

RESEARCH ARTICLE

Dual tableau for a multimodal logic for order of magnitude qualitative reasoning with bidirectional negligibility

J. Golińska-Pilarek[†] and Muñoz-Velasco, E.[‡]

[†]Institute of Philosophy, Warsaw University.

National Institute of Telecommunications. Poland.

E-mail: j.golinska-pilarek@itl.waw.pl

[‡] Dept. Matemática Aplicada. Universidad de Málaga. Spain.

E-mail: emilio@ctima.uma.es

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We present a relational proof system in the style of dual tableaux for the relational logic associated with a multimodal propositional logic for order of magnitude qualitative reasoning with a bidirectional relation of negligibility. We study soundness and completeness of the proof system and we show how it can be used for verification of validity of formulas of the logic.

Keywords: relational logics, dual tableau systems, multimodal propositional logic, order-of-magnitude qualitative reasoning

1. Introduction

The use of models to represent different scientific and engineering situations leads to qualitative reasoning as a good possibility when the traditional numerical methods are limited. Qualitative Reasoning (QR) provides an intermediate level between discrete and continuous models. A form of QR is to manage numerical data in terms of orders of magnitude (see, for example, [12, 14]). Two approaches to order of magnitude reasoning have been identified in [15]: Absolute Order of Magnitude, which is represented by a partition of the real line \mathbb{R} , where each element of \mathbb{R} belongs to a qualitative class and Relative Order of Magnitude, introducing a family of binary order of magnitude relations which establish different comparison relations in \mathbb{R} (e.g., *comparability*, *negligibility* and *closeness*). In general, both models need to be combined to capture all the relevant information.

Several logics have been defined to use QR in different contexts, e.g. spatial and temporal reasoning [1, 16]. In particular, logics dealing with order of magnitude reasoning have been developed in [3, 4] by combining the absolute and relative approaches, that is, defining different qualitative relations by using the intervals provided by a specific absolute order of magnitude model.

In this paper, we focus our attention on the multimodal propositional logic $\mathcal{L}(MQ)^N$ presented in [3], which uses the absolute order of magnitude model with the real line divided into seven intervals to define a binary relation of negligibility.

This negligibility relation is *bidirectional*, that is it allows us to compare positive and negative numbers. Moreover, this relation has good properties with respect to the sum and product of real numbers, which is very useful in the applications (see, for example [12]).

The main difference between our logic approach for order of magnitude reasoning and the given one in [4] is the definition of negligibility and the division of the real line into seven classes. These facts allows us to have more possible comparisons by introducing the classes of *medium* numbers (for more details, see section 4 in [3]).

It is well known that one of the main advantages in the use of the logic formalism is the possibility of having automated deduction systems. For this reason, we present a deduction system in the style of relational dual tableaux for the multimodal logic considered in the paper, for which no other systems are known in the literature. Relational dual tableaux are powerful tools for performing the four major reasoning tasks: verification of validity, verification of entailment, model checking, and verification of satisfaction. We prove that the system presented in the paper enables us to verify validity of formulas of the logic in question.

In the construction of the system, we apply the method known for various non-classical logics, (see e.g., [11]). Firstly, we construct a relational logic appropriate for the multimodal logic $\mathcal{L}(MQ)^N$. Then, we define a validity preserving translation from the language of $\mathcal{L}(MQ)^N$ to the language of the relational logic. Finally, we construct a complete and sound relational proof system for the relational logic appropriate for $\mathcal{L}(MQ)^N$. The relational logic considered in this paper is based on the classical relational logic of binary relations with relational constants 1 and $1'$, which provides a means for proving the identities valid in the class of full relation algebras (see e.g., [8, 11]). The proof system developed in the paper is an extension of the proof system for the classical relational logic. In constructing deduction rules of the system, we follow the general principles of defining relational deduction rules.

The election of this method has many advantages (see [10]): a clear-cut method of generating rules of the system from semantics, the resulting deduction system well suited for automated deduction purposes, a standard and intuitively simple way of proving completeness by constructing a counter-model for a non-provable formula out of its *non closed* decomposition tree in a relational proof system and an almost automatic way of transforming a complete relational proof system into a complete Gentzen calculus system.

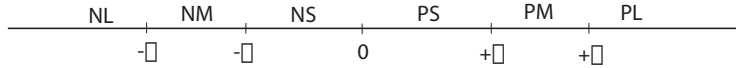
Another approach to relational logics for order of magnitude reasoning has been presented in [5].

The existence of automated deduction systems gives the possibility of implementation. An implementation of a proof system for the classical relational logic can be found at [6]. An implementation of translation procedures from non-classical logics to relational logic is presented in [7]. Focusing our attention on logics for order of magnitude reasoning, a theorem prover for the system introduced in [5] has been given in [2].

The paper is organized as follows: In Section 2, we define the syntax, semantics and the axiomatization of the logic $\mathcal{L}(MQ)^N$. In Section 3, we develop the relational logic appropriate for $\mathcal{L}(MQ)^N$ and a validity preserving translation for it. In section 4, a sound and complete relational proof system is given. Finally, in Section 5, some conclusions and future work are commented.

2. The multimodal logic $\mathcal{L}(MQ)^N$

As we have said in the introduction, we are going to work with the logic $\mathcal{L}(MQ)^N$ presented in [3]. This logic uses an absolute order of magnitude model which considers the real line \mathbb{R} divided into seven equivalence classes using five landmarks, where α, β are two positive real numbers (chosen depending on the context under consideration) such that $\alpha <_{\mathbb{R}} \beta$, being $\leq_{\mathbb{R}}$ the usual order in \mathbb{R} .



The seven intervals are defined by $NL = (-\infty, -\beta)$, $NM = [-\beta, -\alpha)$, $NS = [-\alpha, 0)$, $[0] = \{0\}$, $PS = (0, \alpha]$, $PM = (\alpha, \beta]$ and $PL = (\beta, +\infty)$.

The labels correspond to “negative large”, “negative medium”, “negative small”, “zero”, “positive small”, “positive medium”, and “positive large”, respectively. From this partition of the real line, we define the following negligibility relation. Given $\alpha, \beta \in \mathbb{R}$, such that $0 <_{\mathbb{R}} \alpha <_{\mathbb{R}} \beta$, we say that x is *negligible* with respect to y , in symbols $x N_{\mathbb{R}} y$, iff, we have one of the following possibilities:

$$(i) \quad x = 0 \quad (ii) \quad x \in NS \cup PS \text{ and } y \in NL \cup PL$$

Note that item (i) above corresponds to the intuitive idea that 0 is negligible with respect to any real number and item (ii) corresponds to the intuitive idea that a number *sufficiently small* is negligible with respect to any number *sufficiently large*, independently of the sign of these numbers. For this reason, we say that our negligibility relation is bidirectional.

Using the idea of the previous definition, we construct a logic where the five landmarks $-\beta, -\alpha, 0, \alpha$, and β are replaced, respectively, by the following elements of its language: c_1, c_2, c_3, c_4 , and c_5 , while the negligibility relation N is defined as an accessibility relation obtained from these landmarks.

2.1. Syntax of $\mathcal{L}(MQ)^N$

We consider the language of $\mathcal{L}(MQ)^N$ as a multimodal propositional language with a family of modal operators determined by accessibility relations. Expressions of the language are constructed with symbols from the following pairwise disjoint sets:

- \mathcal{V} a set of propositional variables,
- $\mathcal{C} = \{c_i \mid i \in \{1, \dots, 5\}\}$ a set of specific constants,
- $\{\neg, \wedge, \vee, \rightarrow, \vec{\square}, \overleftarrow{\square}, \square_N, \overline{\square}_N\}$ the set of propositional operations and the specific modal connectives,
 where $\vec{\square}$ and \square_N represent, respectively, the modal connectives for accessibility relations $<$ and N and $\overleftarrow{\square}, \overline{\square}_N$ are their inverses.¹

The set For of $\mathcal{L}(MQ)^N$ -formulas is the smallest set satisfying the following conditions:

$$(1) \quad \mathcal{V} \cup \mathcal{C} \subseteq For,$$

¹As usual in modal logic, we use $\vec{\diamond}, \overleftarrow{\diamond}, \diamond_N, \overline{\diamond}_N$ as abbreviations of $\neg \vec{\square} \neg, \neg \overleftarrow{\square} \neg, \neg \square_N \neg$, and $\neg \overline{\square}_N \neg$, respectively.

(2) If $\varphi, \psi \in For$, then $\neg\varphi, \varphi \wedge \psi, \varphi \vee \psi, \varphi \rightarrow \psi, \vec{\square} \varphi, \overleftarrow{\square} \varphi, \square_N \varphi$ and $\overline{\square}_N \varphi \in For$.

2.2. Semantics of $\mathcal{L}(MQ)^N$

We define the basic concepts of the semantics of our logic.

An $\mathcal{L}(MQ)^N$ -model is a tuple $\mathcal{M} = (U, <, N, m)$, where U is a non-empty set and m is a meaning function satisfying the following conditions:

- (1) $m(p) \subseteq U$ for $p \in \mathcal{V}$,
- (2) $<$ is a strict linear ordering on U , that is, for all $s, s', s'' \in U$ the following conditions are satisfied:
 - (Irref) $s \not< s$,
 - (Trans) if $s < s'$ and $s' < s''$, then $s < s''$,
 - (Lin) $s < s'$ or $s' < s$ or $s = s'$,
- (3) $m(c_i) \in U$ for every $i \in \{1, \dots, 5\}$, and $m(c_i) < m(c_{i+1})$ for every $i \in \{1, \dots, 4\}$,
- (4) N is a relation on U , defined by $N = G_1 \cup G_2 \cup G_3 \subseteq U \times U$, where:
 - $G_1 = \{(s, s') : s = m(c_3)\}$,
 - $G_2 = \{(s, s') : (\lambda \text{ or } \mu) \text{ and } (\gamma \text{ or } \delta) \text{ and } \zeta\}$,
 - $G_3 = \{(s, s') : (\lambda \text{ or } \mu) \text{ and } (\gamma \text{ or } \delta) \text{ and } \eta\}$, being $\lambda := (m(c_2) < s)$, $\mu := (s = m(c_2))$, $\gamma := (s < m(c_4))$, $\delta := (s = m(c_4))$, $\zeta := (s' < m(c_1))$ and $\eta := (m(c_5) < s')$.

Note that item (4) reflects semantically our definition of negligibility.

Let φ be an $\mathcal{L}(MQ)^N$ -formula and let $\mathcal{M} = (U, <, N, m)$ be an $\mathcal{L}(MQ)^N$ -model. The *satisfaction of φ in \mathcal{M} by $s \in U$* , ($(\mathcal{M}, s) \models \varphi$ for short), is defined as usual for propositional connectives. The definition for the direct modal connectives is given as follows:

- $(\mathcal{M}, s) \models \vec{\square} \varphi$ iff for all $s' \in U$, $s < s'$ implies $(\mathcal{M}, s') \models \varphi$,
- $(\mathcal{M}, s) \models \overleftarrow{\square} \varphi$ iff for all $s' \in U$, $s' < s$ implies $(\mathcal{M}, s') \models \varphi$,
- $(\mathcal{M}, s) \models \square_N \varphi$ iff for all $s' \in U$, $(s, s') \in N$ implies $(\mathcal{M}, s') \models \varphi$,
- $(\mathcal{M}, s) \models \overline{\square}_N \varphi$ iff for all $s' \in U$, $(s', s) \in N$ implies $(\mathcal{M}, s') \models \varphi$.

We say that an $\mathcal{L}(MQ)^N$ -formula φ is *satisfiable* if, and only if, there exist an $\mathcal{L}(MQ)^N$ -model \mathcal{M} and $s \in U$ such that $(\mathcal{M}, s) \models \varphi$. An $\mathcal{L}(MQ)^N$ -formula φ is *true* in an $\mathcal{L}(MQ)^N$ -model $\mathcal{M} = (U, m)$ whenever $(\mathcal{M}, s) \models \varphi$ for all $s \in U$. An $\mathcal{L}(MQ)^N$ -formula φ is $\mathcal{L}(MQ)^N$ -*valid*, denoted by $\models \varphi$ whenever it is true in all $\mathcal{L}(MQ)^N$ -models.

2.3. Axiom system for $\mathcal{L}(MQ)^N$

The axiom system for $\mathcal{L}(MQ)^N$ consists of all the tautologies of classical propositional logic together with the following axiom schemata:

Axiom schemata for modal connectives:

$$\mathbf{K1} \vec{\square} (\varphi \rightarrow \psi) \rightarrow (\vec{\square} \varphi \rightarrow \vec{\square} \psi) \quad \mathbf{K2} \varphi \rightarrow \vec{\square} \overline{\square} \varphi \quad \mathbf{K3} \vec{\square} \varphi \rightarrow \vec{\square} \vec{\square} \varphi$$

$$\mathbf{K4} (\vec{\square} (\varphi \vee \psi) \wedge \vec{\square} (\vec{\square} \varphi \vee \vec{\square} \psi) \wedge \vec{\square} (\varphi \vee \vec{\square} \psi)) \rightarrow (\vec{\square} \varphi \vee \vec{\square} \psi)$$

Axiom schemata for constants: (being $i \in \{1, \dots, 5\}$ and $j \in \{1, \dots, 4\}$)

$$\mathbf{C1} \overline{\square} c_i \vee c_i \vee \overline{\square} c_i \quad \mathbf{C2} c_i \rightarrow (\overline{\square} \neg c_i \wedge \overline{\square} \neg c_i) \quad \mathbf{C3} c_j \rightarrow \overline{\square} c_{j+1}$$

Axiom schemata for negligibility connectives:

$$\begin{array}{ll}
\mathbf{N1} \quad \Box_N(\varphi \rightarrow \psi) \rightarrow (\Box_N\varphi \rightarrow \Box_N\psi) & \mathbf{N2} \quad \varphi \rightarrow \Box_N\overline{\Box}_N\varphi \\
\mathbf{N3} \quad (\overline{\Box} \varphi \wedge \varphi \wedge \overline{\Box} \varphi) \rightarrow \Box_N\varphi & \mathbf{N4} \quad (\overline{\Box} c_2 \vee \overline{\Box} c_4) \rightarrow \Box_N(\varphi \wedge \neg\varphi) \\
\mathbf{N5} \quad c_3 \rightarrow (\Box_N\varphi \rightarrow (\overline{\Box} \varphi \wedge \varphi \wedge \overline{\Box} \varphi)) & \mathbf{N6} \quad (\neg c_3 \wedge (c_2 \vee (\overline{\Box} c_2 \wedge \overline{\Box} c_4) \vee c_4)) \rightarrow \Box_N(\overline{\Box} c_1 \vee \overline{\Box} c_5) \\
\mathbf{N7} \quad (\neg c_3 \wedge (c_2 \vee (\overline{\Box} c_2 \wedge \overline{\Box} c_4) \vee c_4)) \rightarrow (\Box_N\varphi \rightarrow (\overline{\Box} (\overline{\Box} c_1 \rightarrow \varphi) \wedge \overline{\Box} (\overline{\Box} c_5 \rightarrow \varphi)))
\end{array}$$

We also consider as axioms the corresponding mirror images of axioms $\mathbf{K1-K4}$, and axioms $\mathbf{N1-N3}$. Moreover, we consider the following Rules of Inference:

(MP) Modus Ponens (R $\overline{\Box}$) If $\vdash \varphi$ then $\vdash \overline{\Box} \varphi$ (R $\overline{\Box}$) If $\vdash \varphi$ then $\vdash \overline{\Box} \varphi$

This system was proved to be complete in [3].

3. The relational logic $\mathcal{R}(MQ)^N$

In this section we present the relational logic $\mathcal{R}(MQ)^N$ appropriate for expressing formulas the multimodal logic $\mathcal{L}(MQ)^N$.

The classical relational logic of binary relations is a logical counterpart to the class RRA of (representable) relation algebras introduced by Tarski. The formulas of the relational language are intended to represent statements saying that two objects are related. The logic $\mathcal{R}(MQ)^N$ is an extension of the classical relational logic. Relational terms of $\mathcal{R}(MQ)^N$ are built from atomic terms with relational operations. The set $\mathbb{R}\mathbf{A}$ of atomic relational terms is the union of the non-empty set $\mathbb{R}\mathbf{V}$ of relational variables and the set $\mathbb{R}\mathbf{C} = \{1, 1', <, N\} \cup \{\Psi_1, \dots, \Psi_5\}$ of relational constants. The set of all relational terms is denoted by $\mathbb{R}\mathbf{T}$. The operations are the Boolean operations of union (\cup), intersection (\cap), and complement ($-$) and the specific relational operations of composition ($P; Q = \{(x, y) : \exists z(xPz \wedge zQy)\}$) and converse ($P^{-1} = \{(x, y) : yPx\}$).

The relational constants include 1 and $1'$, interpreted as the universal relation and an equivalence relation satisfying the extensionality property (i.e., $P; 1' = P = 1'; P$ for any relation P), respectively.

Notice that $1'$ is not necessarily the equality relation.

$\mathcal{R}(MQ)^N$ -formulas are of the form xPy , where P is a relational term and x, y are object symbols. The set $\mathbb{O}\mathbf{S}$ of object symbols is the union of the set $\mathbb{O}\mathbf{V}$ of object variables and the set $\mathbb{O}\mathbf{C} = \{c_1, \dots, c_5\}$ of object constants.

If $\varphi(x, y)$ is a formula xRy , then by $-\varphi(x, y)$ we denote the formula $x-Ry$.

With the $\mathcal{R}(MQ)^N$ -language a class of $\mathcal{R}(MQ)^N$ -models is associated. An $\mathcal{R}(MQ)^N$ -model is a structure $\mathcal{M} = (U, m)$, where U is a non-empty set and m is a meaning function such that $m(1) = U \times U$,

$m(1')$ is an equivalence relation on U such that every relation R on U satisfies $m(1'); R = R; m(1') = R$;

$m(Q) \subseteq U \times U$, for every atomic relational term Q , and the following conditions hold:

(1) $m(<)$ is an irreflexive and transitive relation on U such that for all $s, s', s'' \in U$:

$$(\text{Lin}) \quad (s, s') \in m(<) \text{ or } (s', s) \in m(<) \text{ or } (s, s') \in m(1'),$$

(2) $m(c_i) \in U$ and $(m(c_i), m(c_{i+1})) \in m(<)$, for $i \in \{1, \dots, 5\}$,

(3) $m(\Psi_i) = \{(s, s') \in U \times U : (s, m(c_i)) \in m(1')\}$,

(4) $m(N)$ is a relation on U , defined by $m(N) = G'_1 \cup G'_2 \cup G'_3 \subseteq U \times U$, where:

$$G'_1 = \{(s, s') : (s, m(c_3)) \in m(1')\},$$

$$G'_2 = \{(s, s') : (\lambda' \text{ or } \mu') \text{ and } (\gamma' \text{ or } \delta') \text{ and } \zeta'\},$$

$$G'_3 = \{(s, s') : (\lambda' \text{ or } \mu') \text{ and } (\gamma' \text{ or } \delta') \text{ and } \eta'\}, \text{ being } \lambda' := ((m(c_2), s) \in m(<)), \mu' := ((s, m'(c_2)) \in m(1')), \gamma' := ((s, m(c_4)) \in m(<)), \delta' := ((s, m'(c_4)) \in m(1')), \zeta' := ((s', m(c_1)) \in m(<)) \text{ and } \eta' := ((m(c_5), s') \in m(<)).$$

(5) m' extends to all the compound relational terms as usual, that is:

$$m'(-R) = m(1) \cap -m'(R) \quad m'(R^{-1}) = m'(R)^{-1} \quad m'(R; S) = m'(R); m'(S)$$

$$m'(R \cap S) = m'(R) \cap m'(S) \quad m'(R \cup S) = m'(R) \cup m'(S)$$

Notice that item (4) represents the definition of our negligibility relation.

An $\mathcal{R}(MQ)^N$ -model $\mathcal{M}' = (U', m')$ is said to be *standard* whenever $m'(1')$ is the identity on U' , that is $m'(1') = \{(x, x) : x \in U'\}$ ¹. The class of standard models is denoted by $\mathcal{R}^*(MQ)^N$ and we use in this paper the term *standard model* or $\mathcal{R}^*(MQ)^N$ -*model* indistinctly.

In fact, we prove that $\mathcal{R}(MQ)^N$ -models and standard $\mathcal{R}(MQ)^N$ -models are modally equivalent, i.e., the classes of their valid formulas coincide. A valuation in an $\mathcal{R}(MQ)^N$ -model $\mathcal{M}' = (U', m')$ is a function $v : \mathbb{O}\mathbb{S} \rightarrow U'$ such that $v(c_i) = m'(c_i)$, for every $i \in \{1, \dots, 5\}$. Let xRy be an $\mathcal{R}(MQ)^N$ -formula and let $\mathcal{M}' = (U', m')$ be an $\mathcal{R}(MQ)^N$ -model. A formula xRy is said to be *satisfied in \mathcal{M}' by v* ($\mathcal{M}', v \models xRy$ for short) whenever $(v(x), v(y)) \in m'(R)$. A formula xRy is *true in \mathcal{M}'* if it is satisfied in \mathcal{M}' by all valuations v . xRy is said to be $\mathcal{R}(MQ)^N$ -*valid*, if it is true in all $\mathcal{R}(MQ)^N$ -models. Moreover, a formula is said to be $\mathcal{R}^*(MQ)^N$ -*valid* whenever it is true in all standard models.

Now, we develop the validity preserving *translation function* $t : For \rightarrow \mathbb{RT}$ assigning relational terms to modal formulas. We start with an assignment t' of relational variables to all propositional variables, $t'(p) = R_p$ where $R_p \in \mathbb{RV}$. Then we define:

$$\begin{aligned} t(p) &= t'(p); 1, \text{ for every propositional variable } p \in \mathcal{V}, \\ t(c_i) &= \Psi_i; 1, \text{ for every } i \in \{1, \dots, 5\}. \end{aligned}$$

t extends to all compound $\mathcal{L}(MQ)^N$ -formulas as follows:

$$\begin{aligned} t(\neg\varphi) &= -t(\varphi) & t(\varphi \vee \psi) &= t(\varphi) \cup t(\psi) & t(\varphi \wedge \psi) &= t(\varphi) \cap t(\psi) \\ t(\varphi \rightarrow \psi) &= -t(\varphi) \cup t(\psi) & t(\overline{\square}\varphi) &= -(<; -t(\varphi)) & t(\overline{\square}\varphi) &= -(<^{-1}; -t(\varphi)) \\ t(\square_N\varphi) &= -(N; -t(\varphi)) & t(\square_N\varphi) &= -(N^{-1}; -t(\varphi)) \end{aligned}$$

The translation is defined so that it preserves validity of formulas, that is the following holds:

Theorem 3.1: *For every $\mathcal{L}(MQ)^N$ -formula ψ and for all object variables x and y , we have that ψ is $\mathcal{L}(MQ)^N$ -valid iff $xt(\psi)y$ is $\mathcal{R}(MQ)^N$ -valid.*

4. Relational proof system for $\mathcal{R}(MQ)^N$

The proof system for logic $\mathcal{R}(MQ)^N$ presented in this section belongs to the family of dual tableau systems. Dual tableau systems are founded on Rasiowa-Sikorski deduction system for classical first order logic without identity (see [13]). The aim of Rasiowa and Sikorski was to present a system which, in contrast with the Gentzen system that required the cut rule in the proof of completeness, was cut free. Rasiowa-Sikorski system is a validity checker, i.e., the rules preserve and reflect validity of disjunctions of their premises and conclusions. As shown in [9], Rasiowa-Sikorski proof system for first-order logic with identity is dual to its tableau system. Recall that tableau systems are unsatisfiability checkers, i.e., the rules preserve and reflect unsatisfiability of conjunction of their premises and conclusions. Moreover, it is known that every proof in a dual tableau system can be easily converted into a proof in Gentzen style deduction system (see [10]).

Dual tableau systems are determined by axiomatic sets of formulas and rules which apply to finite sets of formulas. The axiomatic sets take the place of axioms. There are two groups of rules: the *decomposition rules*, which reflect properties of standard relational operations and the *specific rules* which reflect properties of the specific relations imposed in $\mathcal{R}(MQ)^N$ -models. Given a formula, the decomposition rules

¹Note that in standard models $m'(<)$ is a strict linear ordering on U' .

of the system enable us to transform it into simpler formulas, while the specific rules enable us to replace a formula by some other formulas. The rules have the following general form:

$$\text{(rule)} \quad \frac{\Phi(\bar{x})}{\Phi_1(\bar{x}_1, \bar{u}_1, \bar{w}_1) \mid \dots \mid \Phi_n(\bar{x}_n, \bar{u}_n, \bar{w}_n)}$$

where $\Phi(\bar{x})$ is a finite (possibly empty) set of formulas whose object symbols are among the elements of $\text{set}(\bar{x})$, where \bar{x} is a finite sequence of object symbols and $\text{set}(\bar{x})$ is a set of elements of sequence \bar{x} ; every $\Phi_j(\bar{x}_j, \bar{u}_j, \bar{w}_j)$, $1 \leq j \leq n$, is a finite non-empty set of formulas, whose object symbols are among the elements of $\text{set}(\bar{x}_j) \cup \text{set}(\bar{u}_j) \cup \text{set}(\bar{w}_j)$, where $\bar{x}_j, \bar{u}_j, \bar{w}_j$ are finite sequences of object symbols such that $\text{set}(\bar{x}_j) \subseteq \text{set}(\bar{x})$, $\text{set}(\bar{u}_j)$ consists of the variables that may be instantiated to arbitrary object symbols when the rule is applied (usually to the object symbols that appear in the set to which the rule is being applied), $\text{set}(\bar{w}_j)$ consists of the variables that must be instantiated to pairwise distinct new variables (not appearing in the set to which the rule is being applied) and distinct from any variable of sequence \bar{u}_j . A rule of the previous form is the n -fold branching rule, where the j -th branch is the set $\Phi_j(\bar{x}_j, \bar{u}_j, \bar{w}_j)$. A rule can be applied to a finite set of formulas X whenever $\Phi(\bar{x}) \subseteq X$. As a result of an application of a rule of the form (rule) to a finite set X , we obtain the sets $(X \setminus \Phi(\bar{x})) \cup \Phi_j(\bar{x}_j, \bar{u}_j, \bar{w}_j)$, $j = 1, \dots, n$.

We say that an object variable in a rule is *new* whenever it appears in a conclusion of the rule and does not appear in a set to which the rule is applied. The rules of the system presented below guarantee that whenever a node contains xTy or $x-Ty$, for some atomic relational term T , then all of its successors contain this formula as well. Thus, all variables of a given node occur in its successor nodes.

Let $x, y, z \in \mathcal{OS}$ and $R, S \in \mathcal{RT}$. *Decomposition rules* of the system have the following forms, for any object symbol z and for a new object variable w :

$$\begin{array}{lll} (\cup) & \frac{x(R \cup S)y}{xRy, xSy} & (-\cup) \quad \frac{x-(R \cup S)y}{x-Ry \mid x-Sy} \quad (:) \quad \frac{x(R; S)y}{xRz, x(R; S)y \mid zSy, x(R; S)y} \\ (\cap) & \frac{x(R \cap S)y}{xRy \mid xSy} & (-\cap) \quad \frac{x-(R \cap S)y}{x-Ry, x-Sy} \quad (-;) \quad \frac{x-(R; S)y}{x-Rw, w-Sy} \\ (-) & \frac{x--Ry}{xRy} & (-^1) \quad \frac{xR^{-1}y}{yRx} \quad (-^{-1}) \quad \frac{x-R^{-1}y}{y-Rx} \end{array}$$

Let $x, y, z \in \mathcal{OS}$, $R \in \mathcal{RA}$ and $i \in \{1, \dots, 5\}$. *Specific rules* have the following forms, for any object symbol z :

$$\begin{array}{ll} (1'1) & \frac{xRy}{xRz, xRy \mid y1'z, xRy} \quad (1'2) \quad \frac{xRy}{x1'z, xRy \mid zRy, xRy} \\ (\text{Irref} <) & \frac{}{x < x} \quad (\text{Tran} <) \quad \frac{x < y}{x < y, x < z \mid x < y, z < y} \\ (C_i 1) & \frac{}{x\Psi_i y \mid x-\Psi_i y} \quad (C_i 2) \quad \frac{x\Psi_i y}{x\Psi_i y, x1'c_i} \\ (C_i 3) & \frac{x-\Psi_i y}{x-\Psi_i y, x-1'c_i} \quad (\text{N1}) \quad \frac{}{xNy \mid x-Ny} \\ (\text{N2}) & \frac{xNy}{xNy, H_1(x, y) \mid xNy, H_2(x, y) \mid xNy, H_3(x, y)} \quad (\text{N3}) \quad \frac{x-Ny}{x-Ny, -K_1(x, y) \mid \dots \mid x-Ny, -K_7(x, y)} \end{array}$$

where $H_m(x, y)$ and $K_l(x, y)$, $m \in \{1, 2, 3\}$, $l \in \{1, \dots, 7\}$, are defined as follows:

$$\begin{array}{ll} K_1(x, y) := n_1(x, y) & K_2(x, y) := n_2(x, y), n_5(x, y) \\ K_3(x, y) := n_2(x, y), n_4(x, y) & K_4(x, y) := n_3(x, y), n_5(x, y) \\ K_5(x, y) := n_3(x, y), n_4(x, y) & K_6(x, y) := n_6(x, y), n_7(x, y), n_5(x, y) \\ K_7(x, y) := n_6(x, y), n_7(x, y), n_4(x, y) & \end{array}$$

$$\begin{aligned} H_1(x, y) &:= n_1(x, y), n_2(x, y), n_3(x, y), n_6(x, y) \\ H_2(x, y) &:= n_1(x, y), n_2(x, y), n_3(x, y), n_7(x, y) \\ H_3(x, y) &:= n_1(x, y), n_4(x, y), n_5(x, y) \end{aligned}$$

$$-K_l(x, y) := (-n_k(x, y))_{n_k(x, y) \in K_l(x, y)}, \text{ for } l \in \{1, \dots, 7\}$$

being:

$$\begin{aligned} n_1(x, y) &:= c_3 1'x & n_2(x, y) &:= c_4 1'x & n_3(x, y) &:= c_2 1'x & n_4(x, y) &:= c_5 < y \\ n_5(x, y) &:= y < c_1 & n_6(x, y) &:= c_2 < x & n_7(x, y) &:= x < c_4 \end{aligned}$$

Notice that the previous notation respond to the idea of expressing rules (N2) and (N3) in an easy way, following the definition of our negligibility relation.

A finite set of formulas is $\mathcal{R}(MQ)^N$ -axiomatic whenever it includes either of the following subsets, for all $x, y \in \mathcal{OS}$, $i \in \{1, \dots, 4\}$, and for every $R \in \mathbb{RT}$:

$$(Ax1) \{x1'x\} \quad (Ax2) \{x1y\} \quad (Ax3) \{xRy, x-Ry\} \quad (Ax4) \{c_i < c_{i+1}\} \quad (Ax5) \{x < y, y < x, x1'y\}$$

In what follows, we present the sketch of the proof of the soundness and completeness of the system in question.

A finite set of $\mathcal{R}(MQ)^N$ -formulas $\{xR_1y, \dots, xR_ny\}$ is said to be an $\mathcal{R}(MQ)^N$ -set whenever for every $\mathcal{R}(MQ)^N$ -model \mathcal{M}' and for every valuation v in \mathcal{M}' there exists $i \in \{1, \dots, n\}$ such that $\mathcal{M}', v \models xR_iy$. A rule $\frac{\Phi}{\Phi_1 | \dots | \Phi_n}$ is $\mathcal{R}(MQ)^N$ -correct whenever for every finite set X of $\mathcal{R}(MQ)^N$ -formulas $X \cup \Phi$ is an $\mathcal{R}(MQ)^N$ -set iff $X \cup \Phi_j$ is an $\mathcal{R}(MQ)^N$ -set for every $j \in \{1, \dots, n\}$.

Proposition 4.1:

1. The $\mathcal{R}(MQ)^N$ -rules are $\mathcal{R}(MQ)^N$ -correct.
2. The $\mathcal{R}(MQ)^N$ -axiomatic sets are $\mathcal{R}(MQ)^N$ -sets.

An $\mathcal{R}(MQ)^N$ -proof tree for an $\mathcal{R}(MQ)^N$ -formula φ is a tree with this formula at the root of the tree such that each node except the root is obtained by an application of an $\mathcal{R}(MQ)^N$ -rule to its predecessor node and a node does not have successors whenever it is an $\mathcal{R}(MQ)^N$ -axiomatic set.

A branch of an $\mathcal{R}(MQ)^N$ -proof tree is said to be *closed* whenever it contains a node with an $\mathcal{R}(MQ)^N$ -axiomatic set of formulas. A *closed tree* is an $\mathcal{R}(MQ)^N$ -proof tree such that all of its branches are closed. A formula φ is $\mathcal{R}(MQ)^N$ -provable whenever there is a closed proof tree for φ .

As usual in proof theory, a concept of completeness of a non-closed proof tree is needed. Intuitively, completeness of a non-closed tree means that all the rules that can be applied have been applied. By abusing the notation, for any branch b and for any set of formulas X , by $X \in b$ (resp. $X \notin b$) we mean that every formula from X belongs to b (resp. does not belong to b).

Completion Conditions

A branch b of a proof tree is said to be *complete* whenever for all $x, y \in \mathcal{OS}$ it satisfies the following completion conditions:

Cpl(\cup) (resp. Cpl($-\cap$)) If $x(R \cup S)y \in b$ (resp. $x-(R \cap S)y \in b$), then both $xRy \in b$ (resp. $x-Ry \in b$) and $xSy \in b$ (resp. $x-Sy \in b$).

Cpl(\cap) (resp. Cpl($-\cup$)) If $x(R \cap S)y \in b$ (resp. $x-(R \cup S)y \in b$), then either $xRy \in b$ (resp. $x-Ry \in b$) or $xSy \in b$ (resp. $x-Sy \in b$).

Cpl($-$) If $x(-R)y \in b$, then $xRy \in b$.

Cpl($^{-1}$) If $xR^{-1}y \in b$, then $yRx \in b$.

Cpl($^{-1}$) If $x-R^{-1}y \in b$, then $y-Rx \in b$.

Cpl($;$) If $x(R; S)y \in b$, then for every $z \in \mathcal{OS}$, either $xRz \in b$ or $zSy \in b$.

Cpl($;-$) If $x-(R; S)y \in b$, then for some $w \in \mathcal{OV}$, both $x-Rw \in b$ and $w-Sy \in b$.

Cpl($1'1$) If $xRy \in b$ for some $R \in \mathbb{RA}$, then for every $z \in \mathcal{OS}$, either $xRz \in b$ or $y1'z \in b$.

Cpl($1'2$) If $xRy \in b$ for some $R \in \mathbb{RA}$, then for every $z \in \mathcal{OS}$, either $x1'z \in b$ or $zRy \in b$.

Cpl(C_i1) Either $x\Psi iy \in b$ or $x-\Psi iy \in b$.
 Cpl(C_i2) If $x\Psi iy \in b$ then $x1'c_i \in b$.
 Cpl(C_i3) If $x-\Psi iy \in b$ then $x-1'c_i \in b$.
 Cpl(Irref \langle) For every $x \in \mathbb{OS}$, $x < x \in b$.
 Cpl(Tran \langle) If $x < y \in b$, then for every $z \in \mathbb{OS}$, either $x < z \in b$ or $z < y \in b$.
 Cpl(N1) Either $xNy \in b$ or $x-Ny \in b$.
 Cpl(N2) If $xNy \in b$ then there exists $m \in \{1, 2, 3\}$ such that $H_m(x, y) \in b$.
 Cpl(N3) If $x-Ny \in b$ then there exists $l \in \{1, \dots, 7\}$ such that $-K_l(x, y) \in b$.

An $\mathcal{R}(MQ)^N$ -proof tree is said to be *complete* iff all of its non-closed branches are complete. A complete non-closed branch is said to be *open*. It can be easily proved that for every $\mathcal{R}(MQ)^N$ -formula there exists a complete $\mathcal{R}(MQ)^N$ -proof tree for it.

As said in the introduction, there is a standard and intuitively simple way of proving completeness by constructing a counter-model for a non-provable formula out of its *non closed* decomposition tree.

Let b be an open branch of an $\mathcal{R}(MQ)^N$ -proof tree. A *branch structure* \mathcal{M}^b is a structure $\mathcal{M}^b = (U^b, m^b)$ such that $U^b = \mathbb{OS}$, $m^b(c_i) = c_i$, for every $i \in \{1, \dots, 5\}$, $m^b(R) = \{(x, y) \in U^b \times U^b : xRy \notin b\}$, for every $R \in \mathbb{RA}$, and m^b extends to all the compound relational terms as in $\mathcal{R}(MQ)^N$ -models. Observe that for every open branch b of an $\mathcal{R}(MQ)^N$ -proof tree, a branch structure $\mathcal{M}^b = (U^b, m^b)$ is an $\mathcal{R}(MQ)^N$ -model.

Let us consider a valuation v^b in the branch model \mathcal{M}^b defined as $v^b(x) = x$ for every $x \in \mathbb{OS}$. Then we get:

Proposition 4.2: *Let b be an open branch of an $\mathcal{R}(MQ)^N$ -proof tree. For every $\mathcal{R}(MQ)^N$ -formula φ , if $\mathcal{M}^b, v^b \models \varphi$, then $\varphi \notin b$.*

The above proposition can be proved by the induction on the complexity of relational terms by using the completion conditions.

Given $\mathcal{M}^b = (U^b, m^b)$ a branch structure, the *quotient model* is an $\mathcal{R}(MQ)^N$ -model $\mathcal{M}_q^b = (U_q^b, m_q^b)$ such that $U_q^b = \{\|x\| : x \in U^b\}$, where $\|x\|$ is an equivalence class of $m^b(1')$ generated by x , $m_q^b(c_i) = \|c_i\|$, for $i \in \{1, \dots, 5\}$, and $m_q^b(R) = \{(\|x\|, \|y\|) \in U_q^b \times U_q^b : (x, y) \in m^b(R)\}$, for every $R \in \mathbb{RA}$. Since $m^b(1')$ is an equivalence relation satisfying the extensionality property, the definition of $m_q^b(R)$ is correct, that is if $(x, y) \in m^b(R)$ and $(x, z), (y, t) \in m^b(1')$, then $(z, t) \in m^b(R)$. Moreover, the quotient model is a standard $\mathcal{R}(MQ)^N$ -model and for the valuation v_q^b in \mathcal{M}_q^b defined as $v_q^b(x) = \|x\|$, for $x \in \mathbb{OS}$, the following holds:

Proposition 4.3: *For every $\mathcal{R}(MQ)^N$ -formula φ , $\mathcal{M}^b, v^b \models \varphi$ iff $\mathcal{M}_q^b, v_q^b \models \varphi$.*

By the above propositions we obtain the main theorem:

Theorem 4.4 Soundness and Completeness of $\mathcal{R}(MQ)^N$:

For every $\mathcal{R}(MQ)^N$ -formula φ , φ is $\mathcal{R}(MQ)^N$ -provable iff φ is $\mathcal{R}(MQ)^N$ -valid iff φ is $\mathcal{R}^(MQ)^N$ -valid.*

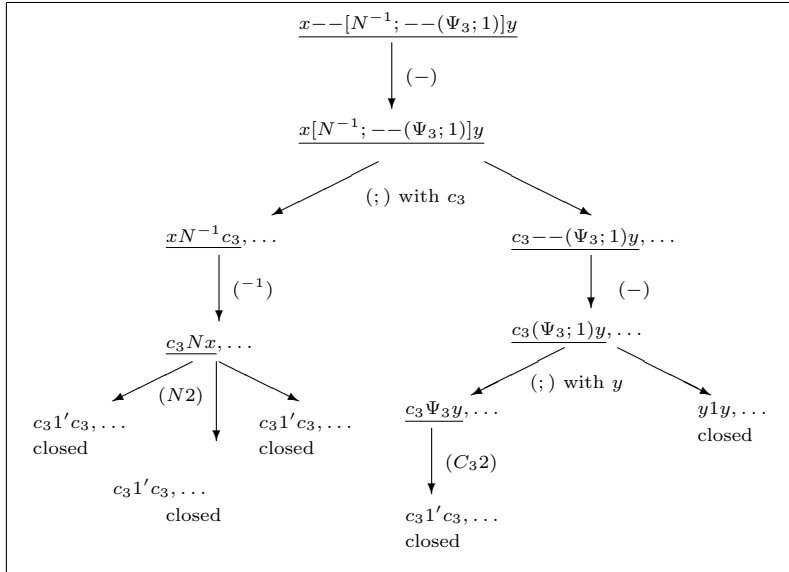
Proof: Soundness of the system follows from Proposition 4.1. Therefore, if a formula φ is $\mathcal{R}(MQ)^N$ -provable, then it is $\mathcal{R}(MQ)^N$ -valid, hence it is also $\mathcal{R}^*(MQ)^N$ -valid. For completeness, assume a formula φ is an $\mathcal{R}^*(MQ)^N$ -valid and suppose it is not $\mathcal{R}(MQ)^N$ -provable. Let b be an open branch of a complete $\mathcal{R}(MQ)^N$ -proof tree for φ . Then, by Proposition 4.2, φ is not satisfied in the branch structure \mathcal{M}^b by v^b . Hence, by Proposition 4.3, it is not satisfied in a standard $\mathcal{R}(MQ)^N$ -model \mathcal{M}_q^b by v_q^b , a contradiction with $\mathcal{R}^*(MQ)^N$ -validity of φ . \square

Finally, by the above and Theorem 3.1 we obtain:

Theorem 4.5 Relational Soundness and Completeness of $\mathcal{L}(MQ)^N$: For every $\mathcal{L}(MQ)^N$ -formula ψ and for all object variables x and y , the following holds:

$$\psi \text{ is } \mathcal{L}(MQ)^N\text{-valid iff } xt(\psi)y \text{ is } \mathcal{R}(MQ)^N\text{-provable}$$

We finish this section with an example of validity checking. Consider the formula $\varphi := \overline{\Diamond}_N c_3$. It is easy to check that this formula is $\mathcal{L}(MQ)^N$ -valid. The translation of φ to the relational term is $t(\varphi) = --(N^{-1}; --(\Psi_3; 1))$. The following picture presents a closed $\mathcal{R}(MQ)^N$ -proof tree. It shows $\mathcal{R}(MQ)^N$ -provability of the relational formula $x\tau(\varphi)y$, and by Theorem 4.5, it proves $\mathcal{L}(MQ)^N$ -validity of φ . In each node of the proof tree, we underline the formula to which a rule has been applied.



5. Conclusions and future work

In this paper, we have introduced a relational proof system in the style of dual tableaux for the relational logic associated with the multimodal propositional logic for order of magnitude qualitative reasoning $\mathcal{L}(MQ)^N$ and we have proved its soundness and completeness. Moreover, we have shown how the proof system can be used for verification of validity and we have given an example of the relational proof of validity for a specific formula.

As a future work, our plan is to adapt the implementation [2] to the specific relational system presented in the paper. We also are thinking about a proof system without cut-like rules (as ((Irref <)), (C_i1) and ((N1))), but at this moment, these rules have been necessary to prove completeness, and none cut-elimination theorem is known. On the other hand, we are planning to investigate the decidability of the logic $\mathcal{L}(MQ)^N$ and, if the answer is positive, to find a decision procedure, that is, a complete and sound relational proof system such that all of its proof trees are finite.

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