Relational approach for a logic for order of magnitude qualitative reasoning with negligibility, non-closeness and distance

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Abstract. We present a relational proof system in the style of dual tableaux for a multimodal propositional logic for order of magnitude qualitative reasoning to deal with relations of negligibility, non-closeness, and distance. This logic enables us to introduce the operation of qualitative sum for some classes of numbers. A relational formalization of the modal logic in question is introduced in this paper, i.e., we show how to construct a relational logic associated with the logic for order-of-magnitude reasoning and its dual tableau system which is a validity checker for the modal logic. For that purpose, we define a validity preserving translation of the modal language into relational language. Then we prove that the system is sound and complete with respect to the relational logic defined as well as with respect to the logic for order of magnitude reasoning. Finally, we show that in fact relational dual tableau does more. It can be used for performing the four major reasoning tasks: verification of validity, proving entailment of a formula from a finite set of formulas, model checking, and verification of satisfaction of a formula in a finite model by a given object.

Keywords: relational logics, dual tableau systems, multimodal propositional logic, order-of-magnitude qualitative reasoning

1 Introduction

Qualitative reasoning (QR) is the area of AI which tries to develop representation and reasoning techniques that enable a program to reason about the behaviour of physical systems, without the kind of precise quantitative information needed by conventional analysis techniques [31]. QR provides an intermediate level between discrete and continuous models [28], when we have to represent continuous aspects of the world, such as space, time, and quantity, which support reasoning with very little information [11].

A form of QR is to manage numerical data in terms of orders of magnitude [8, 9, 19, 24, 27]. Order of magnitude representations stratify values according to some notion of scale, for instance, by including hyperreal numbers [24], numerical thresholds [19], and logarithmic scales [21]. Three issues faced by all these formalisms are the conditions under which many small effects can combine to produce a significant effect, the soundness of the reasoning supported by the formalism, and the efficiency of using them. Order of magnitude reasoning has been developed from two points of view [29]: Absolute Order of Magnitude, which is represented by a partition of the real line $\mathbb{R}$, where each element of $\mathbb{R}$ belongs to a qualitative class, and Relative Order of Magnitude, introducing a family of binary order of magnitude relations which establish different comparison relations in $\mathbb{R}$ (e.g., *comparability*, *negligibility*, and *closeness*). We combine both in our approach, that is, we define different relations using the qualitative classes which appear in a specific absolute order of magnitude model.

The introduction of a logical approach in QR tries to solve the problem about the soundness of the reasoning supported by the formalism and it aims to give some answers about the efficiency of using that. Logics dealing with QR have been defined in many situations [1, 2, 25, 30], for example, for spatial and temporal reasoning. In particular, logics for order of magnitude reasoning have been studied in [4, 5, 7]. In this paper, we focus our attention on the multimodal propositional logic $\mathcal{L}(OM)^{\mathrm{NCD}}$ (from now on, OM for short) presented in [4], which introduces a logic to deal not only with negligibility and order of magnitude relations, but also with non-closeness and distance. This logic enables us to introduce the operation of qualitative sum for some classes of numbers and, in some way, to consider the problem about the conditions under which many small effects can combine to produce a significant effect.

Our definitions of non-closeness and negligibility are based in the election of 5 landmarks. This election was made following the ideas presented in [27, 28] and has many advantages such as the possibility of distinguishing between medium and large numbers. Possible applications of the logic OM can be considered in the field of modelling physical systems where we need to abstract the value domain of continuous variables into a finite set of qualitative values [4, 28].

It is well known that one of the main advantages in the use of the logic formalism is the possibility of having automated deduction systems. For this reason, we present a relational proof system in the style of dual tableaux for the relational logic associated to OM. We prove its soundness and completeness and we show how it can be used for performing the four major reasoning tasks: verification of validity, verification of entailment, model checking, and verification of satisfaction. The relational system presented in the paper is founded on Rasiowa-Sikorski system (RS) for the first-order logic [26] extended with the rules for equality predicate as presented in [16]. The election of this method has many advantages [18]. Namely, it provides a clear-cut method of generating proof rules from the semantics and the resulting deduction system is well suited for automated deduction purposes. Moreover, it provides a standard and intuitively simple way of proving completeness by constructing a model from the syntactic resources of the tree built during the proof search process which falsifies the non-provable formula. It enables an almost automatic way of transforming a complete dual tableau proof tree into a complete Gentzen calculus proof tree. Furthermore, for each particular theory we need only to expand the basic relational logic with specific relational constants and/or operators satisfying the appropriate axioms, then we design specific rules corresponding to given properties of a logic and we adjoin them to the core set of the rules. Hence, we need not implement each deduction system from the scratch, we should only extend the core system with a module corresponding to a specific part of a logic under consideration.

We apply the method known for various non-classical logics [22] in the construction of the system for OM. First, we construct a relational logic $\mathsf{RL}_{\mathsf{OM}}$ appropriate for expressing formulas of the logic OM. For that purpose, we define a validity preserving translation of OM-language into relational language. Then we construct a sound and complete deduction system based on dual tableaux for the relational logic $\mathsf{RL}_{\mathsf{OM}}$ so that it provides a validity checker for the modal logic in question. Finally, we extend this validity checker in order get a system for verifying entailment, model checking and verification of satisfaction. The relational logic $\mathsf{RL}_{\mathsf{OM}}$ is based on the relational logic of binary relations which is a logical counterpart to the class of full relation algebras [17,22]. The proof system developed in the paper is the extension of dual tableau for the relational logic of binary relations originated in [23], see also [17, 22].

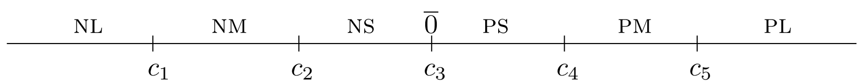
Other approaches to relational logics for order of magnitude reasoning have been presented in [6, 13] The first one only uses 3 qualitative classes, while the second one defines 5. Both papers introduce different notions of negligibility, however they do not consider any relation such as non-closeness nor distance.

Some implementations of these systems have been done. In [10] there is an implementation of the proof system for the classical relational logic and in [12] an implementation of translation procedures from non-classical logics to relational logic is presented. Focusing our attention on logics for order of magnitude reasoning, in [15] a theorem prover for the logic presented in this paper has been developed. Moreover, in [3] an implementation of the logic presented in [6] has been presented.

This paper is organized as follows: In Section 2, we define the syntax, semantics, and the axiomatization of the logic OM. In Section 3, we develop the relational logic appropriate for OM and a validity preserving translation for it. In Section 4 we present a complete relational proof system for logics in question. In Section 5, we show that the presented relational proof system can be used for verification of entailment, model checking, and verification of satisfaction of formulas of the logic OM. Finally, in Section 6, some conclusions and future work are commented.

2 The multimodal logic OM

In this section we present the logic OM introduced in [4]. We consider a strict linearly ordered set $(\mathbb{S}, <)$ [3] divided into seven equivalence classes using five landmarks chosen depending on the context [20, 28]. The system corresponds to the following schematic representation, where $c_i \in \mathbb{S}$, being $i \in \{1, 2, 3, 4, 5\}$ such that $c_j < c_{j+1}$, for all $j \in \{1, 2, 3, 4\}$:



In this paper we consider the following set of qualitative classes:

$$\text{NL} = (-\infty, c_1), \quad \text{NM} = [c_1, c_2) \quad \text{NS} = [c_2, c_3), \quad \overline{0} = \{c_3\}$$

$$\text{PS} = (c_3, c_4], \quad \text{PM} = (c_4, c_5], \quad \text{PL} = (c_5, +\infty)$$

As it could be expected, the labels correspond to "negative large", "negative medium", "negative small", "zero", "positive small", "positive medium", and "positive large", respectively.

After presenting the 'absolute part' of our approach, we introduce the 'relative part' with the concepts of order of magnitude, non-closeness, distance and negligibility. Firstly, we define the relation d to give the intuitive meaning of a constant distance. Let $(\mathbb{S}, <)$ be a strict linearly ordered set which contains the constants c_i, for $i \in \{1, 2, 3, 4, 5\}$ as defined above. We define d as a relation on $\mathbb{S}$ such that, for every $x, y, z, x', y' \in \mathbb{S}$ the following hold:

(i) If $x \,\mathsf{d}\, y$, then $x < y$
(ii) $c_j \,\mathsf{d}\, c_{j+1}$, for $j \in \{1, 2, 3, 4\}$.
(iii) If $x \,\mathsf{d}\, y$ and $x \,\mathsf{d}\, z$, then $y = z$.
(iv) If $x \,\mathsf{d}\, y$, $x' \mathsf{d}\, y'$ and $x < x'$ then $y < y'$.

In the definition above, we assume for simplicity that every pair of consecutive constants are at the same distance. However, the consideration of constants at different distance is a straightforward generalization that does not change the essence of our approach.

We now define the remaining relations on $\mathbb{S}$. Let us consider $(\mathbb{S}, <)$ be defined as above. For every $x, y \in \mathbb{S}$ we define the *order of magnitude* relation such that $x \text{OM} y$ if and only if $x, y \in \text{EQ}$, where EQ denotes a qualitative class, that is, an

[3] For practical purposes, this set could be the real line.

element in the set $\{\text{NL}, \text{NM}, \text{NS}, \overline{0}, \text{PS}, \text{PM}, \text{PL}\}$. Analogously, we define: $x\overline{\text{OM}}y$ whenever x, y do not belong to the same class. The relations of *non-closeness*, NC, and *distance*, D, are defined as follows, where $\mathsf{d}^2 = \mathsf{d} \circ \mathsf{d}$, being $\circ$ the usual composition of relations:

$$\begin{array}{lll} x\text{NC}\, y & \text{if and only if} & \text{either } x\overline{\text{OM}}y \text{ and } x < y \\ & & \text{or there exists } z \in \mathbb{S} \text{ such that } z < y \text{ and } x\,\mathsf{d}\,z, \end{array}$$

$$x\text{D}y \quad \text{if and only if} \quad \text{there exists } z \in \mathbb{S} \text{ such that } z < y \text{ and } x\,\mathsf{d}^2 z.$$

If we assume that $\mathbb{S}$ is a set of real numbers, the intuitive interpretation of the non-closeness relation is that x is non-close to y if and only if either x and y do not have the same order of magnitude or y is obtained from x by adding a positive medium or large number. On the other hand, x is distant from y if and only if y is obtained from x by adding positive large number.

In order to define the negligibility relation, we assume that if a non-zero element x is negligible with respect to y, then either x is distant to y or y is distant to x. Hence, we give the following definition for all $x, y \in \mathbb{S}$: x is *negligible* with respect to y (denoted by $x\text{N}y$) if and only if either of the following holds:

$$(i) \quad x = c_3 \qquad (ii) \quad x \in \text{NS} \cup \text{PS} \text{ and, either } y\text{D}c_2 \text{ or } c_4\text{D}y.$$

Let us observe that item (i) above corresponds to the intuitive idea that zero is negligible wrt any real number and item (ii) corresponds to the intuitive idea that a *sufficiently small* number is negligible wrt any *sufficiently large* number, independently of the sign of these numbers. This definition ensures that if $x \neq c_3$ and $x\text{N}y$, then either $x\text{D}y$ or $y\text{D}x$.

Notice that the relations NC, D and N can be defined in terms of $<, \mathsf{d}$, their inverses, and the constants. For this reason, from now on, we will only consider the last mentioned relations.

Let us introduce now the syntax and semantics of the logic OM. Consider a multimodal propositional language with a family of modal operators determined by accessibility relations. Expressions of the language are constructed with symbols from the following pairwise disjoint sets:

$\mathbb{V}$ - a set of propositional variables;
$\mathbb{C} = \{c_i \mid i \in \{1, 2, \dots, 5\}\}$ - the set of specific constants;
$\{<, \mathsf{d}\}$ - the set of accessibility relational constants;
$\{\neg, \wedge, \vee, \rightarrow\} \cup \{\Box_{\overrightarrow{\mathcal{R}}}, \Box_{\overleftarrow{\mathcal{R}}} : \mathcal{R} \in \{<, \mathsf{d}\}\}$ - the set of propositional operations.

As usual in modal logic, we use $\Diamond_{\overrightarrow{\mathcal{R}}}$, $\Diamond_{\overleftarrow{\mathcal{R}}}$ as abbreviations for $\neg\Box_{\overrightarrow{\mathcal{R}}}\neg$ and $\neg\Box_{\overleftarrow{\mathcal{R}}}\neg$, respectively, where $\overleftarrow{\mathcal{R}}$ is the inverse of relation $\overrightarrow{\mathcal{R}}$, for $\mathcal{R} \in \{<, \mathsf{d}\}$.

The set of OM-*formulas* is the smallest set including $\mathbb{V} \cup \mathbb{C}$ and closed on propositional operations.

For the semantics, we define an OM-*model* as a tuple $\mathcal{M} = (U, m)$, where U is a non-empty set, whose elements are called *states*, and m is a meaning function satisfying the following conditions:

1. $m(p) \subseteq U$, for every $p \in \mathbb{V}$;
2. $m(c) \in U$, for every $c \in \mathbb{C}$;
3. $m(<)$ is a strict linear ordering on U, that is, for all $s, s', s'' \in U$ the following conditions are satisfied:

   (Irref) $(s, s) \notin m(<)$,
   (Tran) if $(s, s') \in m(<)$ and $(s', s'') \in m(<)$, then $(s, s'') \in m(<)$,
   (Lin) $(s, s') \in m(<)$ or $(s', s) \in m(<)$ or $s = s'$;

4. $m(\mathsf{d})$ is a binary relation on U such that for all $s, s', s'', s''' \in U$, we have:

   (i) $m(\mathsf{d}) \subseteq m(<)$,
   (ii) $(m(c_j), m(c_{j+1})) \in m(\mathsf{d})$, for $j \in \{1, 2, 3, 4\}$,
   (iii) If $(s, s'), (s, s'') \in m(\mathsf{d})$, then $s' = s''$,
   (iv) If $(s, s'), (s'', s''') \in m(\mathsf{d})$, and $(s, s'') \in m(<)$, then $(s', s''') \in m(<)$.

Remark 1. Note that item 4 reflects the definition of relation d presented above.

Let φ be an OM-formula and let $\mathcal{M} = (U, m)$ be an OM-model. The *satisfaction of φ in $\mathcal{M}$ by a state s*, $\mathcal{M}, s \models \varphi$ for short, is defined inductively as follows, where $\mathcal{R} \in \{<, \mathsf{d}\}$:

$\mathcal{M}, s \models p$ iff $s \in m(p)$, for any $p \in \mathbb{V}$;
$\mathcal{M}, s \models c$ iff $s = m(c)$, for any $c \in \mathbb{C}$;
$\mathcal{M}, s \models \neg\varphi$ iff not $\mathcal{M}, s \models \varphi$;
$\mathcal{M}, s \models (\varphi \vee \psi)$ iff $\mathcal{M}, s \models \varphi$ or $\mathcal{M}, s \models \psi$;
$\mathcal{M}, s \models (\varphi \wedge \psi)$ iff $\mathcal{M}, s \models \varphi$ and $\mathcal{M}, s \models \psi$;
$\mathcal{M}, s \models (\varphi \rightarrow \psi)$ iff $\mathcal{M}, s \models \neg\varphi \vee \psi$;
$\mathcal{M}, s \models \Box_{\overrightarrow{\mathcal{R}}}\varphi$ iff for all $s' \in U$, $(s, s') \in m(\mathcal{R})$ implies $\mathcal{M}, s' \models \varphi$;
$\mathcal{M}, s \models \Box_{\overleftarrow{\mathcal{R}}}\varphi$ iff for all $s' \in U$, $(s', s) \in m(\mathcal{R})$ implies $\mathcal{M}, s' \models \varphi$

An OM-formula φ is said to be *satisfiable* whenever there exist an OM-model $\mathcal{M}$ and a state $s \in U$ such that $\mathcal{M}, s \models \varphi$. An OM-formula φ is *true* in an OM-model $\mathcal{M} = (U, m)$ whenever φ is satisfied in $\mathcal{M}$ by all states $s \in U$. An OM-formula φ is OM-*valid*, denoted by $\models \varphi$, whenever it is true in all OM-models.

In the subsequent part of this section the axiomatization of the logic OM is presented. From now on, the connectives $\Box_{\overrightarrow{<}}$, $\Box_{\overleftarrow{<}}$, $\Diamond_{\overrightarrow{<}}$, and $\Diamond_{\overleftarrow{<}}$ are denoted by $\overrightarrow{\Box}$, $\overleftarrow{\Box}$, $\overrightarrow{\Diamond}$, $\overleftarrow{\Diamond}$, respectively.

The sound and complete axiom system of OM consists of all tautologies of classical propositional logic together with the following axiom schemata [4]:

Axiom schemata for modal connectives:

K1 $\overrightarrow{\Box}(A \rightarrow B) \rightarrow (\overrightarrow{\Box}A \rightarrow \overrightarrow{\Box}B)$
K2 $A \rightarrow \overrightarrow{\Box}\overleftarrow{\Diamond}A$
K3 $\overrightarrow{\Box}A \rightarrow \overrightarrow{\Box}\overrightarrow{\Box}A$
K4 $\big(\overrightarrow{\Box}(A \vee B) \wedge \overrightarrow{\Box}(\overrightarrow{\Box}A \vee B) \wedge \overrightarrow{\Box}(A \vee \overrightarrow{\Box}B)\big) \rightarrow (\overrightarrow{\Box}A \vee \overrightarrow{\Box}B)$

Axiom schemata for constants:

C1 $\overleftarrow{\Diamond} c_i \vee c_i \vee \overrightarrow{\Diamond} c_i$, where $i \in \{1, \ldots, 5\}$
C2 $c_i \rightarrow (\overleftarrow{\Box} \neg c_i \wedge \overrightarrow{\Box} \neg c_i)$, being $i \in \{1, \ldots, 5\}$

Axiom schemata for specific modal connectives:

d1 $\overrightarrow{\Box} A \rightarrow \Box_{\overrightarrow{\mathsf{d}}} A$.
d2 $\Diamond_{\overrightarrow{\mathsf{d}}} A \rightarrow \Box_{\overrightarrow{\mathsf{d}}} A$
d3 $(\Diamond_{\overrightarrow{\mathsf{d}}} A \wedge \overrightarrow{\Diamond} \Diamond_{\overrightarrow{\mathsf{d}}} B) \rightarrow \overrightarrow{\Diamond}(A \wedge \overrightarrow{\Diamond} B)$
d4 $c_j \rightarrow \Diamond_{\overrightarrow{\mathsf{d}}} c_{j+1}$, where $j \in \{1, \ldots, 4\}$
d5 $\Box_{\overrightarrow{\mathsf{d}}} (A \rightarrow B) \rightarrow (\Box_{\overrightarrow{\mathsf{d}}} A \rightarrow \Box_{\overrightarrow{\mathsf{d}}} B)$
d6 $A \rightarrow \Box_{\overrightarrow{\mathsf{d}}} \Diamond_{\overleftarrow{\mathsf{d}}} A$.

The corresponding mirror images of **K1**–**K4** and **d1**–**d6** are also considered as axioms.

Rules of Inference:

(MP) Modus Ponens for $\rightarrow$
(R$\overrightarrow{\Box}$) If $\vdash A$ then $\vdash \overrightarrow{\Box} A$
(R$\overleftarrow{\Box}$) If $\vdash A$ then $\vdash \overleftarrow{\Box} A$

Remark 2. Notice that axioms **d1**–**d4** reflect syntactic definition of relation d.

3 Relational formalization of OM

The language of the logic $\mathsf{RL_{OM}}$ appropriate for expressing OM-formulas consists of the following pairwise disjoint sets of symbols:

$\mathbb{OV} = \{x, y, z, \ldots\}$ - a countably infinite set of object variables;
$\mathbb{OC} = \{c_i : i \in \{1, \ldots, 5\}\}$ - the set of object constants;
$\mathbb{RV} = \{P, Q, \ldots\}$ - a countably infinite set of binary relational variables;
$\mathbb{RC} = \{1, 1', <, \mathsf{d}\} \cup \{\Psi_i : i \in \{1, \ldots, 5\}\}$ - the set of relational constants;
$\mathbb{OP} = \{-, \cup, \cap, ;, {}^{-1}\}$ - the set of relational operation symbols.

The set of *relational terms* $\mathbb{RT}$ is the smallest set of expressions including the set $\mathbb{RA} = \mathbb{RV} \cup \mathbb{RC}$ of atomic terms and closed with respect to the operation symbols from $\mathbb{OP}$. The set $\mathbb{FR}$ of $\mathsf{RL_{OM}}$-*formulas* (or, simply formulas if it is clear from the context), consists of expressions of the form xPy, where $x, y \in \mathbb{OS} = \mathbb{OV} \cup \mathbb{OC}$ and $P \in \mathbb{RT}$.

An $\mathsf{RL_{OM}}$-*model* is a pair $\mathcal{M} = (U, m)$, where U is a non-empty set and m is a meaning function defined as follows:

1. $m(c) \in U$, for every $c \in \mathbb{OC}$;
2. $m(P) \subseteq U \times U$, for any $P \in \mathbb{RV}$;

3. $m(1')$ is an equivalence relation on U;
4. $m(1'); m(P) = m(P); m(1') = m(P)$, for every $P \in \mathbb{RA}$ (extensionality property);
5. $m(1) = U \times U$;
6. $m(<)$ is a binary relation on U such that for all $s, s', s'' \in U$ the following conditions are satisfied:

   (Irref) $(s, s) \notin m(<)$,
   (Tran) if $(s, s') \in m(<)$ and $(s', s'') \in m(<)$, then $(s, s'') \in m(<)$,
   (Lin) $(s, s') \in m(<)$ or $(s', s) \in m(<)$ or $(s, s') \in m(1')$;
7. $m(\Psi_i) = \{(s, s') \in U \times U : (s, m(c_i)) \in m(1')\}$, for every $i \in \{1, \ldots, 5\}$;
8. $m(\mathsf{d})$ is a binary relation on U such that, for all $s, s', s'', s''' \in U$:
   (i) $m(\mathsf{d}) \subseteq m(<)$,
   (ii) $(m(c_j), m(c_{j+1})) \in m(\mathsf{d})$, for $j \in \{1, \ldots, 4\}$,
   (iii) If $(s, s'), (s, s'') \in m(\mathsf{d})$, then $(s', s'') \in m(1')$,
   (iv) If $(s, s'), (s'', s''') \in m(\mathsf{d})$, and $(s, s'') \in m(<)$, then $(s', s''') \in m(<)$.
9. m extends to all the compound relational terms as follows:
   $m(-P) = m(1) \cap -m(P)$,
   $m(P \cup Q) = m(P) \cup m(Q)$,
   $m(P \cap Q) = m(P) \cap m(Q)$,
   $m(P^{-1}) = m(P)^{-1}$,
   $m(P; Q) = m(P); m(Q)$.

An $\mathsf{RL}_{\mathsf{OM}}$-model $\mathcal{M} = (U, m)$ is said to be *standard* whenever $m(1')$ is the identity on U, that is $m(1') = \{(x, x) : x \in U\}$ [4]. A standard $\mathsf{RL}_{\mathsf{OM}}$-model is referred to as an $\mathsf{RL}^*_{\mathsf{OM}}$-model. A *valuation* in an $\mathsf{RL}_{\mathsf{OM}}$-model $\mathcal{M} = (U, m)$ is a function $v \colon \mathbb{OS} \to U$ such that $v(c) = m(c)$, for every $c \in \mathbb{C}$. Let xPy be an $\mathsf{RL}_{\mathsf{OM}}$-formula and let $\mathcal{M} = (U, m)$ be an $\mathsf{RL}_{\mathsf{OM}}$-model. A formula xPy is said to be *satisfied in $\mathcal{M}$ by v* whenever $(v(x), v(y)) \in m(P)$. A formula xPy is *true* in $\mathcal{M}$ if it is satisfied in $\mathcal{M}$ by all valuations v. xPy is said to be $\mathsf{RL}_{\mathsf{OM}}$-*valid*, if it is true in all $\mathsf{RL}_{\mathsf{OM}}$-models. Moreover, a formula is said to be $\mathsf{RL}^*_{\mathsf{OM}}$-*valid* whenever it is true in all standard models.

The following result is well known:

Proposition 1. *For every $\mathsf{RL}_{\mathsf{OM}}$-formula φ:*

$$\varphi \textit{ is } \mathsf{RL}_{\mathsf{OM}}\textit{-valid iff } \varphi \textit{ is } \mathsf{RL}^*_{\mathsf{OM}}\textit{-valid.}$$

The translation of OM-formulas into relational terms starts with a one-to-one assignment of relational variables to the propositional variables. Let τ' be such an assignment. Then the translation τ of OM-formulas is defined inductively as follows:

$\tau(p) = \tau'(p); 1$, for any propositional variable $p \in \mathbb{V}$;
$\tau(c_i) = \Psi_i; 1$, for any $i \in \{1, \ldots, 5\}$;

[4] Note that in standard models $m(<)$ is a strict linear ordering on U

$\tau(\neg\varphi) = -\tau(\varphi)$;
$\tau(\varphi \vee \psi) = \tau(\varphi) \cup \tau(\psi)$;
$\tau(\varphi \wedge \psi) = \tau(\varphi) \cap \tau(\psi)$;
$\tau(\varphi \rightarrow \psi) = -\tau(\varphi) \cup \tau(\psi)$;

and for $\mathcal{R} \in \{<, \mathsf{d}\}$:

$\tau(\Box_{\overrightarrow{\mathcal{R}}}\varphi) = -(\mathcal{R}; -\tau(\varphi))$;
$\tau(\Box_{\overleftarrow{\mathcal{R}}}\varphi) = -(\mathcal{R}^{-1}; -\tau(\varphi))$.

Notice that every OM-formula φ is associated to the relational term $\tau(\varphi)$, which is a right ideal relation i.e., a relation Q that satisfies $Q = Q; 1$.

The translation τ is defined so that it preserves validity of formulas.

Proposition 2. *Let φ be an OM-formula. Then for every OM-model $\mathcal{M} = (U, m)$ there exists an $\mathsf{RL}^*_{\mathsf{OM}}$-model $\mathcal{M}' = (U, m')$ with the same universe as in $\mathcal{M}$ such that for all $s, s' \in U$ the following holds:*

$$(*)\, \mathcal{M}, s \models \varphi \textit{ iff } (s, s') \in m'(\tau(\varphi)).$$

Proof. Let φ be an OM-formula, let $\mathcal{M} = (U, m)$ be an OM-model. Then we define an $\mathsf{RL}^*_{\mathsf{OM}}$-model $\mathcal{M}' = (U, m')$ as follows:

$m'(1) = U \times U$;
$m'(1')$ is an identity on U;
$m'(\tau(p)) = \{(x, y) \in U \times U : x \in m(p)\}$, for every propositional variable p;
$m'(c_i) = m(c_i)$, for every $i \in \{1, \dots, 5\}$;
$m'(\Psi_i) = \{x \in U' : (x, m'(c_i)) \in m'(1')\} \times U$, for every $i \in \{1, \dots, 5\}$;
$m'(\mathcal{R}) = m(\mathcal{R})$, for $\mathcal{R} \in \{<, \mathsf{d}\}$;
m' extends to all the compound terms as in $\mathsf{RL}_{\mathsf{OM}}$-models.

Clearly, the model defined above is an $\mathsf{RL}^*_{\mathsf{OM}}$-model. Now we prove $(*)$ by induction on the complexity of formulas. Let $s, s' \in U$.

Let $\varphi := p$, for $p \in \mathbb{V}$. Then $\mathcal{M}, s \models p$ iff $s \in m(p)$ iff $(s, s') \in m'(\tau(p))$, since $m'(\tau(p))$ is a right ideal relation.

Let $\varphi := c_i$, for some $i \in \{1, \dots, 5\}$. Then $\mathcal{M}, s \models c_i$ iff $s = m(c_i)$ iff $(s, m(c_i)) \in m'(1')$ iff $(s, s') \in m'(\Psi_i; 1)$ iff $(s, s') \in m'(\tau(c_i))$.

Let $\varphi := \psi \vee \vartheta$. Then $\mathcal{M}, s \models \psi \vee \vartheta$ iff $\mathcal{M}, s \models \psi$ or $\mathcal{M}, s \models \vartheta$ iff (by the induction hypothesis) $(s, s') \in m'(\tau(\psi))$ or $(s, s') \in m'(\tau(\vartheta))$ iff $(s, s') \in m'(\tau(\psi)) \cup m'(\tau(\vartheta))$ iff $(s, s') \in m'(\tau(\psi \vee \vartheta))$.

Let $\varphi := \Box_{\overrightarrow{\mathcal{R}}}\psi$, for some $\mathcal{R} \in \{<, \mathsf{d}\}$. Note that by the induction hypothesis, for all $t, s' \in U$ the following holds: $\mathcal{M}, t \models \psi$ iff $(t, s') \in m'(\tau(\psi))$. Therefore: $\mathcal{M}, s \models \varphi$ iff for all $t \in U$, if $(s, t) \in m(\mathcal{R})$, then $\mathcal{M}, t \models \psi$ iff (by the induction hypothesis) for all $t \in U$, if $(s, t) \in m'(\mathcal{R})$, then $(t, s') \in m'(\tau(\psi))$ iff $(s, s') \in m'(-(\mathcal{R}; -\tau(\psi)))$ iff $(s, s') \in m'(\tau(\varphi))$.

The proofs of the remaining cases are similar. $\square$

Proposition 3. *Let φ be an* OM*-formula. Then for every* $\mathsf{RL}^*_{\mathsf{OM}}$*-model $\mathcal{M}' = (U, m')$ there exists an* OM*-model $\mathcal{M} = (U, m)$ with the same universe as in $\mathcal{M}'$ such that for all $s, s' \in U$ condition $(*)$ of Proposition 2 holds.*

Proof. Let φ be an OM-formula, let $\mathcal{M}' = (U, m')$ be an $\mathsf{RL}^*_{\mathsf{OM}}$-model. Then we define an OM-model $\mathcal{M} = (U, m)$ as follows:

- $m(p) = \{(x \in U : \text{ for some } y \in U,\ (x, y) \in m'(\tau(p))\}$, for every propositional variable p;
- $m(c_i) = m'(c_i)$, for every $i \in \{1, \dots, 5\}$;
- $m(\mathcal{R}) = m'(\mathcal{R})$, for $\mathcal{R} \in \{<, \mathsf{d}\}$.

It is easy to see that the model defined above is an OM-model. Condition $(*)$ can be proved similarly as in Proposition 2. □

Proposition 4. *Let φ be an* OM*-formula. Then for every* OM*-model $\mathcal{M}$ there exists an* $\mathsf{RL}_{\mathsf{OM}}$*-model $\mathcal{M}'$ such that for all object variables x and y the following holds:*

$$(**)\ \mathcal{M} \models \varphi \text{ iff } \mathcal{M}' \models x\tau(\varphi)y.$$

Proof. Let φ be an OM-formula and let $\mathcal{M} = (U, m)$ be an OM-model. Then we define an $\mathsf{RL}_{\mathsf{OM}}$-model $\mathcal{M}' = (U, m')$ as in the proof of Proposition 2. Let x and y be any object variables. Assume $\mathcal{M} \models \varphi$. Suppose there exists a valuation v in $\mathcal{M}'$ such that $\mathcal{M}', v \not\models x\tau(\varphi)y$. Then $(v(x), v(y)) \notin m'(\tau(\varphi))$. However, by Proposition 2, models $\mathcal{M}$ and $\mathcal{M}'$ satisfy: $\mathcal{M}, v(x) \models \varphi$ iff $(v(x), v(y)) \in m'(\tau(\varphi))$. Therefore, $\mathcal{M}, v(x) \not\models \varphi$, and hence $\mathcal{M} \not\models \varphi$, a contradiction Assume $\mathcal{M}' \models x\tau(\varphi)y$. Suppose there exists $s \in U$ such that $\mathcal{M}, s \not\models \varphi$. Let s' be any element of U. By Proposition 2 the following holds: $\mathcal{M}, s \models \varphi$ iff $(s, s') \in m'(\tau(\varphi))$. Let v be a valuation in $\mathcal{M}'$ such that $v(x) = s$ and $v(y) = s'$. Since $\mathcal{M}, s \not\models \varphi$, $(v(x), v(y)) \notin m'(\tau(\varphi))$. Thus, $\mathcal{M}', v \not\models x\tau(\varphi)y$, and hence $\mathcal{M}' \not\models x\tau(\varphi)y$, a contradiction. □

Due to Proposition 3, the following can be proved similarly as Proposition 4.

Proposition 5. *Let φ be an* OM*-formula. Then for every* $\mathsf{RL}^*_{\mathsf{OM}}$*-model $\mathcal{M}'$ there exists an* OM*-model $\mathcal{M}$ such that for all object variables x and y, condition $(**)$ of Proposition 4 holds.*

From Propositions 4 and 5 we obtain the following theorem that shows the semantic relationship between OM and $\mathsf{RL}_{\mathsf{OM}}$:

Theorem 1. *For every* OM*-formula φ and for all object variables x and y the following conditions are equivalent:*

1. *φ is* OM*-valid;*
2. *$x\tau(\varphi)y$ is* $\mathsf{RL}_{\mathsf{OM}}$*-valid.*

Proof. ($\rightarrow$) Let φ be OM-valid. Suppose $x\tau(\varphi)y$ is not $\mathsf{RL}_{\mathsf{OM}}$-valid. Then there exists an $\mathsf{RL}^*_{\mathsf{OM}}$-model $\mathcal{M}$ such that $\mathcal{M} \not\models x\tau(\varphi)y$. By Proposition 5, there is an OM-model $\mathcal{M}'$ such that $\mathcal{M}' \not\models \varphi$, which contradicts the assumption of OM-validity of φ.

($\leftarrow$) Let φ be an OM-formula such that $x\tau(\varphi)y$ is $\mathsf{RL}_{\mathsf{OM}}$-valid. Suppose φ is not OM-valid. Then there exists an OM-model $\mathcal{M}$ such that $\mathcal{M} \not\models \varphi$. By Proposition 4, there exists an $\mathsf{RL}^*_{\mathsf{OM}}$-model $\mathcal{M}'$ such that $\mathcal{M}' \not\models x\tau(\varphi)y$, a contradiction. □

4 Relational dual tableau for $\mathsf{RL}_{\mathsf{OM}}$

The proof system for logic $\mathsf{RL}_{\mathsf{OM}}$ presented in this section belongs to the family of dual tableau systems. Dual tableau systems are determined by axiomatic sets of formulas and rules which apply to finite sets of formulas. The axiomatic sets take the place of axioms. There are two groups of rules: the *decomposition rules*, which reflect definitions of the standard relational operations, and the *specific rules* which reflect the properties of the specific relations assumed in $\mathsf{RL}_{\mathsf{OM}}$-models. The rules have the following general form:

$$(*) \qquad \frac{\Phi}{\Phi_1 \mid \ldots \mid \Phi_n}$$

where $\Phi_1, \ldots, \Phi_n$ are finite non-empty sets of formulas, $n \geq 1$, and Φ is a finite (possibly empty) set of formulas. Φ is called the *premise* of the rule, and $\Phi_1, \ldots, \Phi_n$ are called its *conclusions*. A rule of the form $(*)$ is said to be *applicable* to a set X of formulas whenever $\Phi \subseteq X$. As a result of an application of a rule of the form $(*)$ to a set X, we obtain the sets $(X \setminus \Phi) \cup \Phi_i$, $i = 1, \ldots, n$. As usual, any concrete rule will always be presented in a short form, that is we will omit set brackets. We say that an object variable in a rule is *new* whenever it appears in a conclusion of the rule and does not appear in its premise.

Figure 1 shows the decomposition rules of $\mathsf{RL}_{\mathsf{OM}}$-dual tableau, for all object symbols $x, y \in \mathbb{OS}$ and for all relational terms $P, Q \in \mathbb{RT}$. Specific rules of $\mathsf{RL}_{\mathsf{OM}}$-dual tableau are given in Figure 2, for all object symbols $x, y \in \mathbb{OS}$, for every atomic relational term P, for every $i \in \{1, \ldots, 5\}$, where z, t are any object symbols.
A finite set of $\mathsf{RL}_{\mathsf{OM}}$-formulas is said to be an $\mathsf{RL}_{\mathsf{OM}}$-*axiomatic set* whenever it includes either of the following subsets, for any $x, y \in \mathbb{OS}, P \in \mathbb{RT}$:

(Ax1) $\{x1'x\}$; (Ax2) $\{x1y\}$
(Ax3) $\{xPy, x{-}Py\}$; (Ax4) $\{c_j \,\mathsf{d}\, c_{j+1}\}$, for any $j \in \{1, \ldots, 4\}$
(Ax5) $\{x < y, y < x, x1'y\}$.

A finite set of $\mathsf{RL}_{\mathsf{OM}}$-formulas $\{\varphi_1, \ldots, \varphi_k\}$ is said to be an $\mathsf{RL}_{\mathsf{OM}}$-*set* whenever for every $\mathsf{RL}_{\mathsf{OM}}$-model $\mathcal{M}$ and for every valuation v in $\mathcal{M}$ there exists $i \in \{1, \ldots, k\}$ such that $\mathcal{M}, v \models \varphi_i$. A rule $\frac{\Phi}{\Phi_1 \mid \ldots \mid \Phi_n}$, $n \geq 1$, is $\mathsf{RL}_{\mathsf{OM}}$-*correct* whenever for every finite set X of $\mathsf{RL}_{\mathsf{OM}}$-formulas the following holds: $X \cup \Phi$ is an $\mathsf{RL}_{\mathsf{OM}}$-set if and only if $X \cup \Phi_i$ is an $\mathsf{RL}_{\mathsf{OM}}$-set for every $i \in \{1, \ldots, n\}$.

$(\cup)$	$\dfrac{x(P\cup Q)y}{xPy, xQy}$	$(-\cup)$	$\dfrac{x-(P\cup Q)y}{x-Py \mid x-Qy}$
$(\cap)$	$\dfrac{x(P\cap Q)y}{xPy \mid xQy}$	$(-\cap)$	$\dfrac{x-(P\cap Q)y}{x-Py, x-Qy}$
$(-)$	$\dfrac{x--Py}{xPy}$		
$({}^{-1})$	$\dfrac{xP^{-1}y}{yPx}$	$(-{}^{-1})$	$\dfrac{x-P^{-1}y}{y-Px}$
$(;)$	$\dfrac{x(P;Q)y}{xPz, x(P;Q)y \mid zQy, x(P;Q)y}$	z is any object symbol	
$(-;)$	$\dfrac{x-(P;Q)y}{x-Pw, w-Qy}$	w is a new object variable	

Fig. 1. Decomposition rules

$(1'1)$	$\dfrac{xPy}{xPz, xPy \mid y1'z, xPy}$	$(1'2)$	$\dfrac{xPy}{x1'z, xPy \mid zPy, xPy}$
(Irref<)	$\dfrac{}{x<x}$	(Tran<)	$\dfrac{x<y}{x<y, x<z \mid x<y, z<y}$
(C_i1)	$\dfrac{}{x\Psi_i y \mid x-\Psi_i y}$	(C_i2)	$\dfrac{x\Psi_i y}{x\Psi_i y, x1'c_i}$
(C_i3)	$\dfrac{x-\Psi_i y}{x-\Psi_i y, x-1'c_i}$	(D1)	$\dfrac{x<y}{x\,\mathsf{d}\,y, x<y}$
(D2)	$\dfrac{x1'y}{z\,\mathsf{d}\,x, x1'y \mid z\,\mathsf{d}\,y, x1'y}$	(D3)	$\dfrac{x<y}{z\,\mathsf{d}\,x, x<y \mid t\,\mathsf{d}\,y, x<y \mid z<t, x<y}$

Fig. 2. Specific rules

Due to the semantics we obtain the following:

Proposition 6.

1. *The decomposition rules are* $\mathsf{RL_{OM}}$*-correct.*
2. *The specific rules are* $\mathsf{RL_{OM}}$*-correct.*
3. *The axiomatic sets are* $\mathsf{RL_{OM}}$*-sets.*

Proof. By way of example we prove correctness of the rule (D3). It is easy to see that for every set X of $\mathsf{RL_{OM}}$-formulas, if $X \cup \{x < y\}$ is an $\mathsf{RL_{OM}}$-set, then $X \cup \{z\mathsf{d}x, x < y\}$, $X \cup \{w\mathsf{d}y, x < y\}$, and $X \cup \{z < w, x < y\}$ are $\mathsf{RL_{OM}}$-sets as well. For the other direction, assume that $X \cup \{z\mathsf{d}x, x < y\}$, $X \cup \{w\mathsf{d}y, x < y\}$, and $X \cup \{z < w, x < y\}$ are $\mathsf{RL_{OM}}$-sets. Suppose $X \cup \{x < y\}$ is not an $\mathsf{RL_{OM}}$-set. Then there exist an $\mathsf{RL_{OM}}$-model $\mathcal{M}$ and a valuation v in $\mathcal{M}$ such that for every $\varphi \in X$, $\mathcal{M}, v \not\models \varphi$ and $\mathcal{M}, v \not\models x < y$. Therefore, $(v(x), v(y)) \notin m(<)$. On the other hand, by the assumption, $(v(z), v(x)) \in m(\mathsf{d})$, $(v(w), v(y)) \in m(\mathsf{d})$, and $(v(z), v(w)) \in m(<)$. Thus, by the condition (iv) of $\mathsf{RL_{OM}}$-models, $(v(x), v(y)) \in m(<)$, a contradiction.

The proof of correctness of the remaining rules are similar. □

An $\mathsf{RL_{OM}}$-*proof tree* for xPy is a tree with the following properties:

- the formula xPy is at the root of this tree;
- each node except the root is obtained by an application of an $\mathsf{RL_{OM}}$-rule to its predecessor node;
- a node does not have successors whenever it is an $\mathsf{RL_{OM}}$-axiomatic set.

Remark 3. Due to the forms of the rules for atomic formulas, if a node of an $\mathsf{RL_{OM}}$-proof tree contains an $\mathsf{RL_{OM}}$-formula xPy or $x{-}Py$, for some atomic P, then all of its successors contain this formula as well.

A branch of an $\mathsf{RL_{OM}}$-proof tree is said to be *closed* whenever it contains a node with an $\mathsf{RL_{OM}}$-axiomatic set of formulas. A *closed tree* is an $\mathsf{RL_{OM}}$-proof tree such that all of its branches are closed. A formula xPy is $\mathsf{RL_{OM}}$-*provable* whenever there is a closed proof tree for xPy, which is then referred to as an $\mathsf{RL_{OM}}$-*proof of* xPy.

By Proposition 6, we obtain the soundness of $\mathsf{RL_{OM}}$-dual tableau:

Theorem 2 (Soundness of $\mathsf{RL_{OM}}$). *Let φ be an* $\mathsf{RL_{OM}}$*-formula. If φ is* $\mathsf{RL_{OM}}$*-provable, then it is* $\mathsf{RL_{OM}}$*-valid.*

Since $\mathsf{RL_{OM}}$-validity implies $\mathsf{RL^*_{OM}}$-validity, we obtain the following:

Corollary 1. *If φ is* $\mathsf{RL_{OM}}$*-provable, then it is* $\mathsf{RL^*_{OM}}$*-valid.*

As usual in the proof theory a concept of completeness of a non-closed proof tree is needed. Intuitively, completeness of a non-closed tree means that all the rules that can be applied have been applied. By abusing the notation, for any branch b and for any set of formulas X, by $X \in b$ (resp. $X \notin b$) we mean that every formula from X belongs to b (resp. does not belong to b).

A branch b of a proof tree is said to be *complete* whenever for all $x, y \in \mathbb{OS}$, for all $P, Q \in \mathbb{RT}$, and for every $i \in \{1, \ldots, 5\}$ it satisfies the following completion conditions:

Cpl($\cup$) (resp. Cpl($-\cap$)) If $x(P \cup Q)y \in b$ (resp. $x-(P \cap Q)y \in b$), then both $xPy \in b$ and $xQy \in b$ (resp. $x-Py \in b$ and $x-Qy \in b$);

Cpl($\cap$) (resp. Cpl($-\cup$)) If $x(P \cap Q)y \in b$ (resp. $x-(P \cup Q)y \in b$), then either $xPy \in b$ or $xQy \in b$ (resp. either $x-Py \in b$ or $x-Qy \in b$);

Cpl($-$) If $x(--P)y \in b$, then $xPy \in b$;

Cpl($^{-1}$) If $xP^{-1}y \in b$, then $yPx \in b$;

Cpl($-^{-1}$) If $x-P^{-1}y \in b$, then $y-Px \in b$;

Cpl(;) If $x(P;Q)y \in b$, then for every object symbol z, either $xPz \in b$ or $zQy \in b$;

Cpl($-$;) If $x-(P;Q)y \in b$, then for some object variable w, both $x-Pw \in b$ and $w-Qy \in b$;

Cpl($1'1$) If $xPy \in b$, for some atomic relational term P, then for every object symbol z, either $xRz \in b$ or $y1'z \in b$;

Cpl($1'2$) If $xPy \in b$, for some atomic relational term P, then for every object symbol z, either $x1'z \in b$ or $zPy \in b$;

Cpl(C_i1) Either $x\Psi_i y \in b$ or $x-\Psi_i y \in b$;

Cpl(C_i2) If $x\Psi_i y \in b$, then $x1'c_i \in b$;

Cpl(C_i3) If $x-\Psi_i y \in b$, then $x-1'c_i \in b$.

Cpl(Irref$<$) For every object symbol x, $x < x \in b$;

Cpl(Tran$<$) If $x < y \in b$, then for every object symbol z, either $x < z \in b$ or $z < y \in b$;

Cpl(D1) If $x < y \in b$, then $x \,\mathsf{d}\, y \in b$;

Cpl(D2) If $x1'y \in b$, then for every object symbol z, either $z \,\mathsf{d}\, x \in b$ or $z \,\mathsf{d}\, y \in b$;

Cpl(D3) If $x < y \in b$, then for all object symbols z and t, either $z \,\mathsf{d}\, x \in b$ or $t \,\mathsf{d}\, y \in b$ or $z < t \in b$.

An $\mathsf{RL_{OM}}$-proof tree is said to be *complete* whenever all of its branches are complete. A complete non-closed branch is said to be *open*.

As we said in the introduction, there is a standard and intuitively simple way of proving completeness. Thus, given a tree of a non-provable formula, we construct a model by means of syntactic resources of the tree. Then, we show that the model defined in this way falsifies a non-provable formula.

Let b be an open branch of an $\mathsf{RL_{OM}}$-proof tree. A *branch structure* $\mathcal{M}^b$ is a pair $\mathcal{M}^b = (U^b, m^b)$, such that:

$U^b = \mathbb{OS}$;
$m^b(c_i) = c_i$, for every $i \in \{1, \ldots, 5\}$;
$m^b(P) = \{(x, y) \in U^b \times U^b : xPy \notin b\}$, for every $P \in \mathbb{RA}$;
m^b extends to all the compound relational terms as in $\mathsf{RL_{OM}}$-models.

Proposition 7 (Branch Model Property). *A branch structure $\mathcal{M}^b = (U^b, m^b)$ determined by an open branch of an $\mathsf{RL_{OM}}$-proof tree is an $\mathsf{RL_{OM}}$-model.*

Proof. We need to show that $\mathcal{M}^b$ satisfies conditions 1-9 of $\mathsf{RL_{OM}}$-models. The conditions 1-5 and 9 can be proved in a standard way, as usual in relational dual tableaux (see [14]). Now we prove that $\mathcal{M}^b$ satisfies the conditions 6-8.

For 6, note that by the completion condition Cpl(Irref<), $x < x \in b$ for every $x \in U^b$. Thus $(x, x) \notin m^b(<)$ for every $x \in U^b$, and hence $m^b(<)$ satisfies the condition (Irref). Assume $(x, y) \in m^b(<)$ and $(y, z) \in m^b(<)$, that is $x < y \notin b$ and $y < z \notin b$. Suppose $(x, z) \notin m^b(<)$. Then $x < z \in b$ and by the completion condition Cpl(Tran<), $x < y \in b$ or $y < z \in b$, a contradiction. Therefore $m^b(<)$ satisfies the condition (Tran). Moreover, for all $x, y \in U^b$, $x < y \notin b$ or $y < x \notin b$ or $x1'y \notin b$, since otherwise b would be closed. Thus, $(x, y) \in m^b(<)$ or $(y, x) \in m^b(<)$ or $(x, y) \in m^b(1')$, therefore $m^b(<)$ satisfies the condition (Lin).

For 7, assume $(x, y) \in m^b(\Psi_i)$, that is $x\Psi_i y \notin b$. By Cpl(C_i1), $x{-}\Psi_i y \in b$. Then, by Cpl(C_i3), we have $x{-}1'c_i \in b$. Thus, $x1'c_i \notin b$, since otherwise b would be closed. Therefore, $(x, m^b(c_i)) \in m^b(1')$. Reciprocally, assume $(x, m^b(c_i)) \in m^b(1')$, then $x1'c_i \notin b$. Suppose $(x, y) \notin m^b(\Psi_i)$, that is $x\Psi_i y \in b$. Then, by Cpl(C_i2), $x1'c_i \in b$, a contradiction.

For 8, conditions (i)-(iv) can be proved similarly by using, respectively, completion conditions Cpl(D1)-Cpl(D3). By way of example, we prove (ii) and (iv). Note that since $\{c_i \,\mathsf{d}\, c_{i+1}\}$ is an $\mathsf{RL_{OM}}$-axiomatic set for every $i \in \{1, \ldots, 4\}$, $c_i \mathsf{d} c_{i+1} \notin b$. Thus, $(c_i, c_{i+1}) \in m^b(\mathsf{d})$, for every $i \in \{1, \ldots, 4\}$. Hence, the condition (ii) is satisfied. Now assume that $(z, x) \in m^b(\mathsf{d})$, $(w, y) \in m^b(\mathsf{d})$, and $(z, w) \in m^b(<)$, which means that $z \,\mathsf{d}\, x \notin b$, $w \,\mathsf{d}\, y \notin b$, and $z < w \notin b$. Suppose, $(x, y) \notin m^b(<)$, that is $x < y \in b$. Then, by Cpl(D3), either $z \,\mathsf{d}\, x \in b$ or $w \,\mathsf{d}\, y \in b$ or $z < w \in b$, a contradiction. ☐

Let v^b be a valuation in $\mathcal{M}^b$ defined as $v^b(x) = x$, for every $x \in \mathbb{OS}$. Now we prove the following:

Proposition 8 (Satisfaction in Branch Model Property). *Let b be an open branch of an R proof tree. Then for every $\mathsf{RL_{OM}}$-formula φ the following holds:*

$$(*)\ \text{If } \mathcal{M}^b, v^b \models \varphi,\ \text{then } \varphi \notin b.$$

Proof. The proof is by induction on the complexity of relational terms. For atomic relational terms R, note that $(*)$ follows directly from the definition of $\mathcal{M}$. By way of example, we show that $(*)$ holds for terms of the form $R;S$, where R, S are relational terms. Assume $\mathcal{M}^b, v^b \models x(R;S)y$, that is there exists $z \in U^b$ such that $(x, z) \in m^b(R)$ and $(z, y) \in m^b(S)$. By the induction hypothesis we obtain $xRz \notin b$ and $zSy \notin b$. Suppose $x(R;S)y \in b$. By the completion condition Cpl(;), for every $z \in U^b$, either $xRz \in b$ or $zSy \in b$, a contradiction. ☐

Given a branch structure $\mathcal{M}^b = (U^b, m^b)$, we define the *quotient model* $\mathcal{M}^b_q = (U^b_q, m^b_q)$ as follows:

> $U^b_q = \{\|x\| : x \in U^b\}$, where $\|x\|$ is an equivalence class of $m^b(1')$ generated by x;
> $m^b_q(c_i) = \|c_i\|$, for every $i \in \{1, \ldots, 5\}$;
> $m^b_q(P) = \{(\|x\|, \|y\|)) \in U^b_q \times U^b_q : (x, y) \in m^b(R)\}$, for every $P \in \mathbb{RA}$;
> m^b_q extends to all the compound relational terms as in $\mathsf{RL}_{\mathsf{OM}}$-models.

Remark 4. Since $m^b(1')$ is an equivalence relation satisfying the extensionality property, the definition of $m^b_q(P)$ is correct, that is, the following condition is satisfied: If $(x, y) \in m^b(R)$ and $(x, z), (y, t) \in m^b(1'),$ then $(z, t) \in m^b(P)$.

It is easy to see that $m^b_q(1')$ is the identity on U^b_q. Therefore we obtain the following:

Proposition 9. *Let b be an open branch of an $\mathsf{RL}_{\mathsf{OM}}$-proof tree. The quotient model $\mathcal{M}^b_q = (U^b_q, m^b_q)$ is a standard $\mathsf{RL}_{\mathsf{OM}}$-model.*

Let v^b_q be a valuation in $\mathcal{M}^b_q$ such that $v^b_q(x) = \|x\|$, for every $x \in \mathbb{OS}$. Then the following can be easily proved:

Proposition 10. *Let b be an open branch of an $\mathsf{RL}_{\mathsf{OM}}$-proof tree. Then for every $\mathsf{RL}_{\mathsf{OM}}$-formula φ the following holds: $\mathcal{M}^b, v^b \models \varphi$ if and only if $\mathcal{M}^b_q, v^b_q \models \varphi$.*

The above propositions enable us to prove the completeness of a relational dual tableau for $\mathsf{RL}_{\mathsf{OM}}$.

Theorem 3 (Completeness of $\mathsf{RL}_{\mathsf{OM}}$). *Let φ be an $\mathsf{RL}_{\mathsf{OM}}$-formula. If φ is $\mathsf{RL}^*_{\mathsf{OM}}$-valid, then it is $\mathsf{RL}_{\mathsf{OM}}$-provable.*

Proof. Assume φ is $\mathsf{RL}^*_{\mathsf{OM}}$-valid. Suppose φ is not $\mathsf{RL}_{\mathsf{OM}}$-provable, that is there is no closed $\mathsf{RL}_{\mathsf{OM}}$-proof tree for φ. Let b be an open branch of a complete $\mathsf{RL}_{\mathsf{OM}}$-proof tree for φ. Since $\varphi \in b$, by Proposition 8, the branch model $\mathcal{M}^b$ does not satisfy φ. By Proposition 10, φ is not satisfied in the quotient model $\mathcal{M}^b_q$ by v^b_q. Since $\mathcal{M}^b_q$ is a standard $\mathsf{RL}_{\mathsf{OM}}$-model (by Proposition 9), φ is not $\mathsf{RL}^*_{\mathsf{OM}}$-valid, a contradiction. □

By Theorems 2, 3, and Corollary 1 we obtain the following main theorem:

Theorem 4 (Soundness and Completeness of $\mathsf{RL}_{\mathsf{OM}}$). *Let φ be an $\mathsf{RL}_{\mathsf{OM}}$-formula. Then the following conditions are equivalent:*

1. *φ is $\mathsf{RL}_{\mathsf{OM}}$-valid;*
2. *φ is $\mathsf{RL}^*_{\mathsf{OM}}$-valid;*
3. *φ is $\mathsf{RL}_{\mathsf{OM}}$-provable.*

Finally, by the above and Theorem 1 we get:

Theorem 5 (Soundness and Completeness of OM).
Let φ be an OM-formula. Then for all object variables x and y the following conditions are equivalent:

1. *φ is OM-valid;*
2. *$x\tau(\varphi)y$ is $\mathsf{RL}_{\mathsf{OM}}$-provable.*

Example
Consider OM-formula $\varphi = \Diamond_{\overrightarrow{\mathsf{d}}} p \to \Box_{\overrightarrow{\mathsf{d}}} p$ (axiom **d2**). The translation of φ to the relational term is $\tau(\varphi) = -(\mathsf{d}; (P; 1)) \cup -(\mathsf{d}; -(P; 1))$, where $\tau'(p) = P$, for some relational variable P. Figure 3 presents a closed $\mathsf{RL}_{\mathsf{OM}}$-proof tree for $x\tau(\varphi)y$ which, by Theorem 5, proves OM-validity of φ. In each node of the tree, we underline the formula to which a rule has been applied, and we indicate only those formulas that are essential for the construction of the tree.

5 Entailment, model checking and verification of satisfaction

The relational logic can be used to verify the entailment in the logic OM. The method is based on the following fact. Let $Q_1, \ldots, Q_n, Q$ be binary relations on a set U and let $1 = U \times U$. It is known that $Q_1 = 1, \ldots, Q_n = 1$ imply $Q = 1$ iff $(1; -(Q_1 \cap \ldots \cap Q_n); 1) \cup Q = 1$. It follows that for every $\mathsf{RL}_{\mathsf{OM}}$-model $\mathcal{M}$, $\mathcal{M} \models xQ_1y, \ldots, \mathcal{M} \models xQ_ny$ imply $\mathcal{M} \models xQy$ iff $\mathcal{M} \models x(1; -(Q_1 \cap \ldots \cap Q_n); 1) \cup Q)y$ which means that entailment in $\mathsf{RL}_{\mathsf{OM}}$ can be expressed in its language. This method can be used for verification of entailment in OM-logic. Namely, OM-formulas $\varphi_1, \ldots, \varphi_n$ imply an OM-formula φ iff $x(1; -(\tau(\varphi_1) \cap \ldots \cap \tau(\varphi_n)); 1) \cup \tau(\varphi))y$ is $\mathsf{RL}_{\mathsf{OM}}$-valid, for all object variables x and y. This, in turn, is equivalent to $\mathsf{RL}_{\mathsf{OM}}$-provability of the formula $x(1; -(\tau(\varphi_1) \cap \ldots \cap \tau(\varphi_n)); 1) \cup \tau(\varphi))y$.

The relational logic can also be used for model checking in finite OM-models. Let $\mathcal{M} = (U, m)$ be a fixed OM-model with a finite universe U and let φ be an OM-formula. It is easy to prove that there exists an $\mathsf{RL}_{\mathsf{OM}}$-model $\mathcal{N} = (U, n)$ with a finite universe of the same cardinality as U such that for all object variables x and y, the problem $\mathcal{M} \models \varphi$ is equivalent to the problem $\mathcal{N} \models x\tau(\varphi)y$. For the latter, we consider an instance $\mathsf{RL}_{\mathcal{N},\psi}$ of the logic $\mathsf{RL}_{\mathsf{OM}}$, where ψ denotes $x\tau(\varphi)y$. Its language provides a code of model $\mathcal{N}$ and formula ψ, and in its models the syntactic elements of ψ are interpreted as in the model $\mathcal{N}$. This coding leads to a relational logic which has precisely one model. Therefore its proof system enables us to verify the truth of ψ in model $\mathcal{N}$.

The vocabulary of the logic $\mathsf{RL}_{\mathcal{N},\psi}$ consists of the following pairwise disjoint sets: a countable infinite set of object variables; a finite set $\{c_{\mathtt{a}} : \mathtt{a} \in U\} \cup \mathbb{C}$ of object constants such that constants $c_{\mathtt{a}}$ uniquely name elements of model $\mathcal{M}$ in such a way that if $\mathtt{a} \neq \mathtt{b}$, then $c_{\mathtt{a}} \neq c_{\mathtt{b}}$; a set $\{Q : Q$ is an atomic subterm of

$$\underline{x-(\mathsf{d};(P;1)) \cup -(\mathsf{d};-(P;1))y}$$

$$\Big\downarrow \ (\cup)$$

$$\underline{x-(\mathsf{d};(P;1))y}, \underline{x-(\mathsf{d};-(P;1))y}$$

$$\Big\downarrow \ (-;)\times 2 \text{ and } (-),\ w_1,\ w_2 \text{ new variables}$$

$$x-\mathsf{d}w_1,\ \underline{w_1-(P;1)y},\ x-\mathsf{d}\,w_2,\ w_2(P;1)y$$

$$\Big\downarrow \ (-;),\ w_3 \text{ new variable}$$

$$x-\mathsf{d}w_1,\ w_1-Pw_3,\ w_3-1y,\ x-\mathsf{d}\,w_2,\ \underline{w_2(P;1)y}$$

$$(;) \text{ with } w_3$$

$$w_3 1y, \ldots \quad \text{closed} \qquad\qquad x-\mathsf{d}\,w_1, w_1-Pw_3, x-\mathsf{d}\,w_2, \underline{w_2Pw_3}, \ldots$$

$$(1'2) \text{ with } w_1$$

$$w_1Pw_3,\ w_1-Pw_3, \ldots \quad \text{closed} \qquad\qquad x-\mathsf{d}\,w_1, x-\mathsf{d}\,w_2, \underline{w_2 1' w_1}, \ldots$$

$$(\text{D2}) \text{ with } x$$

$$x-\mathsf{d}\,w_2, x\,\mathsf{d}\,w_2, \ldots \quad \text{closed} \qquad\qquad x-\mathsf{d}\,w_1, x\,\mathsf{d}\,w_1, \ldots \quad \text{closed}$$

Fig. 3. An $\mathsf{RL}_{\mathsf{OM}}$-proof of axiom **d2**

$\tau(\varphi)\} \cup \{1, 1'\}$ of relational constants; and the set $\{-, \cup, \cap, ;, ^{-1}\}$ of relational operations. An $\mathsf{RL}_{\mathcal{N},\psi}$-model is a pair $\mathcal{N}' = (U', n')$, where

$U' = U$;
$n'(c_{\mathtt{a}}) = \mathtt{a}$, for any object constant $c_{\mathtt{a}}$;
$n'(c) = n(c)$, for every $c \in \mathbb{C}$;
$n'(Q) = n(Q)$, for any atomic subterm Q of $\tau(\varphi)$;
$n'(1)$, $n'(1')$ are defined as in $\mathsf{RL}_{\mathsf{OM}}$-models;
n' extends to all the compound terms as in $\mathsf{RL}_{\mathsf{OM}}$-models.

A valuation in $\mathcal{N}'$ is a function v assigning object symbols to elements of U' such that $v(c) = n'(c)$, for every object constant c. Observe that any valuation v in model $\mathcal{N}'$ restricted to object variables and object constants from $\mathbb{C}$ is a valuation in model $\mathcal{N}$. Moreover, the above definition implies that for every atomic subterm Q of $\tau(\varphi)$ and for all object variables x and y, $\mathcal{N}, v \models xQy$ iff $\mathcal{N}', v \models xQy$. Therefore, it is easy to prove that $n'(\tau(\varphi)) = n(\tau(\varphi))$. Since the universe and the interpretation of all the syntactic elements of the $\mathsf{RL}_{\mathcal{N},\psi}$-language are fixed, such an $\mathsf{RL}_{\mathcal{N},\psi}$-model $\mathcal{N}'$ is unique. Therefore, $\mathsf{RL}_{\mathcal{N},\psi}$-validity is equivalent to the truth in a single $\mathsf{RL}_{\mathcal{N},\psi}$-model $\mathcal{N}'$, that is the following holds:

Proposition 11. *Let $\mathcal{N}$ and ψ be as above. Then for every $\mathsf{RL}_{\mathcal{N},\psi}$-formula ϑ, the following statements are equivalent:*

1. *$\mathcal{N} \models \vartheta$;*
2. *ϑ is $\mathsf{RL}_{\mathcal{N},\psi}$-valid.*

In particular, for $\vartheta := \psi$ this theorem states that validity of ψ in $\mathsf{RL}_{\mathcal{N},\psi}$-logic is equivalent to the truth of ψ in model $\mathcal{N}$. Consequently, it is equivalent to the truth of φ in $\mathcal{M}$.

The relational dual tableau for $\mathsf{RL}_{\mathcal{N},\psi}$ consists of the rules and axiomatic sets of $\mathsf{RL}_{\mathsf{OM}}$-system adapted to the language of $\mathsf{RL}_{\mathcal{N},\psi}$, and. moreover, the specific rules and axiomatic sets presented in Figures 4 and 5, respectively. The rule $(-Q\mathtt{ab})$ expresses a definition of a relation $-Q$, for any atomic subterm Q of $\tau(\varphi)$. The rule $(1')$ expresses that the object variables represent elements of the universe of model $\mathcal{M}$. The rule $(\mathtt{a} \neq \mathtt{b})$ says that different elements of model $\mathcal{M}$ are represented by different object constants. Axiomatic sets provide a code of atomic subterms of $\tau(\varphi)$.

Theorem 6 (Soundness and Completeness of $\mathsf{RL}_{\mathcal{N},\psi}$). *Let $\mathcal{N}$ and ψ be as above. Then for every $\mathsf{RL}_{\mathcal{N},\psi}$-formula ϑ the following conditions are equivalent:*

1. *ϑ is $\mathsf{RL}_{\mathcal{N},\psi}$-provable;*
2. *ϑ is $\mathsf{RL}_{\mathcal{N},\psi}$-valid.*

The proof of the above theorem can be found in [14]. Due to Theorem 6 and Proposition 11 we obtain the following:

For any atomic subterm Q of $\tau(\varphi)$ and for any object symbols x, y:

$(-Q\mathtt{ab})$ $$\frac{x-Qy}{x1'c_{\mathtt{a}}, x-Qy \mid y1'c_{\mathtt{b}}, x-Qy}$$

for any $\mathtt{a}, \mathtt{b} \in U$ such that $(\mathtt{a}, \mathtt{b}) \notin n(Q)$

$(1')$ $$\frac{}{x-1'c_1 \mid \ldots \mid x-1'c_{\mathtt{n}}}$$

where $\{c_1, \ldots, c_{\mathtt{n}}\}, \mathtt{n} \geq 1$, is the set of all new object constants

$(\mathtt{a} \neq \mathtt{b})$ $$\frac{}{c_{\mathtt{a}}1'c_{\mathtt{b}}}$$

for any $\mathtt{a}, \mathtt{b} \in U$ such that $\mathtt{a} \neq \mathtt{b}$

Fig. 4. Specific rules of $\mathsf{RL}_{\mathcal{N},\psi}$-dual tableau

Theorem 7 (Model Checking). *Let $\mathcal{M}$, φ, ψ, and $\mathcal{N}$ be as above. Then the following statements are equivalent:*

1. $\mathcal{M} \models \varphi$;
2. ψ *is* $\mathsf{RL}_{\mathcal{N},\psi}$*-provable.*

Similarly, we can use the relational system for the verification of satisfaction in a fixed finite OM-model. Let $\mathcal{M} = (U, m)$ be a finite OM-model, let $\mathtt{a}$ be an element of U and let φ be an OM-formula. It can be proved that there exist an $\mathsf{RL}_{\mathsf{OM}}$-model $\mathcal{N} = (U, n)$ with the same universe as in $\mathcal{M}$ and a valuation $v_{\mathtt{a}}$ in $\mathcal{N}$ such that $v_{\mathtt{a}}(x) = \mathtt{a}$, and $\mathcal{M}, \mathtt{a} \models \varphi$ iff $\mathcal{N}, v_{\mathtt{a}} \models x\tau(\varphi)y$. A proof of this fact can be found in [14]. Then for all $\mathtt{b} \in U$, the problem '$\mathcal{M}, \mathtt{a} \models \varphi$?' is equivalent to the problem '$(\mathtt{a}, \mathtt{b}) \in n(\tau(\varphi))$?'. In order to obtain the relational formalization of the latter we use the logic $\mathsf{RL}_{\mathcal{N},\psi}$ defined above, where ψ denotes $x\tau(\varphi)y$. The following is proved in [14]:

Proposition 12. *Let $\mathcal{M}$, φ, and $\mathtt{a}$ be as above. Then for every $b \in U$, the following statements are equivalent:*

1. $\mathcal{M}, \mathtt{a} \models \varphi$;
2. $(\mathtt{a}, \mathtt{b}) \in n(\tau(\varphi))$;
3. $c_{\mathtt{a}}\tau(\varphi)c_{\mathtt{b}}$ *is* $\mathsf{RL}_{\mathcal{N},\psi}$*-valid.*

By the above proposition and Theorem 6 we obtain the following:

Theorem 8 (Satisfaction). *Let $\mathcal{M}$, φ, and $\mathtt{a}$ be as above. Then for every $b \in U$, the following statements are equivalent:*

1. $\mathcal{M}, \mathtt{a} \models \varphi$;
2. $c_{\mathtt{a}}\tau(\varphi)c_{\mathtt{b}}$ *is* $\mathsf{RL}_{\mathcal{N},\psi}$*-provable.*

For every atomic subterm Q of $\tau(\varphi)$ and for any $\mathtt{a}, \mathtt{b} \in U$:

$\{c_{\mathtt{a}} Q c_{\mathtt{b}}\}$, if $(\mathtt{a}, \mathtt{b}) \in n(Q)$;

$\{c_{\mathtt{a}} {-Q} c_{\mathtt{b}}\}$, if $(\mathtt{a}, \mathtt{b}) \notin n(Q)$.

Fig. 5. $\mathsf{RL}_{\mathcal{N},\psi}$-axiomatic sets

6 Conclusions and future work

In this paper, we have introduced a relational proof system in the style of dual tableaux for the relational logic associated with the multimodal propositional logic for order of magnitude qualitative reasoning OM. We have proved its soundness and completeness. Moreover, we have shown how this proof system can be used for verification of validity, verification of entailment, model checking, and verification of satisfaction in finite models.

The goal for the future is to study the decidability of the logic OM. In the case of positive answer it is natural to look for a decision procedure for the logic OM based on relational dual tableau.

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References

1. Bennett, B., *Modal logics for qualitative spatial reasoning.*, Bull. of the IGPL 3, 1–22, 1995.
2. Bennett, B., Cohn, A.G., Wolter, F. and Zakharyaschev, M., *Multi-Dimensional Modal Logic as a Framework for Spatio-Temporal Reasoning.*, Applied Intelligence 17(3), 239–251, 2002.
3. Burrieza, A., Mora, A., Ojeda-Aciego, M., and Orłowska, E., *Implementing a relational system for order-of-magnitude reasoning.* International Journal of Computer Mathematics. To appear, 2009.
4. Burrieza, A., Muñoz-Velasco, E., and Ojeda-Aciego, M., *A Logic for Order of Magnitude Reasoning with Negligibility, Non-closeness and Distance*, Lecture Notes in Artifical Intelligence 4788: 210-219, 2007.

5. Burrieza, A., Muñoz-Velasco, E., and Ojeda-Aciego, M., *Order of magnitude reasoning with bidirectional negligibility.* Lecture Notes in Artifical Intelligence 4177: 370-378, 2006.
6. Burrieza, A., Ojeda-Aciego, M., and Orłowska, E., *Relational approach to order-of-magnitude reasoning.* Lecture Notes in Computer Science 4342: 105-124, 2006.
7. Burrieza, A., and Ojeda-Aciego, M., *A multimodal logic approach to order of magnitude qualitative reasoning with comparability and negligibility relations.*, Fundamenta Informaticae 68: 21–46, 2005.
8. Dague, P., *Symbolic reasoning with relative orders of magnitude.*, in: Proc. 13th Intl. Joint Conference on Artificial Intelligence, Morgan Kaufmann, 1509–1515, 1993.
9. Dague, P., *Numeric reasoning with relative orders of magnitude.*, in: Proc. 11th National Conference on Artificial Intelligence,The AAAI Press/The MIT Press, 541–547, 1993.
10. Dallien, J. and MacCaull, W., *RelDT: A relational dual tableaux automated theorem prover.* `http://www.logic.stfx.ca/reldt/`.
11. K.D. Forbus. *Qualitative Reasoning.* In A.B. Tucker, editor, The Computer Science and Engineering Handbook, pages 715–733. CRC Press, 1996.
12. Formisano, A. and Orłowska, E. and Omodeo, E., *A PROLOG tool for relational translation of modal logics: A front-end for relational proof systems.*, TABLEAUX 2005 Position Papers and Tutorial Descriptions, B. Beckert (ed.), Fachberichte Informatik No 12, Universitaet Koblenz-Landau, 1–10, 2005. `http://www.di.univaq.it/TARSKI/transIt/`.
13. Golińska-Pilarek, J. and Muñoz-Velasco, E., *Dual tableau for a multimodal logic for order of magnitude qualitative reasoning with bidirectional negligibility.* International Journal of Computer Mathematics. To appear 2009.
14. Golińska-Pilarek, J., and Orłowska, E., *Dual Tableaux: Foundations, Methodology, Case Studies.* A draft of the book 2009.
15. Golińska-Pilarek, J., Mora, A. and Muñoz-Velasco, E., *An ATP of a Relational Proof System for Order of Magnitude Reasoning with Negligibility, Non-Closeness and Distance.* Lecture Notes in Artificial Intelligence Vol. 5351, pp. 128-139. Springer, 2008.
16. Golińska-Pilarek, J., and Orłowska, E., *Tableaux and Dual Tableaux: Transformation of proofs.*, *Studia Logica* 85, 291–310, 2007.
17. Golińska-Pilarek, J., and Orłowska, E., *Relational logics and their applications.*, in: H. de Swart, E. Orłowska, M. Roubens, and G. Schmidt, *Theory and Applications of Relational Structures as Knowledge Instruments II*, Lecture Notes in Artificial Intelligence 4342: 125–161, 2006.
18. Konikowska, B., *Rasiowa-Sikorski deduction systems in computer science applications.*, Theoretical Computer Science 286, 323–366, 2002.
19. Mavrovouniotis, M.L., and Stephanopoulos, G., *Reasoning with orders of magnitude and approximate relations.*, Proc. 6th National Conference on Artificial Intelligence, The AAAI Press/The MIT Press, 1987.
20. Missier, A., Piera, N. and Travé, L. *Order of Magnitude Algebras: a Survey.* Revue d'Intelligence Artificielle 3(4):95–109, 1989.
21. Nayak, P. *Causal Approximations.* Artificial Intelligence 70. 277-334, 1994.
22. Orłowska, E., *Relational proof systems for modal logics.*, in: Wansing, H. (ed.), Proof Theory of Modal Logics, Kluwer, 55-77, 1996.

23. Orłowska, E., *Relational interpretation of modal logics.*, in: H. Andréka, D. Monk and I. Nemeti (eds.), Algebraic Logic, Col. Math. Soc. J. Bolyai 54, North Holland, Amsterdam, 443–471, 1988.
24. Raiman, O., *Order of magnitude reasoning.*, Artificial Intelligence 51 , 11–38, 1991.
25. Randell, D., Cui, Z. and Cohn, A., *A spatial logic based on regions and connections.*, Proc. of the 3rd International Conference on Principles of Knowledge Representation and Reasoning (KR'92), 165–176, 1992.
26. Rasiowa, H., and Sikorski, R., *The Mathematics of Metamathematics.*, Polish Scientific Publishers, Warsaw 1963.
27. Sánchez, M., Prats, F., and Piera, N., *Una formalizacin de relaciones de comparabilidad en modelos cualitativos.*, Boletín de la AEPIA (Bulletin of the Spanish Association for AI) 6, 15–22, 1996.
28. Travé-Massuyès, L., Ironi, L. and Dague P., *Mathematical Foundations of Qualitative Reasoning.*, AI Magazine, American Asociation for Artificial Intelligence, 91–106, 2003.
29. Travé-Massuyès, L., Prats, F., Sánchez, M., and Agell, M., *Consistent relative and absolute order-of-magnitude models.*, in: Proc. Qualitative Reasoning Conference, 2002.
30. Wolter, F. and Zakharyaschev, M., *Qualitative spatio-temporal representation and reasoning: a computational perspective.*, in G. Lakemeyer and B. Nebel (eds.), Exploring Artificial Intelligence in the New Millenium, Morgan Kaufmann, 2002.
31. Iwasaki, Y. *Qualitative reasoning.* `http://www.aaai.org/AITopics/html/qual.html`.