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**RESEARCH ARTICLE**

# Functional systems in the context of temporal×modal logics with indexed flows

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We consider logics which define different properties of functions –such as injectivity, surjectivity, monotonicity, etc.– in the context of temporal×modal logic. In this type of logics, the possible worlds semantics is modified by considering each world as a temporal flow and using accessibility functions to represent the connection among them. This approach is adequate to model interactions between processes with clocks that can be either synchronized or not. We study the definability and give indexed axiomatic systems for these properties.

**Keywords:** temporal logic, modal logic, logic in computer science, definability, $T \times W$-logics, functional axiomatic systems.

**AMS Subject Classification**: 03B44, 03B45, 03B70

## 1. Introduction

Modal and Temporal logics have been traditionally combined with different purposes from both a theoretical and practical point of view. Several approaches that combine (linear and branched) time with different type of modalities [20] (spatial, epistemic, etc.) have been arising due to the requirements of Theoretical Computer Science and Artificial Intelligence. Some examples are: spatio-temporal reasoning [15], parallel processes [18], security protocols [9], verification of multi-agent systems [10, 13], distributed systems [14], design of architectures of reasoning with mental attitudes [17], cooperation and planning [1, 19], semantics of messages [16], etc. About the importance and motivation of combination of temporal and other modalities, see also [2, 11].

Our interest in the field of logics which time and modality is to develop a bi-dimensional semantic approach that we call *functional*. Basically, this functional approach is based on possible worlds semantics, each world with its own (linear) temporal flow. The most significant feature of this approach is the use of functions (called *accessibility functions*) to connect the temporal flows in a frame, called *functional frame* [5–7]. Technically, functional frames constitute a generalization of Kamp-frames [6] by establishing more complex comparisons among different temporal flows.

From a practical point of view, our approach is adequate to model interactions between processes with clocks that can be either synchronized or not. For example, in distributed systems [12] we can consider our temporal flows as different processes

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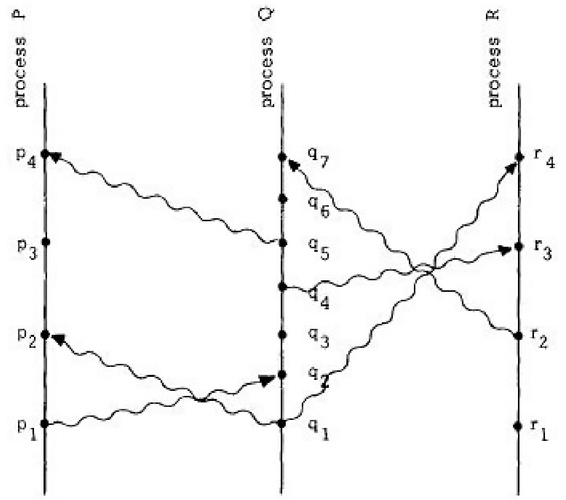
consisting of collections of (linearly ordered) events. An event can be either the execution of a subprogram or the execution of a single machine instruction, depending upon the application. Any process can communicate to each other by sending messages using our accessibility functions.

In the figure below, we have 3 processes $P, Q$ and $R$ such that their events are denoted $p_i, q_i$ and $r_i$, respectively. For example, the communication from $Q$ to $R$ is given by a partial function such that the images of $q_1, q_4$ are $r_4, r_3$, respectively.




From a theoretical point of view, the use of accessibility functions leads us to deal with different properties, such as injectivity, surjectivity, monotonicity, etc. As a consequence of this, we have to provide formal systems (*functional systems*) in order to manage these properties.

In [6], some results on definability for basic properties of functions were established and the proof of completeness for systems which define totality and a special class of partial functions, called uniform domain, was carried out. Moreover, the incompleteness of the system which defines the property of being total and injective was proved.

This approach was enriched in [5] by introducing indexes as names of temporal flows to represent the worlds where our accessibility functions have their images, that is, by using names only for specific temporal flows in which we are interested in establishing a contact.

The aim of this work is to make a progress in the study of functional systems. In fact, we give a complete axiomatic system for every property studied, such as to be constant, injective, monotone, etc. The only system which remains incomplete is the one for surjective functions. This reveals how fruitful the use of indexed languages can be because, as said above, an analogous system was proved to be incomplete without using indexes.

This paper is organized as follows: In Section 2, the language and the functional semantics of the temporal×modal logic $\mathcal{L}^{\mathcal{F}}_{(T\times W)}$-$\mathfrak{I}$ (from now on, for simplicity, $\mathcal{L}^{\mathfrak{I}}$) is sketched and the definability of properties of partial functions such as being non-total, constant, injective, surjective and increasing is given. Moreover, axiomatic systems dealing with these properties of functions are introduced. In Section 3, we study the soundness and completeness of the systems given in the previous section. Finally, some conclusions and prospectives of future work are given in Section 4.

## 2. The logics $\mathcal{L}^{\mathfrak{I}}$

This section is devoted to the language and semantics of the temporal×modal logic $\mathcal{L}^{\mathfrak{I}} = (L^{\mathfrak{I}}, \mathcal{M}^{\mathfrak{I}})$, where $\mathfrak{I}$ is a denumerable nonempty set of indexes, $L^{\mathfrak{I}}$ denotes the language and $\mathcal{M}^{\mathfrak{I}}$ models for $L^{\mathfrak{I}}$. The choice of a particular set of indexes $\mathfrak{I}$ determines a specific logic $\mathcal{L}^{\mathfrak{I}}$.

**Definition 2.1:** Given a (nonempty) denumerable set of indexes $\mathfrak{I}$, the alphabet of the language $L^{\mathfrak{I}}$ consists of:

(1) A denumerable set $\mathcal{V}$ of propositional variables (atoms).
(2) The logic constants $\top$ ("truth") and $\bot$ ("falsity"), and the Boolean connectives $\neg$, $\wedge$, $\vee$, and $\rightarrow$.
(3) The Priorean temporal connectives $F$ ("at some future time") and $P$ ("at some past time").
(4) A family of unary modal connectives of the form $<i>$, for $i \in \mathfrak{I}$ ("A is true in flow $i$, at the image of the reference instant").

Well-formed formulas (wffs) are generated by the construction rules of classical propositional logic, adding the following rule: If $A$ is a wff, then $FA$, $PA$ and $<i>A$ (with $i \in \mathfrak{I}$) are wffs.

Now, for each $i \in \mathfrak{I}$, we can also introduce the connective $[\,i\,]$ defined (as usual) by $[\,i\,]A =_{def} \neg <i> \neg A$ which has the following non-existential meaning:

"*If there exists some image of the reference instant in flow i, then A is true at such an image*".

**Definition 2.2:** An **ind-functional frame** for $L^{\mathfrak{I}}$ is a tuple $\Sigma^{\mathfrak{I}} = (W, \mathcal{T}, \mathcal{F})$ such that:

(1) $W$ is a nonempty set of labels (for a set of temporal flows).
(2) $\mathcal{T}$ is a nonempty set of strict linear orders pairwise disjoint labelled by $W$, that is, $\mathcal{T} = \{(T_w, <_w) \mid w \in W\}$ such that:
- $T_w \neq \varnothing$ and $<_w$ is a strict linear order on $T_w$, for all $w \in W$, called **temporal flow**.
- if $w \neq w'$, then $T_w \cap T_{w'} = \varnothing$, for all $w, w' \in W$

(3) $\mathcal{F}$ is a set of functions, called **accessibility functions**, such that:
a) Each function in $\mathcal{F}$ is a nonempty partial function from $T_w$ to $T_{w'}$, for some $w \in W$ and some $w' \in W \cap \mathfrak{I}$.
b) For an arbitrary pair $(w, w') \in W \times (W \cap \mathfrak{I})$, there is (in $\mathcal{F}$) at most one accessibility function from $T_w$ to $T_{w'}$, denoted by $\overset{w\ w'}{\longrightarrow}$

Notice that the definition of $\mathcal{T}$ depends only on $W$, whereas $\mathcal{F}$ depends on both $W$ and $\mathfrak{I}$. Thus, in order to ensure that modal connectives are able to represent the image of any accessibility function, this image should be a temporal flow labeled by an element of $W \cap \mathfrak{I}$. In particular, this means that if no temporal flow is named by any element of $\mathfrak{I}$, then $W \cap \mathfrak{I}$ would be empty. In this case, the set of accessibility functions $\mathcal{F}$ would be empty as well.

**Definition 2.3:** Let $\Sigma^{\mathfrak{I}} = (W, \mathcal{T}, \mathcal{F})$ be an ind-functional frame.
The elements of the disjoint union $t_w \in \bigcup_{w \in W} T_w$ are called **coordinates** and we will refer to $\mathcal{C}oord_{\Sigma^{\mathfrak{I}}} = \bigcup_{w \in W} T_w$ as the set of coordinates of $\Sigma^{\mathfrak{I}}$.

We now introduce some notation and terminology:

- If $(A, \leq)$ is a nonempty linearly ordered set and $a \in A$:
$[a, \rightarrow) = \{a' \in A \mid a \leq a'\}$ $(a, \rightarrow) = \{a' \in A \mid a < a'\}$

$(\leftarrow, a] = \{a' \in A \mid a' \leq a\}$; $(\leftarrow, a) = \{a' \in A \mid a' < a\}$.

- If $f : A \longrightarrow B$ is a nonempty partial function from $A$ to $B$, $Dom(f)$ represents the domain of $f$ and $X \subseteq A$, we define, as usual, $f(X) = \{f(x) \mid x \in X \cap Dom(f)\}$. Specifically, if $a \notin Dom(f)$, then $f(\{a\}) = \varnothing$.
- If $(A, \leq)$ and $(B, \leq)$ are nonempty linearly ordered sets, $f : A \longrightarrow B$ a nonempty partial function and $f(\{a\}) = \varnothing$, then:
$(\leftarrow, f(\{a\})) = (\leftarrow, f(\{a\})] = (f(\{a\}), \rightarrow) = [f(\{a\}), \rightarrow) = \varnothing$.

**Definition 2.4:** An **ind-functional model** for $L^{\mathfrak{I}}$ is a tuple $(\Sigma, h)$, where $\Sigma = (W, \mathcal{T}, \mathcal{F})$ is an ind-functional frame and $h$ is a function, called **functional interpretation**, assigning to each atom $p \in \mathcal{V}$ a subset of $\mathcal{C}oord_\Sigma$. The functional interpretation $h$ is recursively extended to a function (still denoted by $h$) defined for all the formulas of $L^{\mathfrak{I}}$, by interpreting the constants and Boolean connectives in a standard way and satisfying the following conditions:

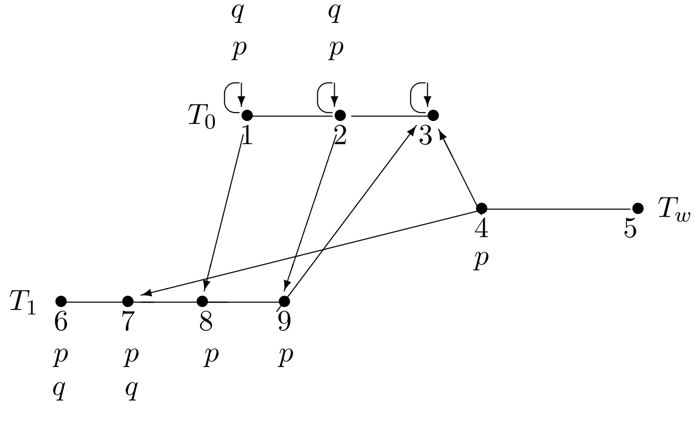
- $h(FA) = \{t_w \in \mathcal{C}oord_{\Sigma^{\mathfrak{I}}} \mid (t_w, \rightarrow) \cap h(A) \neq \varnothing\}$.
- $h(PA) = \{t_w \in \mathcal{C}oord_{\Sigma^{\mathfrak{I}}} \mid (\leftarrow, t_w) \cap h(A) \neq \varnothing)\}$.
- $h(<i>A) = \{t_w \in \mathcal{C}oord_{\Sigma^{\mathfrak{I}}} \mid \xrightarrow{w\ i} \in \mathcal{F} \textit{ and } \xrightarrow{w\ i}(\{t_w\}) \cap h(A) \neq \varnothing\}$.

As a consequence, the semantic definition of $[\,i\,]$ is the following:

$$h([\,i\,]A) = \{t_w \in \mathcal{C}oord_{\Sigma^{\mathfrak{I}}} \mid \xrightarrow{w\ i} \notin \mathcal{F}\} \cup \{t_w \in \mathcal{C}oord_{\Sigma^{\mathfrak{I}}} \mid \xrightarrow{w\ i} \in \mathcal{F} \textit{ and } \xrightarrow{w\ i}(\{t_w\}) \subseteq h(A)\}.$$

**Definition 2.5:** We say that a formula $A$ is **satisfiable** if there exists an ind-functional model $\mathcal{M} = (\Sigma^{\mathfrak{I}}, h)$ for $L^{\mathfrak{I}}$ and a $t_w \in \mathcal{C}oord_{\Sigma^{\mathfrak{I}}}$ such that $t_w \in h(A)$; in this case, we also say that $A$ is **true at** $t_w$. $A$ is said to be **false at** $t_w \in Coord_{\Sigma^{\mathfrak{I}}}$ if $t_w \notin h(A)$. In this case, we also say that $\mathcal{M}^{\mathcal{F}}$ is a **countermodel of** $A$. $A$ is said to be **valid in the ind-functional model** $(\Sigma^{\mathfrak{I}}, h)$ if $h(A) = \mathcal{C}oord_{\Sigma^{\mathfrak{I}}}$. If $A$ is valid in every ind-functional model on $\Sigma^{\mathfrak{I}}$, then $A$ is said to be **valid in the ind-functional frame** $\Sigma^{\mathfrak{I}}$. If $A$ is valid in every ind-functional frame, then $A$ is said to be **valid**. Let $\mathbb{K}$ be a class of ind-functional frames; then $A$ is said to be **valid in** $\mathbb{K}$ if $A$ is valid in every ind-functional frame $\Sigma^{\mathfrak{I}}$ such that $\Sigma^{\mathfrak{I}} \in \mathbb{K}$ .

**Example 2.6** Let us consider an ind-functional frame $\Sigma^{\mathfrak{I}} = (W, \mathcal{T}, \mathcal{F})$ such that $\mathfrak{I} = \{0, 1\}$, $W = \{w, 0, 1\}$, $\mathcal{T} = \{(T_0, <_0), (T_w, <_w), (T_1, <_1)\}$, being $T_0 = \{1, 2, 3\}$, $T_w = \{4, 5\}$, $T_1 = \{6, 7, 8, 9\}$; $<_w, <_0$ , $<_1$ the restrictions of the usual strict linear order in the real line, and $\mathcal{F} = \{\xrightarrow{0\ 0}, \xrightarrow{w\ 0}, \xrightarrow{w\ 1}, \xrightarrow{0\ 1}, \xrightarrow{1\ 0}\}$ such that $\xrightarrow{0\ 0} = \{(1, 1), (2, 2), (3, 3)\}$, $\xrightarrow{w\ 0} = \{(4, 3)\}$, $\xrightarrow{w\ 1} = \{(4, 7)\}$, $\xrightarrow{0\ 1} = \{(1, 8), (2, 9)\}$ and $\xrightarrow{1\ 0} = \{(9, 3)\}$. Consider also a ind-functional model $(\Sigma^{\mathfrak{I}}, h)$ such that $h(p) = \{1, 2, 4, 6, 7, 8, 9\}$ and $h(q) = \{1, 2, 6, 7\}$, being $p$ and $q$ two atoms of our language. In the picture below, we represent the previous ind-functional model:



As a consequence of the previous definitions, we give the coordinates where some formulas are true in the previous model:

| **Formula** | **True at** |
|---|---|
| $[1]p$ | every coordinate |
| $<0>p$ | $1_0, 2_0$ |
| $P<0>[1]p$ | $2_0, 3_0, 5_w$ |
| $F<0>p \wedge H<1>\neg q$ | $1_0$ |
| $G<0>H(p \wedge q)$ | $1_0, 2_0, 3_0, 8_1, 9_1, 5_w$ |
| $p \wedge <1>(Hp \wedge Gp \wedge q)$ | $4_w$ |
| $H(p \rightarrow <0>P<1>Hq)$ | $1_0, 6_1, 4_w, 5_w$ |

### 2.1. *Definability of properties of functions in $\mathcal{L}^{\mathfrak{I}}$*

In this section we present a sketch of definability proof of the properties of functions discussed in this work.
The following theorem provides a characterization of several properties of functions.

**Theorem 2.7:** *If $(A, \leq)$ and $(B, \leq)$ are nonempty linearly ordered sets and $f: A \longrightarrow B$ is a nonempty partial function, then we have that:*

*(1) $f$ is constant if, and only if, for all $a \in Dom(f)$, we have:*

$$f((\leftarrow, a)) \cup f((a, \rightarrow)) \subseteq \{f(a)\} \qquad \textbf{(CONSTANT)}$$

*(2) $f$ is injective if, and only if, for all $a \in Dom(f)$, we have:*

$$f((\leftarrow, a)) \cup f((a, \rightarrow)) \subseteq (\leftarrow, f(a)) \cup (f(a), \rightarrow) \qquad \textbf{(INJECTIVE)}$$

*(3) $f$ is surjective if, and only if, for all $a \in A$, we have:*

$$(\leftarrow, f(\{a\})) \cup (f(\{a\}), \rightarrow) \subseteq f((\leftarrow, a)) \cup f((a, \rightarrow)) \qquad \textbf{(SURJECTIVE)}$$

*(4) $f$ is increasing if, and only if, for all $a \in Dom(f)$, we have:*

$$f((a, \rightarrow)) \subseteq [f(a), \rightarrow) \qquad \textbf{(INCREASING)}$$

*(5) $f$ is a total function if, and only if, for any $a \in A$ we have that:*

$$f((\leftarrow, a)) \cup f((a, \rightarrow)) \subseteq (\leftarrow, f(\{a\})] \cup (f(\{a\}), \rightarrow) \qquad \textbf{(TOTAL)}$$

**Definition 2.8:** Let $\mathbb{J}$ be a class of ind-functional frames and $\mathbb{K} \subseteq \mathbb{J}$. We say that $\mathbb{K}$ is $\mathcal{L}^{\mathfrak{I}}$**- definable in** $\mathbb{J}$ if there exists a set $\Gamma$ of formulas in $L^{\mathfrak{I}}$ such that for every ind-functional frame $\Sigma^{\mathfrak{I}} \in \mathbb{J}$ we have that $\Sigma^{\mathfrak{I}} \in \mathbb{K}$ if, and only if, every formula of $\Gamma$ is valid in $\Sigma^{\mathfrak{I}}$. If $\mathbb{J}$ is the class of *all* ind-functional frames, we say that $\mathbb{K}$ is $\mathcal{L}^{\mathfrak{I}}$**-definable**.

Let $P$ be a property of functions (i.e., injectivity, surjectivity, etc.) and $\mathbb{K}$ the class of all ind-functional frames whose functions have the property $P$. We say that $P$ is $\mathcal{L}^{\mathfrak{I}}$**-definable** if $\mathbb{K}$ is $\mathcal{L}^{\mathfrak{I}}$**-definable**.

In order to establish the definability of the desired properties of functions, we introduce the following sets of formulas:

| $\alpha$ | $S^{\alpha}$ |
|---|---|
| non-total | $(Non\text{-}Tot)$-$ind$: $\{P[i]\bot \vee [i]\bot \vee F[i]\bot \mid i \in \mathfrak{I}\}$ |
| constant | $(Cons)$-$ind$: $\{\langle i\rangle A \to (H[i]A \wedge G[i]A) \mid i \in \mathfrak{I}\}$ |
| injective | $(Inj)$-$ind$: $\{\langle i\rangle (HA \wedge GA) \to (H[i]A \wedge G[i]A) \mid i \in \mathfrak{I}\}$ |
| surjective | $(Surj)$-$ind$: $\{(H[i]A \wedge G[i]A) \to [i](HA \wedge GA) \mid i \in \mathfrak{I}\}$ |
| increasing | $(Inc)$-$ind$: $\{\langle i\rangle (A \wedge GA) \to G[i]A \mid i \in \mathfrak{I}\}$ |
| decreasing | $(Dec)$-$ind$: $\{\langle i\rangle (A \wedge HA) \to G[i]A \mid i \in \mathfrak{I}\}$ |
| strictly increasing | $(Str\text{-}Inc)$-$ind$: $\{\langle i\rangle GA \to G[i]A \mid i \in \mathfrak{I}\}$ |
| stricty decreasing | $(Str\text{-}Dec)$-$ind$: $\{\langle i\rangle HA \to G[i]A \mid i \in \mathfrak{I}\}$ |

The following result can be obtained by using Theorem 2.7 and our semantics.

**Theorem 2.9:** *For every property $\alpha$ given in the previous table, the class of ind-functional frames $\{(W,\mathcal{T},\mathcal{F}) \mid \mathcal{F}$ is a class of functions with the property $\alpha\}$ is $\mathcal{L}^{\mathfrak{I}}$-definable by the set of formulas $S^{\alpha}$.*

**Proof:** We focus our attention on the property of injectivity. The rest of cases can be treated similarly. Let $\Sigma^{\mathfrak{I}} = (W,\mathcal{T},\mathcal{F})$ be a frame such that every accessibility function is injective and let us consider $\xrightarrow{w\ i} \in \mathcal{F}$. From Theorem 2.7, for every $t_w \in Dom(\xrightarrow{w\ i})$, we have that:

$$(*) \qquad \xrightarrow{w\ i}((\leftarrow, t_w) \cup (t_w, \rightarrow)) \subseteq (\xrightarrow{w\ i}(t_w), \rightarrow) \cup (\leftarrow, \xrightarrow{w\ i}(t_w))$$

It suffices to prove the validity of $\langle i\rangle (HA \wedge GA) \to (H[i]A \wedge G[i]A)$ in $\Sigma^{\mathfrak{I}}$. Consider a model $(\Sigma^{\mathfrak{I}}, h)$ and a coordinate $t_w$ such that $t_w \in h(\langle i\rangle (HA \wedge GA))$, this means that there exists $\xrightarrow{w\ i} \in \mathcal{F}$ such that $t_w \in Dom(\xrightarrow{w\ i})$. As a result:

$$(\xrightarrow{w\ i}(t_w), \rightarrow) \cup (\leftarrow, \xrightarrow{w\ i}(t_w)) \subseteq h(A)$$

Hence, from $(*)$, we have $\xrightarrow{w\ i}((\leftarrow, t_w) \cup (t_w, \rightarrow)) \subseteq h(A)$, which leads, by using $t_w \in h(G[i]A)$ if, and only if, $\xrightarrow{w\ i}((t_w, \rightarrow)) \subseteq h(A)$ and $t_w \in h(H[i]A)$ if, and only if, $\xrightarrow{w\ i}((\leftarrow, t_w)) \subseteq h(A)$, to $t_w \in h((H[i]A \wedge G[i]A))$ and proves the validity of the set $(Inj)$-$ind$ in every frame with injective accessibility functions.

Reciprocally, suppose a frame $\Sigma^{\mathfrak{I}} = (W,\mathcal{T},\mathcal{F})$ such that there exists a non injective accessibility function $\xrightarrow{w\ i} \in \mathcal{F}$. By using again Theorem 2.7, we have that there exists a coordinate $t_w \in Dom(\xrightarrow{w\ i})$ such that:

$$\xrightarrow{w\ i}((\leftarrow, t_w) \cup (t_w, \rightarrow)) \not\subseteq (\xrightarrow{w\ i}(t_w), \rightarrow) \cup (\leftarrow, \xrightarrow{w\ i}(t_w))$$

In order to prove that $\langle i\rangle (HA \wedge GA) \to (H[i]A \wedge G[i]A)$ is not valid in $\Sigma^{\mathfrak{I}} = (W,\mathcal{T},\mathcal{F})$, we take $A = p$ being $p$ any atom, and consider the model $(\Sigma^{\mathfrak{I}}, h)$ such that $h(A) = (\xrightarrow{w\ i}(t_w), \rightarrow) \cup (\leftarrow, \xrightarrow{w\ i}(t_w))$. In this model, it is straightforward to prove that our formula is not valid.

□

**Remark 1:** The previous result is given for partial functions. It is easy to prove a similar result for total functions, that is, those which are defined for every coordinate in the corresponding temporal flow. In all cases except surjectivity, we have only to change boxes by diamonds in the second member of every formula in order to ensure the existence of the desired image. For instance, the set of formulas:

$$(Tot\text{-}Inj)\text{-}ind \quad \{\langle i\rangle (HA \wedge GA) \to (H\langle i\rangle A \wedge G\langle i\rangle A)) \mid i \in \mathfrak{I}\}$$

defines the property of being total and injective. In the case of surjectivity, the set of formulas which defines this property is the union of the set $(Surj)$-*ind* considered above with $\{<i>(HA \wedge A \wedge GA) \rightarrow (H <i> A \wedge G <i> A)) \mid i \in \mathfrak{I}\}$.

### 2.2. *Functional systems for partial functions*

In this section, we introduce the minimal system for partial functions and then several extensions of it dealing with non-total, constant, injective, surjective and increasing functions. The axioms of system $\mathcal{S}^{\mathfrak{I}}$ -*Par* are the following:

(1) Those of the minimal system of propositional linear temporal logic $\mathcal{K}_l$.
(2) For each $i \in \mathfrak{I}$, $[\,i\,](A \rightarrow B) \rightarrow ([\,i\,]A \rightarrow [\,i\,]B)$.
(3) For each $i \in \mathfrak{I}$, $<i>A \rightarrow [\,i\,]A$. (**Axiom of Functionality**)
(4) $(\lambda <i>A \wedge \lambda' <i>B) \rightarrow \lambda <i>(A \wedge (PB \vee B \vee FB))$. (**Axiom of Confluence**)

where:

$$\begin{cases} \lambda = \gamma_1 <j_1> \gamma_2 \ldots <j_n> \gamma_{n+1}, & n \in \mathbb{N},\ \gamma_i \in \{F, P, \epsilon\},\ j_i \in \mathfrak{I} \\ \lambda' = \gamma'_1 <k_1> \gamma'_2 \ldots <k_m> \gamma'_{m+1}, & m \in \mathbb{N},\ \gamma'_i \in \{F, P, \epsilon\},\ k_i \in \mathfrak{I} \end{cases}$$

and $\epsilon$ denotes the empty chain.

**Remark 2 :** Axiom (3) establishes functionality ensuring the uniqueness of the image. Axiom (4) assures that it is possible to access to the same temporal flow by using different paths.

The **inference rules** of $\mathcal{S}^{\mathfrak{I}}$-*Par* are the following, being $i \in \mathfrak{I}$:
$(MP)$ $A, A \rightarrow B \vdash B$; $(RG)$ $A \vdash GA$; $(RH)$ $A \vdash HA$; $(N_i)$ $A \vdash [\,i\,]A$
The syntactical concepts of *proof*, *theorem*, etc., are defined as usual.

Let us define now the different extensions of the system $\mathcal{S}^{\mathfrak{I}}$-*Par* to deal with the properties which have been considered previously.

| | | |
|---|---|---|
| $\mathcal{S}^{\mathfrak{I}}$-*Non-Tot* | = | $\mathcal{S}^{\mathfrak{I}}$-*Par* + (*Non-Tot*)-*ind* |
| $\mathcal{S}^{\mathfrak{I}}$-*Cons* | = | $\mathcal{S}^{\mathfrak{I}}$-*Par* + (*Cons*)-*ind* |
| $\mathcal{S}^{\mathfrak{I}}$-*Inj* | = | $\mathcal{S}^{\mathfrak{I}}$-*Par* +(*Inj*)-*ind* |
| $\mathcal{S}^{\mathfrak{I}}$-*Surj* | = | $\mathcal{S}^{\mathfrak{I}}$-*Par* + (*Surj*)-*ind* |
| $\mathcal{S}^{\mathfrak{I}}$-*Inc* | = | $\mathcal{S}^{\mathfrak{I}}$-*Par* + (*Inc*)-*ind* |
| $\mathcal{S}^{\mathfrak{I}}$-*Dec* | = | $\mathcal{S}^{\mathfrak{I}}$-*Par* + (*Dec*)-*ind* |
| $\mathcal{S}^{\mathfrak{I}}$-*Str-Inc* | = | $\mathcal{S}^{\mathfrak{I}}$-*Par* + (*Str-Inc*)-*ind* |
| $\mathcal{S}^{\mathfrak{I}}$-*Str-Dec* | = | $\mathcal{S}^{\mathfrak{I}}$-*Par* + (*Str-Dec*)-*ind* |

**Remark 3 :** We can obtain the corresponding axiomatic system for total and injective functions by using the set of formulas (*Tot-Inj*)-*ind* given in Remark **1**, that is, $\mathcal{S}^{\mathfrak{I}}$-*Par* + (*Tot-Inj*)-*ind*. Similarly for the rest of properties.

## 3. Soundness and Completeness

In this section we study the soundness and completeness of all of the previous systems. We state that all systems except $\mathcal{S}^{\mathfrak{I}}$-*Surj* are complete. This reveals how fruitful the use of indexed languages can be, because as said above, a system for injective functions was proved to be incomplete without using indexes in [6].

The soundness of all the systems is straightforward: we only need to prove the validity of the axioms and that the rules preserve validity.

Regarding completeness, a *step-by-step* proof (see, for example, [3, 4]) can be given in the following terms: Given any consistent formula $A$, we have to prove that $A$ is satisfiable. With this purpose, the step-by-step method defines a frame $\Sigma^{\Im}$ and a function $f_{\Sigma^{\Im}}$ which assigns a maximal consistent set (mc-set) to each coordinate, such that $A \in f_{\Sigma^{\Im}}(t_w)$ for some coordinate $t_w$. The process to build such a frame is recursive, and successive extensions of frames are defined until $\Sigma^{\Im}$ is obtained. Finally, to give a model which satisfies $A$, we consider that the mc-set assigned to each coordinate is the set of formulas that are true at this coordinate. We start the construction with a finite frame $\Sigma_0^{\Im} = (W_0, \mathcal{T}_0, \mathcal{F}_0)$, where: $W_0 = \{w_0\}$, $\mathcal{T}_0 = \{(\{t_{w_0}\}, \varnothing)\}$ and $\mathcal{F}_0 = \varnothing$. Moreover, we define $f_{\Sigma_0^{\Im}}(t_{w_0}) = \Gamma$, being $\Gamma$ a maximal consistent set containing $A$. Now, we obtain a denumerable sequence of finite frames $\Sigma_0^{\Im}, \Sigma_1^{\Im}, \ldots, \Sigma_n^{\Im}, \ldots$ whose union is $\Sigma^{\Im}$, and a denumerable sequence of corresponding functions, $f_{\Sigma_0^{\Im}}, f_{\Sigma_1^{\Im}}, \ldots, f_{\Sigma_n^{\Im}}, \ldots$, whose union is $f_{\Sigma^{\Im}}$. This construction is generated by the existential formulas (i.e., those of the form $FB$, $PB$ or $<i> B$) which may appear initially in $\Gamma$ and, as a consequence, in the rest of mc-sets associated to the coordinates introduced in the process. As an example, consider constructed $\Sigma_n^{\Im}$, and assume an existential formula $<i>B \in f_{\Sigma_n^{\Im}}(t_w)$. If we were in the case that there is no $t_i = \xrightarrow{w\,i}(t_w)$ such that $B \in f_{\Sigma_n^{\Im}}(t_i)$, then the construction is developed in such a way that guarantees a solution for this problem. In effect, it is generated a finite frame $\Sigma_m^{\Im}$, extension of $\Sigma_n^{\Im}$, in which there exists some $t_i = \xrightarrow{w\,i}(t_w)$ such that $B \in f_{\Sigma_m^{\Im}}(t_i)$ (where $\xrightarrow{w\,i}$ has the desired properties and $f_{\Sigma_m^{\Im}}$ is an extension of $f_{\Sigma_n^{\Im}}$). The resulting frame $\Sigma^{\Im}$ as defined, should have the required properties for the system considered; for example, if the system is $S^{\Im}$-$Inj$, all the accessibility functions in $\Sigma^{\Im}$ have to be injective. The technical problems which arise from the use of any different property of accessibility functions need special attention. Due to lack of space, the formal details are omitted in this paper, but can be seen in [8].

Thus, we have the following result:

**Theorem 3.1 :** *(***Soundness and Completeness***)*
*Let be $\beta \in \{Non\text{-}Tot, Cons, Inj, Inc, Dec, Str\text{-}Inc, Str\text{-}Dec\}$, then we have:*

- *Every theorem of $\mathcal{S}^{\Im}$-$\beta$ is valid in the class of frames:*

$$\{(W, \mathcal{T}, \mathcal{F}) \mid \mathcal{F} \text{ is a class of functions with the property } \beta\}.$$

- *Every valid formula in the class of frames:*

$$\{(W, \mathcal{T}, \mathcal{F}) \mid \mathcal{F} \text{ is a class of functions with the property } \beta\}$$

*is a theorem of $\mathcal{S}^{\Im}$-$\beta$.*

Finally, we focus now our attention on the incompleteness of the system $\mathcal{S}^{\Im}$-$Surj$. We prove that there is no a class of *ind*-functional frames for this system (i.e., a class where every theorem is valid) with respect to $S^{\Im}$-$Surj$ is complete. For this, it suffices to show that $S^{\Im}$-$Surj$ is not complete with respect to the class of *all* *ind*-functional frames for this system. This class is precisely the class of all *ind*-functional frames where every function is surjective. The previous statement is a consequence of the soundness of $S^{\Im}$-$Surj$, given in Theorem 3.1, and the fact that the class of all *ind*-functional frames where every function is surjective is definable by the set $(Surj)$-$ind$, given in Theorem 2.9. Hence, the incompleteness of $S^{\Im}$-$Surj$ is obtained by means of the following results:

**(1)** Consider $i \in \Im$. Let $X$ be the scheme of formula:

$$<i> F(A \wedge FB) \rightarrow \Big( P(<i> A \wedge P <i> B) \vee (P <i> A \wedge F <i> B) \vee F(<i> A \wedge F <i> B) \vee P(P <i> A \wedge <i> B) \vee (F <i> A \wedge P <i> B) \vee F(F <i> A \wedge <i> B)\Big);$$

then, every instance of $X$ is valid in the class of all *ind*-functional frames where every function is surjective.

**(2)** There exists a (non *ind*-functional) model $\mathcal{M}$ such that:
a) Every theorem of $\mathcal{S}^{\mathfrak{I}}$-$Surj$ is valid in $\mathcal{M}$.
b) There exists an instance of $X$ which is not valid in $\mathcal{M}$.

**Proof of (1)**: Firstly, we shall prove that every instance of $X$ is valid in the following class of *ind*-functional frames

$$\mathbb{K} = \{\Sigma^{\mathfrak{I}} = (W, \mathcal{T}, \mathcal{F}) \mid \mathcal{F} \text{ is a class of surjective functions}\}.$$

Let $\Sigma^{\mathfrak{I}} \in \mathbb{K}$ and let $(\Sigma^{\mathfrak{I}}, h)$ be any *ind*-functional model on $\Sigma^{\mathfrak{I}}$.
Suppose $t_w \in h(<i> (F(A \wedge FB)))$, then there are $t_i <_i t'_i <_i t''_i$ in $T_i$ such that $t_i = \stackrel{w\,i}{\longrightarrow} (t_w)$, $t'_i \in h(A)$ and $t''_i \in h(B)$. Since $\stackrel{w\,i}{\longrightarrow}$ is surjective, there are $t'_w, t''_w \in T_w$, such that $t'_w \neq t_w$ and $t''_w \neq t_w$ and whose images are, respectively, $t'_i$ and $t''_i$. Now, we have the following possibilities with respect to $t'_w$ and $t''_w$:

- $t'_w, t''_w \in (t_w, \rightarrow)$, then $t_w \in h(F(<i> A \wedge F <i> B) \vee F(F <i> A \wedge <i> B))$.
- $t'_w, t''_w \in (\leftarrow, t_w)$, then $t_w \in h(P(<i> A \wedge P <i> B) \vee P(P <i> A \wedge <i> B))$.
- One of these $t'_w$, $t''_w$ belongs to $(t_w, \rightarrow)$ and the other one belongs to $(\leftarrow, t_w)$, then $t_w \in h((P <i> A \wedge F <i> B) \vee (F <i> A \wedge P <i> B))$.

In all three cases $t_w \in h(X)$ and, finally, we have that $X$ is valid in the class $\mathbb{K}$.

**Proof of (2)**: We proceed to construct the required model $\mathcal{M}$ and we choose an instance of $X$ which is not valid in $\mathcal{M}$.
Fix $i \in \mathfrak{I}$ and $\Psi^{\mathfrak{I}} = (W, \mathcal{T}, \mathcal{F})$ be a tuple associated to $\mathfrak{I}$ such that:

- $W = \{w, i\}$.
- $\mathcal{T} = \{(T_w, <_w), (T_i, R_i)\}$, where
  - $T_w = \{1_w, 2_w\}, <_w = \{(1_w, 2_w)\}$
  - $T_i = \{3_i\}, R_i = T_i \times T_i$.
- $\mathcal{F} = \{f_{wi}\}$ where $f_{wi} : T_w \rightarrow T_i$ with $f_{wi}(1_w) = f_{wi}(2_w) = 3_i$.

Notice that $\Psi^{\mathfrak{I}}$ is not an *ind*-functional frame, as expected, because the order $R_i$ is not a strict linear order.
Now, we define $\mathcal{M} = (\Psi^{\mathfrak{I}}, h)$, where $h$ is an application defined as follows:

$$h : \mathcal{V} \longrightarrow 2^{\{1_w, 2_w, 3_i\}}$$

so that $h(p) = \mathcal{C}oord_{\Psi^{\mathfrak{I}}}$ for all $p \in \mathcal{V}$. Therefore, the semantics of the connective $<i>$ is $h(<i> A) = \{t_w \in \mathcal{C}oord_{\Psi^{\mathfrak{I}}} \mid f_{w0}(\{t_w\}) \cap h(A) \neq \varnothing\}$. For (2.a), it is not difficult to check that all the axioms of the system $\mathcal{S}^{\mathcal{F}}_{(T \times W) - \mathfrak{I}}$ -$Surj$ are valid in $\mathcal{M}$ and rules preserves validity, so all the theorems of $\mathcal{S}^{\mathfrak{I}}$ -$Surj$ are equally valid in $\mathcal{M}$. On the other hand, for proving (2.b), we shall see that $1_w \notin h(X)$. Consider the instance of $X$, $A$ is $p$ and $B$ is $q$. Its antecedent is true at $1_w$. To check this, notice that each atom is true everywhere, thus $T_i \subseteq h(p) \cap h(q)$, so $3_i \in h(F(p \wedge Fq))$ and finally $1_w \in h(<i> F(p \wedge Fq))$. On the other hand, the consequent of that instance is false at $1_w$ as we can see:

- Every alternative to the consequent with a formula beginning with the connective $P$ is immediately false at $1_w$, because this is an initial point in $T_w$. Thus, all the following formulas are false at that coordinate:

$$P(<i> p \wedge P <i> q), P <i> p \wedge F <i> q, P(<i> q \wedge P <i> p), P <i> q \wedge F <i> p.$$

- The other alternatives to the consequent, namely, $F(<i> p \wedge F <i> q)$ and $F(<i> q \wedge F <i> p)$, are false at $1_w$ because $(T_w, <_w)$ is a strict linear order with only two elements.

Hence, it has been proved that $1_w \notin h(X)$ and, therefore, that $X$ is not valid in $\mathcal{M}$. This ends the proof of the incompleteness of $\mathcal{S}^{\Im}$-$Surj$.

## 4. Conclusions and Future Work

We have studied functional axiomatic systems dealing with several properties of functions by using names for temporal flows. These systems can be applied to deal with interactive systems not necessarily synchronized. The use of these indexed languages has led to the completeness of systems (such as the one for injectivity) which were proved to be incomplete by using non-indexed connectives. The only system which remains incomplete is the one for surjective functions. This result shows that our indexed language may be not enough to obtain the completeness of this system. A natural extension of this approach could be to add another index to indicate the domain of the accessibility function.

Other future works that we have planned are related to the study of the decidability of these logics and the design of theorem provers.

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