a fragment of basic linear-time temporal logic; **c1** and **c2** state the existence and the unicity of the milestones in a frame, respectively; **c3-c7** state the ordering of these milestones. Axioms **p1** and **p2** state the existence and unicity, respectively, of proximity intervals; **p3** states that all points denoted by a proximity constant form an interval; **p4** states that every proximity constant denotes some proximity interval. **m1** states that the length of a qualitative class QC fully covers a given proximity interval. **m2** is specific to deal with INF, and states that this class is totally covered by a proximity interval (in combination with **m1**, this axiom implies that INF constitutes itself a proximity interval.) **m3-m4** enable the representation of closeness and negligibility in terms of white connectives and constants; this allows us to use, from now on, only white connectives and constants.

Rules of inference:

(MP) Modus Ponens for $\rightarrow$.
(N$\overrightarrow{\Box}$) If $\vdash A$ then $\vdash \overrightarrow{\Box}A$.
(N$\overleftarrow{\Box}$) If $\vdash A$ then $\vdash \overleftarrow{\Box}A$.

The syntactical notions of *theorem* and *proof* for $MQ^{\mathcal{P}}$ are defined as usual.

Soundness is straightforward, since it is easy to check that all the axioms are valid formulas and the inference results preserve validity.

The completeness follows by the step-by-step method, which is a Henkin-style proof, see [3]. The idea is to show that for any consistent formula A, a model for A can be built, and this is done by successive finite approximations.

Theorem 1 (Completeness). *If A is valid formula of $\mathcal{L}(MQ)^{\mathcal{P}}$, then A is a theorem of $MQ^{\mathcal{P}}$.*

Some words for Luis

– Te conocí hace muchos años y hemos coincidido pocas veces; pero la simpatía que despertaste en mí desde el primer momento no ha hecho sino crecer con los años. Hay algo que me llama la atención especialmente de tu persona, y es cómo has fusionado un sentido del humor jovial con el rigor intelectual. Espero, además, que permanezca eso en ti siempre y que podamos realizar tareas en común en los tiempos venideros, ya que la actividad que realizamos nunca se acaba, sólo se interrumpe.
  Querido Luis, quiero expresarte con estas pocas palabras mi admiración y cariño. Alfredo
– Admiro mucho tu trabajo y, aunque hemos coincidido poco en persona, he leído muchos de tus artículos En particular, tus trabajos relacionados con las cláusulas de Horn para lógicas modales han servido de inspiración para mi investigación actual sobre fragmentos sub-proposicionales para lógicas temporales de intervalos.
  En el Workshop realizado en Málaga el año pasado, pude asistir a tu charla y a tus comentarios en las charlas de los demás, incluida la mía. Me impresionó tu claridad y amplitud de ideas, así como la forma de expresarlos, proponiendo ideas nuevas y cuestiones muy interesantes.
  Espero que sigas vinculado a este mundo de la investigación y nos sigas regalando tu sabiduría. Gracias y enhorabuena. Emilio
– Coincidimos por primera vez hace casi veinticinco años (¡ya ha llovido!, incluso en Málaga). Durante una de nuestras visitas al Imperial College, en la obligada parada en el pub antes de buscar sitio para cenar, se formó un grupo en el que se hablaba, al menos, tres idiomas diferentes. ¿Quién estaba en la intersección? Por supuesto, Luis, al que yo contemplada embelesado, viendo cómo alternaba inglés, francés y español sin mayor problema, en función de la lengua de su

interlocutor. Lo curioso de tal capacidad es que, parece ser, Luis parece disponer de una versión científica, que le posibilita observar un problema desde distintos puntos de vista, de manera aparentemente simultánea, y obteniendo soluciones siempre originales y profundas.
Posteriormente, llegué a conocer algo más su faceta personal y disfrutar con su campechanía (stricto sensu) y con su especialísimo sentido del humor.
¿Qué más puedo decir? Que siempre ha sido un placer coincidir contigo en distintos eventos todos estos años y que, por supuesto, deseo que sigamos coincidiendo, más frecuentemente si cabe , ¿por qué no?, en tu etapa post-Festschrift.
Un fuerte abrazo. Manolo.

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