Analyzing completeness of axiomatic functional systems for temporal×modal logics

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In previous works, we presented a modification of the usual possible world semantics by introducing an independent temporal structure in each world and using accessibility functions to represent the relation among them. Different properties of the accessibility functions (being injective, surjective, increasing, etc.) have been considered and axiomatic systems (called functional) which define these properties have been given. Only a few of these systems have been proved to be complete. The aim of this paper is to make a progress in the study of completeness for functional systems. For this end, we use indexes as names for temporal flows and give new proofs of completeness. Specifically, we focus our attention on the system which defines injectivity, because the system which defines this property without using indexes was proved to be incomplete in previous works. The only system considered which remains incomplete is the one which defines surjectivity, even if we consider a sequence of natural extensions of the previous one.

1 Introduction

The combination of time and modality appears in the seminal work of Prior [21] in order to study determinism, by establishing different type of semantics, called *Peircean* and *Ockhamist*, where time is conceived as a tree, each point having one linear past but branching towards the future. The different branches represent alternative possibilities whereas the unique past represents the necessity of the past, the inevitability of past facts or the historical necessity [14, 24, 27]. There are other approaches that also combine time and modality to deal with the historical necessity but not using trees. They are two-dimensional, that is, possible worlds have a temporal compound [25]. Usual approaches in this way are $T \times W$-frames, where the order of time is the same in all possible worlds; and Kamp-frames, a generalization of $T \times W$-frames, where each world has its own order of time. Moreover, Kamp-frames are technically equivalent to a generalization of trees, called *bundled trees*.

In this area, we have developed a semantic approach (called *functional*), based on possible world semantics, by considering each world provided with its own flow of time. We use *accessibility functions* to connect these flows in a frame (called *functional frame* [9–12]) which is a generalization of the Kamp-frames [9]. This approach follows the tradition of applying non-standard logics in mathematics. We represent basic properties of functions, such as being injective, surjective, increasing, etc., by means of a combination of modal and temporal logics. Moreover, we focus our attention in the completeness of the formal systems which represent these properties. Recent uses of modal logic in mathematical theories follow the two major ideas that dominate the landscape of modal logic application in mathematics: Gödel's Provability Semantics and Tarski's Topologycal Semantics. For instance, in [4–6], the use of modal logic in Geometry and/or Topology has been studied and the relationship between modal logics and coalgebras has been considered in [15, 19].

As said in [3], modal logic offers a quantifier free language in order to express mathematical properties. In our case, we use modal logics to represent the properties of accessibility functions mentioned above. We have studied the definability of these properties and the proof of completeness for a system which defines the property

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of totality has been given in [9]. Moreover, the incompleteness of the system which defines the property of being total and injective was proved. This approach has been enriched in [10, 12] by introducing indexes as names of temporal flows to represent the worlds where our accessibility functions have their images. The use of names for possible worlds has its origin in works of Prior, that can be considered the beginning of the *Hybrid Logics*, where these names are used in an extended way, with very important advantages (see e.g. [2]). In our case, the first result obtained with the introduction of these indexes was a proof of completeness for a minimal system for partial functions.

The objective of this paper is to improve the completeness results of functional axiomatic systems. The main result is the proof of completeness for the system which defines injectivity. This fact shows the advantages of using indexed languages, because as said above, an analogous system was proved to be incomplete without using indexes. The completeness proof for each system is not a trivial extension of the given one for partial functions in [10], because of the technical problems which arise from the use of any different property need special attention. For this reason, we give also an Appendix with the completeness result of the system for increasing functions. The rest of the properties can be treated in a similar way, by taking into account the specificity of each one. In addition, we show the incompleteness of the system which defines surjectivity and we also prove the incompleteness of a sequence of systems obtained from the previous one by adding successively new axioms which appear in a natural way.

From the practical point of view, several works that combine (linear and branched) time with different type of modalities (spatial, epistemic, etc.) have been arising due to the requirements of Theoretical Computer Science and Artificial Intelligence. Some examples of this are: spatio-temporal reasoning [18], parallel processes [23], security protocols [13], verification of multi-agent systems [16], distributed systems [17], design of architectures of reasoning with mental attitudes [22], cooperation and planning [26], semantics of messages [20], etc. Our approach is adequate to model interactions between processes with clocks that can be either synchronized or not.

This paper is organized as follows: In Section 2, the language and semantics of the temporal×modal logic $\mathcal{L}^{\mathcal{F}}_{(T\times W)}$-$\mathfrak{I}$ (from now, for simplicity, $\mathcal{L}^{\mathfrak{I}}$), introduced in [10, 12] is sketched and the definability of properties of partial functions such as being non-total, constant, injective, surjective and increasing is given. Moreover, axiomatic systems for an indexed language dealing with these properties of functions are introduced. In Section 3, we prove the soundness and completeness of the system for injective functions and Section 4 is devoted to prove the incompleteness of the system which defines surjectivity. Finally, some conclusions and prospectives of future work are given in Section 5, while in Section 6 an Appendix with the more important details about the proof of the completeness of the system for increasing functions is given.

2 The logics $\mathcal{L}^{\mathfrak{I}}$

Let us consider the temporal×modal logic $\mathcal{L}^{\mathfrak{I}} = (L^{\mathfrak{I}}, \mathcal{M}^{\mathfrak{I}})$, where $\mathfrak{I}$ is a denumerable nonempty set of indexes, $L^{\mathfrak{I}}$ denotes the language and $\mathcal{M}^{\mathfrak{I}}$ a set of models for $L^{\mathfrak{I}}$. Given a set of indexes $\mathfrak{I}$, the alphabet of the language $L^{\mathfrak{I}}$ consists of a denumerable set $\mathcal{V}$ of propositional variables (atoms); the logic constants $\top$ ("truth") and $\bot$ ("falsity"), and the Boolean connectives $\neg$, $\wedge$, $\vee$, and $\rightarrow$; the Priorean temporal connectives F ("at some future time") and P ("at some past time") and a family of unary modal connectives of the form $<i>$, for $i \in \mathfrak{I}$ ("A is true in flow i, at the image of the reference instant"). Well-formed formulas (wffs) are generated by the construction rules of classical propositional logic, adding the following rule: If A is a wff, then FA, PA and $<i>A$ (with $i \in \mathfrak{I}$) are wffs. Now, for each $i \in \mathfrak{I}$, we can also introduce the connectives G, H and $[\,i\,]$ defined as usual.

Definition 2.1 An ind**-functional frame** for $L^{\mathfrak{I}}$ is a tuple $\Sigma^{\mathfrak{I}} = (W, \mathcal{T}, \mathcal{F})$ such that:

1. W is a nonempty set of labels (for a set of temporal flows).
2. $\mathcal{T} = \{(T_w, <_w) \mid w \in W\}$ is a non empty set such that:
   - $T_w \neq \varnothing$ and $<_w$ is a strict linear order on T_w, for all $w \in W$;
   - if $w \neq w'$, then $T_w \cap T_{w'} = \varnothing$, for all $w, w' \in W$.
3. $\mathcal{F}$ is a set of non-empty functions, called **accessibility functions**, such that:
   (a) each function in $\mathcal{F}$ is a partial function from T_w to $T_{w'}$, for some $w \in W$ and some $w' \in W \cap \mathfrak{I}$;

(b) for an arbitrary pair $(w, w') \in W \times (W \cap \mathfrak{I})$, there is (in $\mathcal{F}$) at most one accessibility function from T_w to $T_{w'}$, denoted by $\overset{w\ w'}{\longrightarrow}$.

Definition 2.2 Let $\Sigma^{\mathfrak{I}} = (W, \mathcal{T}, \mathcal{F})$ be an ind-functional frame. The elements of the following disjoint union $\bigcup_{w \in W} T_w$ are called **coordinates** and we will refer to $\mathcal{C}oord_{\Sigma}$ as the set of coordinates of $\Sigma^{\mathfrak{I}}$.

We now introduce some notation and terminology:

- If $(A, \leq)$ is a nonempty linearly ordered set and $a \in A$:
  $[a, \rightarrow) = \{a' \in A \mid a \leq a'\}$; $(a, \rightarrow) = \{a' \in A \mid a < a'\}$. Similarly we define $(\leftarrow, a]$ and $(\leftarrow, a)$.
- If $f : A \longrightarrow B$ is a nonempty partial function from A to B, $Dom(f)$ represents the domain of f and $X \subseteq A$, we define, as usual, $f(X) = \{f(x) \mid x \in X \cap Dom(f)\}$. Particularly, if $a \notin Dom(f)$, then $f(\{a\}) = \varnothing$.
- If $(A, \leq)$ and $(B, \leq)$ are nonempty linearly ordered sets, $f : A \longrightarrow B$ a nonempty partial function and $f(\{a\}) = \varnothing$, then: $(\leftarrow, f(\{a\})) = (\leftarrow, f(\{a\})] = (f(\{a\}), \rightarrow) = [f(\{a\}), \rightarrow) = \varnothing$.

Definition 2.3 An **ind-functional model** for $L^{\mathfrak{I}}$ is a tuple (Σ, h), where $\Sigma = (W, \mathcal{T}, \mathcal{F})$ is an ind-functional frame and h is a function, called functional interpretation, assigning to each atom $p \in \mathcal{V}$ a subset of $\mathcal{C}oord_{\Sigma}$. The functional interpretation h is recursively extended to a function (still denoted by h) defined for all the formulas of $L^{\mathfrak{I}}$, by interpreting the constants and the Boolean in a standard way and satisfying the following conditions:

- $h(FA) = \{t_w \in \mathcal{C}oord_{\Sigma^{\mathfrak{I}}} \mid (t_w, \rightarrow) \cap h(A) \neq \varnothing\}$; $\quad h(PA) = \{t_w \in \mathcal{C}oord_{\Sigma^{\mathfrak{I}}} \mid (\leftarrow, t_w) \cap h(A) \neq \varnothing)\}$;
- $h(<i>A) = \{t_w \in \mathcal{C}oord_{\Sigma^{\mathfrak{I}}} \mid \overset{w\ i}{\longrightarrow} \in \mathcal{F}\ and\ \overset{w\ i}{\longrightarrow}(\{t_w\}) \cap h(A) \neq \varnothing\}$.

The concepts of *satisfiability*, *validity* and other semantical notions are introduced as usual.

2.1 Definability of properties of functions in $\mathcal{L}^{\mathfrak{I}}$

In this section, we present a sketch of definability of the properties of functions discussed in this work.

Definition 2.4 Let $\mathbb{J}$ be a class of ind-functional frames and $\mathbb{K} \subseteq \mathbb{J}$. We say that $\mathbb{K}$ is $\mathcal{L}^{\mathfrak{I}}$ - **definable** in $\mathbb{J}$ by a set of formulas Γ if for every ind-functional frame $\Sigma^{\mathfrak{I}} \in \mathbb{J}$ we have that $\Sigma^{\mathfrak{I}} \in \mathbb{K}$ iff every formula of Γ is valid in $\Sigma^{\mathfrak{I}}$. If $\mathbb{J}$ is the class of all ind-functional frames, we say that $\mathbb{K}$ is $\mathcal{L}^{\mathfrak{I}}$ **-definable** by Γ.

We introduce now the following sets of formulas and notation:

α	S^{α}
non total	$(Non\text{-}Tot)\text{-}ind$: $\{P[i]\bot \vee [i]\bot \vee F[i]\bot \mid i \in \mathfrak{I}\}$
constant	$(Cons)\text{-}ind$: $\{<i>A \rightarrow G[i]A \mid i \in \mathfrak{I}\}$
injective	$(Inj)\text{-}ind$: $\{<i>(HA \wedge GA) \rightarrow (H[i]A \wedge G[i]A) \mid i \in \mathfrak{I}\}$
surjective	$(Surj)\text{-}ind$: $\{(H[i]A \wedge G[i]A) \rightarrow [i](HA \wedge GA) \mid i \in \mathfrak{I}\}$
increasing	$(Inc)\text{-}ind$: $\{<i>(A \wedge GA) \rightarrow G[i]A \mid i \in \mathfrak{I}\}$
decreasing	$(Dec)\text{-}ind$: $\{<i>(A \wedge HA) \rightarrow G[i]A \mid i \in \mathfrak{I}\}$
strictly increasing	$(Str\text{-}Inc)\text{-}ind$: $\{<i>GA \rightarrow G[i]A \mid i \in \mathfrak{I}\}$
stricty decreasing	$(Str\text{-}Dec)\text{-}ind$: $\{<i>HA \rightarrow G[i]A \mid i \in \mathfrak{I}\}$

We can now give the following result:

Theorem 2.5 *The following class of ind-functional frames is $\mathcal{L}^{\mathfrak{I}}$ -definable by the set of formulas S^{α}.*

$$\{(W, \mathcal{T}, \mathcal{F}) \mid \mathcal{F}\ is\ a\ class\ of\ \alpha\text{-}functions\}$$

2.2 Functional systems for partial functions

In this section, we introduce the minimal system for partial functions and then several extensions of it dealing with non-total, constant, injective, surjective and increasing functions. The axioms of system $\mathcal{S}^{\mathfrak{I}}$ -Par are:

1. Those of the minimal system of propositional linear temporal logic $\mathcal{K}_l$.
2. For each $i \in \mathfrak{I}$, $[\,i\,](A \to B) \to ([\,i\,]A \to [\,i\,]B)$
3. For each $i \in \mathfrak{I}$, $<i>A \to [\,i\,]A$. **(Axiom of Functionality)**
4. $(\lambda <i>A \wedge \lambda' <i>B) \to \lambda <i> (A \wedge (PB \vee B \vee FB))$

   (Axiom of Confluence)

   where: $\begin{cases} \lambda = \gamma_1 <j_1> \gamma_2 \dots <j_n> \gamma_{n+1}, & n \in \mathbb{N},\ \gamma_i \in \{F, P, \epsilon\},\ j_i \in \mathfrak{I} \\ \lambda' = \gamma'_1 <k_1> \gamma'_2 \dots <k_m> \gamma'_{m+1}, & m \in \mathbb{N},\ \gamma'_i \in \{F, P, \epsilon\},\ k_i \in \mathfrak{I} \end{cases}$

   and ϵ denotes the empty chain.

The **inference rules** of $\mathcal{S}^{\mathfrak{I}}$-$Par$ are the following:

$$(MP)\ \ A,\, A \to B \vdash B; \quad (RG)\ \ A \vdash GA; \quad (RH)\ \ A \vdash HA; \quad (N_i)\ \ A \vdash [\,i\,]A \quad \text{(for each } i \in \mathfrak{I}\text{)}.$$

Remark 2.6 Axiom (3) establishes functionality ensuring the uniqueness of the image. Axiom (4) assures that we can go to the same temporal flow from distinct paths.

The syntactical concepts of *proof*, *theorem*, etc., are defined as usual.

In order to deal with every property α considered previously, we define the extensions of the system $\mathcal{S}^{\mathfrak{I}}$-$Par$ by adding the set of formulas S^{α}. For example, the system $\mathcal{S}^{\mathfrak{I}}$-$Inj$ will be the extension of $\mathcal{S}^{\mathfrak{I}}$-$Par$ by adding the set of formulas (Inj)-ind and similarly for the rest of properties. All of the previous systems are complete except $\mathcal{S}^{\mathfrak{I}}$-$Surj$. For the sake of simplicity, we will focus our attention on the proof of completeness of the system $\mathcal{S}^{\mathfrak{I}}$-$Inj$, because the corresponding system for total injective functions with non indexed connectives was proved to be incomplete in [9]. Similar proofs can be given for $\mathcal{S}^{\mathfrak{I}}$-$Non$-$Tot$, $\mathcal{S}^{\mathfrak{I}}$-$Cons$, $\mathcal{S}^{\mathfrak{I}}$-$Str$-$Inc$ and $\mathcal{S}^{\mathfrak{I}}$-$Str$-$Dec$. Moreover, due to its peculiarity, we present an Appendix at the end of this paper with the most important details about the proof of completeness of the system $\mathcal{S}^{\mathfrak{I}}$-$Inc$. Similar ideas as given in the Appendix can be applied to the system $\mathcal{S}^{\mathfrak{I}}$-$Dec$.

3 Soundness and Completeness of $\mathcal{S}^{\mathfrak{I}}$-$Inj$

The proof of soundness is straightforward. Hence, we will focus our attention on completeness. Specifically, we will provide a proof of completeness by using the *step-by-step* method (see, for example, [7] and [8] for modal and temporal systems; and [10] and [9] for functional systems). In this section, we consider the system $\mathcal{S}^{\mathfrak{I}}$-$Inj$, however easy modifications would lead us to obtain the completeness of the system for total injective functions $\mathcal{S}^{\mathfrak{I}}$-$Tot$-$Inj$ [1]. As said above, a similar system was proved to be incomplete without using indexes. In this section, we introduce firstly some general definitions and theorems in this type of proof, then we give the specific results for our system for injective functions. As we will see, these completeness proofs are not trivial extensions of the given one for partial functions, because the technical problems which arise from the use of any different property need special attention.

3.1 Maximally consistent sets in functional systems

Given any system considered in this paper, we will denote by $\mathcal{MC}$ the family of maximally consistent sets (from now on, mc-sets, denoted by $\Gamma_1, \Gamma_2 \dots$) of any system considered in this paper. Some familiarity with the basic properties of mc-sets is assumed. In this section, we introduce some relations in $\mathcal{MC}$ which are very useful for the completeness proofs. To begin with, we consider the following definition that gives the intuitive idea of connecting coordinates in different temporal flows. It will be useful later in the proof of completeness.

[1] This system is defined by extending $\mathcal{S}^{\mathfrak{I}}$-$Par$ with $\{<i> (HA \wedge GA) \to (H <i> A \ \wedge\ G <i> A) \mid i \in \mathfrak{I}\}$.

Definition 3.1 Let $\Sigma^{\Im} = (W, \mathcal{T}, \mathcal{F})$ be an ind-functional frame and $t_w, t'_{w'} \in \mathcal{C}oord_{\Sigma^{\Im}}$.

1. $t_w \prec_i^{\sim T} t_{w'}$ iff either $w' = w$ or $w' = i \in \mathcal{I} \cap W$ and there are some $t^1_w \in T_w$ and $t^2_{w'} \in T_{w'}$ such that $\xrightarrow{w\ i} (t^1_w) = t^2_{w'}$.

2. $t_w \searrow t_{w'}$ iff either $t_w \prec_i^{\sim T} t'_{w'}$ or there are $n > 1,\ i_1, \ldots, i_n \in \Im$ and $t^1_{w_1}, \ldots, t^{n-1}_{w_{n-1}} \in \mathcal{C}oord_{\Sigma^{\Im}}$ such that $t_w \prec_{i_1}^{\sim T} t^1_{w_1} \prec_{i_2}^{\sim T} t^2_{w_2} \prec_{i_3} \ldots \prec_{i_{n-1}}^{\sim T} t^{n-1}_{w_{n-1}} \prec_{i_n}^{\sim T} t_{w'}$.

The previous definition has its analogous for mc-sets. Since their role is identical in both cases, they will be denoted by the same symbols, namely, $\prec_i^{\sim T}$ and $\searrow$. The context will always make it clear which one we refer to.

Definition 3.2 Let $i \in \Im$, we have the following definitions:

1. $\Gamma_1 \prec_T \Gamma_2$ iff $\{A \mid GA \in \Gamma_1\} \subseteq \Gamma_2$. Moreover, $\Gamma_1 \sim_T \Gamma_2$ iff $\big(\Gamma_1 \prec_T \Gamma_2,\ \text{or}\ \Gamma_2 \prec_T \Gamma_1,\ \text{or}\ \ \Gamma_1 = \Gamma_2\big)$.

2. $\Gamma_1 \prec_i \Gamma_2$ iff $\varnothing \neq \{A \mid <i> A \in \Gamma_1\} \subseteq \Gamma_2$.

3. $\Gamma_1 \prec_i^{\sim T} \Gamma_2$ iff either $\Gamma_1 \sim_T \Gamma_2$ or there are $\Gamma_3, \Gamma_4 \in \mathcal{MC}$ such that $\Gamma_1 \sim_T \Gamma_3, \Gamma_3 \prec_i \Gamma_4$ and $\Gamma_4 \sim_T \Gamma_2$.

4. $\Gamma_1 \searrow \Gamma_2$ iff either for some $i \in \Im$, $\Gamma_1 \prec_i^{\sim T} \Gamma_2$ or for some $n \geq 1$, there are $i_1, \ldots, i_n \in \Im$ and $\Omega_1, \ldots, \Omega_n \in \mathcal{MC}$ such that $\Gamma_1 \prec_i^{\sim T} \Omega_1 \prec_{i_1}^{\sim T} \Omega_2 \prec_{i_2}^{\sim T} \ldots \prec_{i_n}^{\sim T} \Gamma_2$.

We present now a result which holds for all functional systems under consideration.

Proposition 3.3 *The following properties are satisfied:*

1. *Any consistent set of formulas can be extended to an mc-set (Lindenbaum's Lemma).*

2. *If $FA \in \Gamma_1$, there is $\Gamma_2 \in \mathcal{MC}$ such that $\Gamma_1 \prec_T \Gamma_2$ and $A \in \Gamma_2$. Similarly for PA.*

3. *Let $i \in \Im$, if $<i>A \in \Gamma_1$ there is $\Gamma_2 \in \mathcal{MC}$ such that $\Gamma_1 \prec_i \Gamma_2$ and $A \in \Gamma_2$.*

4. *If $\Gamma_1 \prec_T \Gamma_2$ and $\Gamma_2 \prec_T \Gamma_3$, then $\Gamma_1 \prec_T \Gamma_3$.*

5. *If $\Gamma_1 \prec_T \Gamma_2$ and $\Gamma_1 \prec_T \Gamma_3$, then $\Gamma_2 \sim_T \Gamma_3$. Similarly, if $\Gamma_2 \prec_T \Gamma_1$ and $\Gamma_3 \prec_T \Gamma_1$, then $\Gamma_2 \sim_T \Gamma_3$.*

6. *$\Gamma_1 \sim_T \Gamma_2$ iff there is $\gamma \in \{F, P, \epsilon\}$ such that $\{\gamma A \mid A \in \Gamma_2\} \subseteq \Gamma_1$.*

7. *For every $i \in \Im$, we have that $\Gamma_1 \prec_i \Gamma_2$ iff $\{A \mid [i]A \in \Gamma_1\} \subseteq \Gamma_2$ iff $\{<i> A \mid A \in \Gamma_2\} \subseteq \Gamma_1$.*

8. *If $i \in \Im$, then $\Gamma_1 \prec_i^{\sim T} \Gamma_2$ iff either there exists $\gamma \in \{F, P, \epsilon\}$ such that $\{\gamma A \mid A \in \Gamma_2\} \subseteq \Gamma_1$ or there are $\gamma_1, \gamma_2 \in \{F, P, \epsilon\}$ such that $\{\gamma_1 <i> \gamma_2 A \mid A \in \Gamma_2\} \subseteq \Gamma_1$.*

Theorem 3.4 *$\Gamma_1 \searrow \Gamma_2$ iff one of the following conditions is satisfied:*

(a) *there exists $\gamma \in \{F, P, \epsilon\}$ such that $\{\gamma A \mid A \in \Gamma_2\} \subseteq \Gamma_1$;*

(b) *there are $\gamma_1, \ldots \gamma_{n+1} \in \{F, P, \epsilon\}$ and $i_1, \ldots, i_n \in \Im$, with $n \geq 1$, such that:*

$$\{\gamma_1 <i_1> \gamma_2 \ldots <i_n> \gamma_{n+1} A \mid A \in \Gamma_2\} \subseteq \Gamma_1$$

The following result, called *Diamond Theorem*, was proved in [10]. It is recalled here because it will be relevant for the rest of the paper.

Theorem 3.5 *Let $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{MC}$ such that:*

1. *$\Gamma_1 \searrow \Gamma_2$ and $\Gamma_1 \searrow \Gamma_3$.*

2. *There are $i \in \Im$ and $\Omega_1 \in \mathcal{MC}$ such that* $\left\{ \begin{array}{l} 2.1)\ \ \Gamma_2 \prec_i \Omega_1 \\ 2.2)\ \{A \mid <i>A \in \Gamma_3\} \neq \varnothing \end{array} \right.$

Then, there exists $\Gamma_4 \in \mathcal{MC}$ *such that* $\Gamma_2 \searrow \Gamma_4$ *and* $\Gamma_3 \searrow \Gamma_4$*. Specifically, there exists* $\Omega_2 \in \mathcal{MC}$ *such that* $\Gamma_3 \prec_i \Omega_2$ *and* $\Omega_2 \sim_T \Omega_1$*.*

As usual in the *step-by-step* completeness method, we introduce a function (called *trace*) that associates elements of $\mathcal{MC}$ to coordinates in an ind-functional frame which will allow us to construct the desired model.

Definition 3.6 Let $\Sigma^{\mathfrak{I}} = (W, \mathcal{T}, \mathcal{F})$ be an ind-functional frame for a language $L^{\mathfrak{I}}$. A **trace** of $\Sigma^{\mathfrak{I}}$ is a function $\Phi_{\Sigma^{\mathfrak{I}}} : \mathcal{C}oord_{\Sigma^{\mathfrak{I}}} \longrightarrow 2^{L^{\mathfrak{I}}}$ such that, for all $t_w \in \mathcal{C}oord_{\Sigma^{\mathfrak{I}}}$, the set $\Phi_{\Sigma^{\mathfrak{I}}}(t_w)$ is an mc-set.

We introduce now the properties of the trace function in a standard way.

Definition 3.7 Let $\Phi_{\Sigma^{\mathfrak{I}}}$ be a trace of an ind-functional frame $\Sigma^{\mathfrak{I}}$. Then $\Phi_{\Sigma^{\mathfrak{I}}}$ is called:
temporally coherent if, for all $t_w, t'_w \in \mathcal{C}oord_{\Sigma^{\mathfrak{I}}}$: if $t'_w \in (t_w, \rightarrow)$, then $\Phi_{\Sigma^{\mathfrak{I}}}(t_w) \prec_T \Phi_{\Sigma^{\mathfrak{I}}}(t'_w)$;
ind-modally coherent if, for all $t_w, t_i \in \mathcal{C}oord_{\Sigma^{\mathfrak{I}}}$ with $i \in W \cap \mathfrak{I}$:

$$\text{if } t_i = \stackrel{w\,i}{\longrightarrow}(t_w), \text{ then } \Phi_{\Sigma^{\mathfrak{I}}}(t_w) \prec_i \Phi_{\Sigma^{\mathfrak{I}}}(t_i);$$

coherent if it is temporally coherent and ind-modally coherent.
prophetic if it is temporally coherent and, moreover, for all $A \in L^{\mathfrak{I}}$ and all coordinate $t_w \in \mathcal{C}oord_{\Sigma^{\mathfrak{I}}}$:
(1) if $FA \in \Phi_{\Sigma^{\mathfrak{I}}}(t_w)$, there exists $t'_w \in (t_w, \rightarrow)$ such that $A \in \Phi_{\Sigma^{\mathfrak{I}}}(t'_w)$;
historic if it is temporally coherent and, moreover, for all $A \in L^{\mathfrak{I}}$ and all coordinate $t_w \in \mathcal{C}oord_{\Sigma^{\mathfrak{I}}}$:
(2) if $PA \in \Phi_{\Sigma^{\mathfrak{I}}}(t_w)$, there exists $t'_w \in (\leftarrow, t_w)$ such that $A \in \Phi_{\Sigma^{\mathfrak{I}}}(t'_w)$;
ind-possibilistic if it is ind-modally coherent and, moreover, for all $A \in L^{\mathfrak{I}}$, all coordinate $t_w \in \mathcal{C}oord_{\Sigma^{\mathfrak{I}}}$ and all $i \in W \cap \mathfrak{I}$:
(3) if $<\!i\!>A \in \Phi_{\Sigma^{\mathfrak{I}}}(t_w)$, there exists $t_i = \stackrel{w\,i}{\longrightarrow}(t_w)$ such that $A \in \Phi_{\Sigma^{\mathfrak{I}}}(t_i)$.

Definition 3.8 We say that a conditional sentence of the form (1) (resp., (2) or (3)) used in Definition 3.7 is called a **prophetic** (resp., **historic** or **ind-possibilistic**) **conditional for** $\Phi_{\Sigma^{\mathfrak{I}}}$. In general, we will use the expression **conditional for** $\Phi_{\Sigma^{\mathfrak{I}}}$ to mean that it is a prophetic, historic or possibilistic conditional for $\Phi_{\Sigma^{\mathfrak{I}}}$. An ind-functional trace, $\Phi_{\Sigma^{\mathfrak{I}}}$, is called **full** if it is prophetic, historic, and ind-possibilistic.

The following Proposition links the properties of mc-sets to the coherence of traces. It can be easily proved by combining Definition 3.1 and Definition 3.7.

Proposition 3.9 *Let* $\Phi_{\Sigma^{\mathfrak{I}}}$ *be a coherent trace, for all* $t_w, t_{w'} \in \mathcal{C}oord_{\Sigma^{\mathfrak{I}}}$*, we have that* $t_w \searrow t'_{w'}$ *implies* $\Phi_{\Sigma^{\mathfrak{I}}}(t_w) \searrow \Phi_{\Sigma^{\mathfrak{I}}}(t'_{w'})$*.*

In this method for proving completeness, it is necessary to ensure the satisfiability of the different classes of existential formulas (i.e., $<\!i\!>A$, FA and PA) which may appear in the mc-sets associated to each coordinate. In order to do so, we present various types of conditionals, in the style of Burgess [8] for temporal logic.

Definition 3.10 Let $\Phi_{\Sigma^{\mathfrak{I}}}$ be a trace of an ind-functional frame $\Sigma^{\mathfrak{I}} = (W, \mathcal{T}, \mathcal{F})$.

- Consider a prophetic conditional: If $FA \in \Phi_{\Sigma^{\mathfrak{I}}}(t_w)$, there is $t'_w \in (t_w, \rightarrow)$ such that $A \in \Phi_{\Sigma^{\mathfrak{I}}}(t'_w)$.

  We say that it is *active*, if $FA \in \Phi_{\Sigma^{\mathfrak{I}}}(t_w)$, but there is no $t'_w \in (t_w, \rightarrow)$ such that $A \in \Phi_{\Sigma^{\mathfrak{I}}}(t'_w)$. On the other hand, we say that it is *exhausted* if there exists a coordinate $t'_w \in (t_w, \rightarrow)$ such that $A \in \Phi_{\Sigma^{\mathfrak{I}}}(t'_w)$.

  The case for a historic conditional is defined in a similar way.

- Given an ind-possibilistic conditional: If $<\!i\!>A \in \Phi_{\Sigma^{\mathfrak{I}}}(t_w)$, there exists $t_i = \stackrel{w\,i}{\longrightarrow}(t_w)$ such that $A \in \Phi_{\Sigma^{\mathfrak{I}}}(t_i)$, we say that it is *active* if $<\!i\!>A \in \Phi_{\Sigma^{\mathfrak{I}}}(t_w)$, but there is no $t_i = \stackrel{w\,i}{\longrightarrow}(t_w)$ such that $A \in \Phi_{\Sigma^{\mathfrak{I}}}(t_i)$. On the other hand, the conditional is *exhausted* if there exists $t_i = \stackrel{w\,i}{\longrightarrow}(t_w)$ such that $A \in \Phi_{\Sigma^{\mathfrak{I}}}(t_i)$.

The proof of the following result, called *Trace Lemma*, is straightforward by induction on complexity of any formula A.

Lemma 3.11 *Let* $\Phi_{\Sigma^{\mathfrak{I}}}$ *be a full trace of an* ind*-functional frame* $\Sigma^{\mathfrak{I}}$*. Let* h *be an* ind*-functional interpretation assigning to each propositional variable,* p*, the set* $h(p) = \{t_w \in \mathcal{C}oord_{\Sigma^{\mathfrak{I}}} \mid p \in \Phi_{\Sigma^{\mathfrak{I}}}(t_w)\}$*. Then, for any formula A, we have* $h(A) = \{t_w \in \mathcal{C}oord_{\Sigma^{\mathfrak{I}}} \mid A \in \Phi_{\Sigma^{\mathfrak{I}}}(t_w)\}$*.*

In order to prove the completeness of our systems, we need two special classes of ind-functional frames. The first one corresponds to the intuitive idea that any indexed flow contains the range of some function, while the second one means that there exists a temporal flow which is connected to the other ones.

Definition 3.12 An ind-functional frame $\Sigma^{\Im} = (W, \mathcal{T}, \mathcal{F})$ is **admissible** if it satisfies: for all $i \in W \cap \Im$, we have $\xrightarrow{w\ i} \in \mathcal{F}$ for some $w \in W$. On the other hand, $\Sigma^{\Im}$ is **rooted** iff there exists some T_w in $\Sigma^{\Im}$ such that, for all $T_{w'}$ in $\Sigma^{\Im}$, $t_w \in T_w$ and $t_{w'} \in T_{w'}$, it holds $t_w \searrow t_{w'}$.

3.2 Specific results for $\mathcal{S}^{\Im}$-Inj

First of all, we give a family of theorems in this system.

Lemma 3.13 *Every formula of the following set is a theorem of* $\mathcal{S}^{\Im}$-Inj*:*

$$\{(<i>\top \wedge (P<i>A \vee F<i>A)) \rightarrow <i>(PA \vee FA) \mid i \in \mathcal{I}\}$$

We now use the previous Lemma to prove the following two results which give *a sort of injectivity* for mc-sets.

Proposition 3.14 *If* $\Gamma_1 \prec_T \Gamma_2$*,* $\Gamma_1 \prec_i \Gamma_3$ *and* $\Gamma_2 \prec_i \Gamma_4$*, then either* $\Gamma_3 \prec_T \Gamma_4$ *or* $\Gamma_4 \prec_T \Gamma_3$.

Proof. Let us suppose $\Gamma_1 \prec_T \Gamma_2$, $\Gamma_1 \prec_i \Gamma_3$ and $\Gamma_2 \prec_i \Gamma_4$. If $\Gamma_3 \not\prec_T \Gamma_4$ and $\Gamma_4 \not\prec_T \Gamma_3$, then there are formulas A and B such that $GA \wedge \neg B \in \Gamma_3$ and $\neg A \wedge GB \in \Gamma_4$. Let $\alpha = \neg A \wedge GB$; then, given that $\Gamma_1 \prec_T \Gamma_2$ and $\Gamma_2 \prec_i \Gamma_4$, we have $F<i>\alpha \in \Gamma_1$. On the other hand, since $\Gamma_1 \prec_i \Gamma_3$ we obtain $<i>\top \in \Gamma_1$. Now by using Lemma 3.13, we obtain $<i>(P\alpha \vee F\alpha) \in \Gamma_1$, and again by $\Gamma_1 \prec_i \Gamma_3$, we have $P\alpha \vee F\alpha \in \Gamma_3$, which easily leads us to a contradiction. □

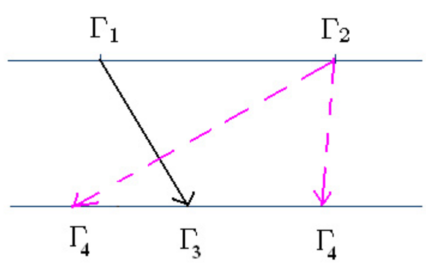
The following result can be proved easily, by using the previous one and Proposition 3.3(2).

Proposition 3.15 *Consider* $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{MC}$*, then we have:*

1. *If* $\Gamma_1 \prec_T \Gamma_2$*,* $<i>\top \in \Gamma_2$ *and* $\Gamma_1 \prec_i \Gamma_3$*, then there exists* $\Gamma_4 \in \mathcal{MC}$ *such that* $\Gamma_2 \prec_i \Gamma_4$ *and also either* $\Gamma_3 \prec_T \Gamma_4$ *or* $\Gamma_4 \prec_T \Gamma_3$.

2. *If* $\Gamma_1 \prec_T \Gamma_2$*,* $<i>\top \in \Gamma_1$ *and* $\Gamma_2 \prec_i \Gamma_3$*, then there exists* $\Gamma_4 \in \mathcal{MC}$ *such that* $\Gamma_1 \prec_i \Gamma_4$ *and also either* $\Gamma_3 \prec_T \Gamma_4$ *or* $\Gamma_4 \prec_T \Gamma_3$.

The following picture gives the intuitive idea about the previous result. Our purpose is to preserve the injectivity:



The previous Propositions will be the key to prove the following result, called *Exhausting Lemma*, which is fundamental to prove the completeness of our system. In order to ensure a better reading, we give the proof of the Exhausting Lemma after the main Theorem of Completeness.

Theorem 3.16 *Let* $\Phi_{\Sigma_1^{\Im}}$ *be a coherent trace of a finite, injective* [2]*, admissible and rooted* ind*-functional frame* $\Sigma_1^{\Im}$*, and let* (α) *be a conditional for* $\Phi_{\Sigma_1^{\Im}}$ *which is active. Then there is a finite, injective, admissible and rooted* ind*-functional frame* $\Sigma_2^{\Im}$*, extension of* $\Sigma_1^{\Im}$*, and a coherent trace* $\Phi_{\Sigma_2^{\Im}}$ *of* $\Sigma_2^{\Im}$*, extension of* $\Phi_{\Sigma_1^{\Im}}$*, such that* (α) *is a conditional for* $\Phi_{\Sigma_2^{\Im}}$ *which is exhausted.*

[2] that is, every accessibility function is injective.

Theorem 3.17 *If a formula $A \in L^{\Im}$ is valid in the class of* ind*-functional frames:*

$$\{(W, \mathcal{T}, \mathcal{F}) \mid \mathcal{F} \text{ is a class of injective functions}\}$$

then A is a theorem of $\mathcal{S}^{\Im}$-Inj.

Proof. It suffices to show that given a consistent formula A, this formula is satisfiable. To accomplish this task, we shall construct a model $\mathcal{M} = (\Sigma^{\Im}, h)$ where $\Sigma^{\Im}$ is an injective ind-functional frame.

We begin with a finite ind-functional frame $\Sigma_0^{\Im} = (W_0, \mathcal{T}_0, \mathcal{F}_0)$, where:

$$W_0 = \{w_0\}, \text{ with } w_0 \notin \Im; \quad \mathcal{T}_0 = \{(\{t_{w_0}\}, \varnothing)\}; \quad \mathcal{F}_0 = \varnothing$$

The corresponding trace $\Phi_{\Sigma_0^{\Im}}$, defined as $\Phi_{\Sigma_0^{\Im}}(t_{w_0}) = \Gamma_0$, being Γ_0 an mc- set containing A (Lindenbaum's Lemma). It is straightforward that the an ind-functional $\Sigma_0^{\Im}$ is a finite, injective, admissible and rooted ind-functional frame and that its trace $\Phi_{\Sigma_0^{\Im}}$ is coherent. Now, we want to obtain a denumerable chain of finite, injective, admissible and rooted ind-functional frames $\Sigma_0^{\Im}, \Sigma_1^{\Im}, \dots, \Sigma_n^{\Im} \dots$ whose union is the ind-functional frame $\Sigma^{\Im}$, and a denumerable sequence of corresponding traces, $\Phi_{\Sigma_0^{\Im}}, \Phi_{\Sigma_1^{\Im}}, \dots, \Phi_{\Sigma_n^{\Im}}, \dots$, whose union is $\Phi_{\Sigma^{\Im}}$. For this goal, we provide an enumeration $A_0, A_1, \dots$ of all the existential formulas of $L^{\Im}$, i.e., of the form FB, PB, $<i>B$, in which every formula occurs infinitely many times. Once we have constructed the base case of both $\Sigma_0^{\Im}$ and $\Phi_{\Sigma_0^{\Im}}$, we proceed inductively as follows: assume that $\Sigma_n^{\Im} = (W_n, \mathcal{T}_n, \mathcal{F}_n)$ and $\Phi_{\Sigma_n^{\Im}}$ (with $n \geq 0$) are defined. If no conditionals are active, then we stop the construction of finite ind-functional frames, and $\Sigma^{\Im} = \Sigma_n^{\Im}$ and $\Phi_{\Sigma^{\Im}} = \Phi_{\Sigma_n^{\Im}}$. Otherwise, we have to define $\Sigma_{n+1}^{\Im}$ and $\Phi_{\Sigma_{n+1}^{\Im}}$. For this purpose, consider the finite set $\mathcal{C}oord_{\Sigma_n^{\Im}}$ and the existential formula A_n of the enumeration characterized above. Let $C \subseteq \mathcal{C}oord_{\Sigma_n^{\Im}}$ be the set of coordinates, c, such that "$A_n \in \Phi_{\Sigma_n^{\Im}}(c)$" is the antecedent of an active conditional. Then, we have two possible situations:

1. If $C = \varnothing$, then we establish $\Sigma_{n+1}^{\Im} = \Sigma_n^{\Im}$ and $\Phi_{\Sigma_{n+1}^{\Im}} = \Phi_{\Sigma_n^{\Im}}$, and continue the process considering the existential formula A_{n+1}.
2. If $C = \{c_1, \dots, c_m\}$, for each $c_i \in C$, we consider the active conditional $\mathfrak{C}_i$ whose antecedent is just $A_n \in \Phi_{\Sigma_n^{\Im}}(c_i)$. So, by Theorem 3.16 (Exhausting Lemma), we obtain a sequence $\Sigma_{n_1}^{\Im}, \dots, \Sigma_{n_m}^{\Im}$ of finite, injective, admissible and rooted ind-functional frames such that $\Sigma_n^{\Im} \subseteq \Sigma_{n_1}^{\Im} \subseteq \dots \subseteq \Sigma_{n_m}^{\Im}$, and a corresponding sequence of coherent traces $\Phi_{\Sigma_{n_1}^{\Im}}, \dots, \Phi_{\Sigma_{n_m}^{\Im}}$ such that $\Phi_{\Sigma_n^{\Im}} \subseteq \Phi_{\Sigma_{n_1}^{\Im}} \subseteq \dots \subseteq \Phi_{\Sigma_{n_m}^{\Im}}$, so that each conditional active $\mathfrak{C}_i$ (for $1 \leq i \leq m$) is exhausted after constructing $\Sigma_{n_i}^{\Im}$ and $\Phi_{\Sigma_{n_i}^{\Im}}$. Now, we set $\Sigma_{n+1}^{\Im} = \Sigma_{n_m}^{\Im}$ and $\Phi_{\Sigma_{n+1}^{\Im}} = \Phi_{\Sigma_{n_m}^{\Im}}$.

The above process, ensures that:

- $\Sigma^{\Im}$ is an injective ind-functional frame.
- $\Phi_{\Sigma^{\Im}}$ is coherent, because each trace of the corresponding frames in the sequence, that is, $\Phi_{\Sigma_0^{\Im}} \subseteq \Phi_{\Sigma_1^{\Im}} \subseteq \dots \subseteq \Phi_{\Sigma_n^{\Im}} \dots$, is guaranteed to be coherent. Moreover, $\Phi_{\Sigma^{\Im}}$ is prophetic, historic and ind-possibilistic and, therefore, it is also full. In effect, although a given trace $\Phi_{\Sigma_k^{\Im}}$ in that sequence is not guaranteed to be prophetic and historic and ind-possibilistic, by applying the Exhausting Lemma, any conditional which is active for $\Phi_{\Sigma_k^{\Im}}$ will become exhausted.

Now, we define the model $(\Sigma^{\Im}, h)$, where $h(p) = \{t_w \in \mathcal{C}oord_{\Sigma^{\Im}} \mid p \in \Phi_{\Sigma}^{\Im}(t_w)\}$, and the Trace Lemma ensures the satisfiability of the formula A under consideration. This completes the proof. □

Finally, let us prove the Exhausting Lemma.

Proof. Let $\Phi_{\Sigma_1^{\Im}}$ be a coherent trace of a finite, injective, admissible and rooted ind-functional frame, $\Sigma_1^{\Im} = (W_1, \mathcal{T}_1, \mathcal{F}_1)$, and let (α) be a conditional for $\Phi_{\Sigma_1^{\Im}}$ which is active. We want to construct a finite, injective, admissible and rooted ind-functional frame, $\Sigma_2^{\Im} = (W_2, \mathcal{T}_2, \mathcal{F}_2)$, extension of $\Sigma_1^{\Im}$, and also to define a coherent trace $\Phi_{\Sigma_2^{\Im}}$, extension of $\Phi_{\Sigma_1^{\Im}}$, in which the conditional (α) is exhausted. If (α) is either a prophetic or historic conditional, then such a construction is carried out by following the standard way in temporal logic (see [8]). So, let us consider only the case in which (α) is an ind-possibilistic conditional.

Assume $i \in \Im$ and let (α) be the following active ind-possibilistic conditional for $\Phi_{\Sigma^{\Im}}$: If $<i>A \in \Phi_{\Sigma_1^{\Im}}(t_w)$, there is $t_i = \stackrel{w\,i}{\longrightarrow} (t_w)$ such that $A \in \Phi_{\Sigma_1^{\Im}}(t_w)$. Thus, we have that $< i > A \in \Phi_{\Sigma_1^{\Im}}(t_w)$, but there is no $t_i = \stackrel{w\,i}{\longrightarrow} (t_w)$ such that $A \in \Phi_{\Sigma_1^{\Im}}(t_w)$. Now, we have to consider two cases: **(I)** $i \notin W_1$ and **(II)** $i \in W_1$.

(I) If $i \notin W_1$, then, by item 3 of Proposition 3.3, there exists an mc-set, Γ, such that $\Phi_{\Sigma_1^{\Im}}(t_w) \prec_i \Gamma$ and $A \in \Gamma$. Now, we need a new temporal flow labeled with i, T_i, which requires extending W_1 and, also, introducing a new coordinate t_i associated with Γ so that $t_i = \stackrel{w\,i}{\longrightarrow} (t_w)$. Thus, $\Sigma_2^{\Im} = (W_2, \mathcal{T}_2, \mathcal{F}_2)$, extension of $\Sigma_1^{\Im}$, and $\Phi_{\Sigma_2^{\Im}}$, extension of $\Phi_{\Sigma_1^{\Im}}$ are defined as follows:

- $W_2 = W_1 \cup \{i\}$;
- $\mathcal{T}_2 = \mathcal{T}_1 \cup \{(T_i, <_i)\}$, where $(T_i, <_i) = (\{t_i\}, \varnothing)$;
- $\mathcal{F}_2 = \mathcal{F}_1 \cup \{\stackrel{w\,i}{\longrightarrow}\}$, where $\stackrel{w\,i}{\longrightarrow} = \{(t_w, t_i)\}$;
- $\Phi_{\Sigma_2^{\Im}} = \Phi_{\Sigma_1^{\Im}} \cup \{(t_i, \Gamma)\}$.

The ind-functional frame $\Sigma_2^{\Im}$, as defined, is indeed a finite ind-functional frame. More specifically, since $(T_i, <_i)$ is linear, it is immediate that the linearity of the temporal flows is preserved and the introduction of $\stackrel{w\,i}{\longrightarrow}$, as defined, preserves condition (3) in Definition 2.1 about the set of functions. Moreover, since i is the unique label introduced, the properties of $\Sigma_2^{\Im}$ of being admissible and rooted are also preserved. Furthermore, $\Sigma_2^{\Im}$ is an injective ind-functional frame, since the only new flow is T_i and only t_w has an image in T_i. It is also immediate that $\Phi_{\Sigma_2^{\Im}}$ is coherent.

(II) If $i \in W_1$, we consider the following situations: (II.1) $\stackrel{w\,i}{\longrightarrow} \in \mathcal{F}_1$; (II.2) $\stackrel{w\,i}{\longrightarrow} \notin \mathcal{F}_1$

(II.1) In this case, let t_i be the minimum[3] of $\stackrel{w\,i}{\longrightarrow} (T_w)$ and t'_w a coordinate such that $t_i = \stackrel{w\,i}{\longrightarrow} (t'_w)$. Then, by coherence of $\Phi_{\Sigma_1^{\Im}}$, we have $\Phi_{\Sigma_1^{\Im}}(t'_w) \prec_i \Phi_{\Sigma_1^{\Im}}(t_i)$ and, as $t'_w \in (\leftarrow, t_w) \cup (t_w, \rightarrow)$, by coherence again, we have $\Phi_{\Sigma_1^{\Im}}(t_w) \prec_T \Phi_{\Sigma_1^{\Im}}(t'_w)$ or $\Phi_{\Sigma_1^{\Im}}(t'_w) \prec_T \Phi_{\Sigma_1^{\Im}}(t_w)$. Now, by Proposition 3.15, there exists an mc-set Γ such that $\Phi_{\Sigma_1^{\Im}}(t_w) \prec_i \Gamma$ and either $\Phi_{\Sigma_1^{\Im}}(t_i) \prec_T \Gamma$ or $\Gamma \prec_T \Phi_{\Sigma_1^{\Im}}(t_i)$. Assume $\Phi_{\Sigma_1^{\Im}}(t_i) \prec_T \Gamma$, the other possibility can be treated similarly. Then, if we consider the number s of successors of t_i, we have two posibilities: (II.1.a) $s = 0$; (II.1.b) $s > 0$.

(II.1.a) If $s = 0$, then a new coordinate t'_i, i.e., $t'_i \notin \mathcal{C}oord_{\Sigma_1^{\Im}}$, is introduced to be associated to Γ and, thus, we have to extend $\Sigma_1^{\Im}$ and $\Phi_{\Sigma_1^{\Im}}$ as follows:

- $W_2 = W_1$;
- $\mathcal{T}_2 = (\mathcal{T}_1 - \{(T_i, <_i)\}) \cup \{(T'_i, <'_i)\}$, where $T'_i = T_i \cup \{t'_i\}$ and $<'_i = <_i \cup \{(t_i, t'_i)\} \cup \{(t^*_i, t'_i) \mid t^*_i <_i t_i\}$;
- $\mathcal{F}_2 = (\mathcal{F}_1 - \{\stackrel{w\,i}{\longrightarrow}\}) \cup \{\stackrel{w\,i}{\longrightarrow}'\}$, where $\stackrel{w\,i}{\longrightarrow}' = \stackrel{w\,i}{\longrightarrow} \cup \{(t_w, t'_i)\}$; (*)
- $\Phi_{\Sigma_2^{\Im}} = \Phi_{\Sigma_1^{\Im}} \cup \{(t'_i, \Gamma)\}$. (**)

Notice that $\Sigma_2^{\Im}$, as defined, is an ind-functional frame. Also it is admissible and rooted, since so is $\Sigma_1^{\Im}$, and $W_2 = W_1$. For proving the injectivity of the functions in $\mathcal{F}_2$, there is only a new coordinate, namely $t'_i = \stackrel{w\,i}{\longrightarrow}' (t_w)$, being $\stackrel{w\,i}{\longrightarrow}'$ the extension of the injective function $\stackrel{w\,i}{\longrightarrow} \in \mathcal{F}_1$. On the other hand, if we take into account Proposition 3.3(4), the proof of the coherence of $\Phi_{\Sigma_2^{\Im}}$ is straightforward.

(II.1.b) If $s > 0$, let t^1_i be the immediate successor of t_i. Then, since we have, by hypothesis, that $\Phi_{\Sigma_1^{\Im}}(t_i) \prec_T \Gamma$ and, by coherence, that $\Phi_{\Sigma_1^{\Im}}(t_i) \prec_T \Phi_{\Sigma_1^{\Im}}(t^1_i)$, it follows from item 5 of Proposition 3.3 that some of the following cases hold: (A) $\Gamma = \Phi_{\Sigma_1^{\Im}}(t^1_i)$; (B) $\Gamma \prec_T \Phi_{\Sigma_1^{\Im}}(t^1_i)$; (C) $\Phi_{\Sigma_1^{\Im}}(t^1_i) \prec_T \Gamma$.

(A) If $\Gamma = \Phi_{\Sigma_1^{\Im}}(t^1_i)$ holds, there are two situations to be considered if injectivity is to be preserved:

[3] In fact, we could consider any coordinate. We have chosen the minimum for the sake of intelligibility.

$$(A_1) \quad t_i^1 \notin \overset{w\ i}{\longrightarrow}(T_w) \quad \text{and} \quad (A_2) \quad t_i^1 \in \overset{w\ i}{\longrightarrow}(T_w).$$

(A$_1$) In this case, we only have to extend the accessibility function $\overset{w\ i}{\longrightarrow}$ in order to ensure that the image of t_w is t_i^1.

It is straightforward that $\Sigma_2^{\Im}$ is a finite, admissible and rooted ind-functional frame and that $\Phi_{\Sigma_2^{\Im}}$ is coherent. On the other hand, injectivity is preserved because $\overset{w\ i}{\longrightarrow}$ is injective and $t_i^1 \notin \overset{w\ i}{\longrightarrow}(T_w)$.

(A$_2$) If $t_i^1 \in \overset{w\ i}{\longrightarrow}(T_w)$, then $t_i^1 = \overset{w\ i}{\longrightarrow}(t_w^1)$, for some $t_w^1 \in T_w$. Thus, by Proposition 3.14, applied to $\Phi_{\Sigma_1^{\Im}}(t_w)$, $\Phi_{\Sigma_1^{\Im}}(t_w^1)$, $\Phi_{\Sigma_1^{\Im}}(t_i^1)$ and Γ, we get again at least one of the two remaining possibilities considered, namely: (B) $\Gamma \prec_T \Phi_{\Sigma_1^{\Im}}(t_i^1)$; (C) $\Phi_{\Sigma_1^{\Im}}(t_i^1) \prec_T \Gamma$.

(B) If $\Gamma \prec_T \Phi_{\Sigma_1^{\Im}}(t_i^1)$, we have $\Phi_{\Sigma_1^{\Im}}(t_i) \prec_T \Gamma \prec_T \Phi_{\Sigma_1}(t_i^1)$ and we introduce a new coordinate, t_i', to be associated to Γ and located between t_i and t_i^1. $\mathcal{F}_2$ and $\Phi_{\Sigma_2^{\Im}}$ are defined as in case (II.1.a), i.e., as in (*) and (**), respectively.

It is easy to see that $\Sigma_2^{\Im}$ and $\Phi_{\Sigma_2^{\Im}}$ satisfy the required properties as in previous cases.

(C) If $\Phi_{\Sigma_1^{\Im}}(t_i^1) \prec_T \Gamma$, then we have to consider the immediate successor of t_i^1, if any. If it does not exist, then we reason, with respect to t_i^1, as in case (II.1.a). Otherwise, let t_i^2 be the immediate successor of t_i^1; then, the process for t_i^1 can be repeated for t_i^2.

By iterating this process, at most s times, we get the desired result.

Finally, the case (II.2) has to be considered:

(II.2) Let us suppose that $\overset{w\ i}{\longrightarrow}$ is not defined in $\mathcal{F}_1$. Since $\Sigma_1^{\Im}$ is admissible, there will be some temporal flow, $T_{w'}$, with $w' \neq w$ and $\overset{w'\ i}{\longrightarrow} \in \mathcal{F}_1$. Let t_i be the minimum of $\overset{w'\ i}{\longrightarrow}(T_{w'})$ and consider $t_{w'}$ such that $\overset{w'\ i}{\longrightarrow}(t_{w'}) = t_i$. Thus, $\Phi_{\Sigma_1^{\Im}}(t_{w'}) \prec_i \Phi_{\Sigma_1^{\Im}}(t_i)$. Now, by definition of $\Sigma_1^{\Im}$ again (as it is rooted), we have three subcases:

$$\begin{cases} \text{(II.2.a)} & \Phi_{\Sigma_1^{\Im}}(t_w) \searrow \Phi_{\Sigma_1^{\Im}}(t_{w'}); \\ \text{(II.2.b)} & \Phi_{\Sigma_1^{\Im}}(t_{w'}) \searrow \Phi_{\Sigma_1^{\Im}}(t_w); \\ \text{(II.2.c)} & \text{there exists a flow } T_{w''}, \text{ with } w'' \neq w \text{ and } w'' \neq w', \text{ and} \\ & \text{there exists } t_{w''} \in T_{w''} \text{ such that:} \\ & \quad \Phi_{\Sigma_1^{\Im}}(t_{w''}) \searrow \Phi_{\Sigma_1^{\Im}}(t_w) \text{ and } \Phi_{\Sigma_1^{\Im}}(t_{w''}) \searrow \Phi_{\Sigma_1^{\Im}}(t_{w'}). \end{cases}$$

(II.2.a): Given that $\Phi_{\Sigma_1^{\Im}}(t_w) \searrow \Phi_{\Sigma_1^{\Im}}(t_w)$, by the Diamond Theorem, there exists an *mc*-set Γ such that $\Phi_{\Sigma_1^{\Im}}(t_w) \prec_i \Gamma$ and $\Gamma \sim_T \Phi_{\Sigma_1^{\Im}}(t_i)$. Thus, we may have, once again, one of the following three situations:

$$\text{(II.2.a.1) } \Gamma = \Phi_{\Sigma_1^{\Im}}(t_i); \qquad \text{(II.2.a.2) } \Phi_{\Sigma_1^{\Im}}(t_i) \prec_T \Gamma; \qquad \text{(II.2.a.3) } \Gamma \prec_T \Phi_{\Sigma_1^{\Im}}(t_i).$$

Here, we can proceed as in case (II.1)[4] by considering the number of successors (or predecessors) of t_i. Thus, since in (II.2.a.1), (II.2.a.2), and (II.2.a.3) it is the case that $\overset{w\ i}{\longrightarrow} \notin \mathcal{F}_1$, in the extensions of $\mathcal{F}_1$ we will obtain that $\mathcal{F}_2 = \mathcal{F}_1 \cup \{\overset{w\ i}{\longrightarrow}\}$. More specifically, in subcase (II.2.a.1) we will obtain $\overset{w\ i}{\longrightarrow} = \{(t_w, t_i)\}$, whereas in subcases (II.2.a.2) and (II.2.a.3), we will have that $\overset{w\ i}{\longrightarrow}(t_w)$ is situated, respectively, on the right and on the left of t_i.

(II.2.b): Given that $\Phi_{\Sigma_1^{\Im}}(t_{w'}) \searrow \Phi_{\Sigma_1^{\Im}}(t_{w'})$, the Diamond Theorem ensures that there exists an *mc*-set Γ such that $\Phi_{\Sigma_1^{\Im}}(t_w) \prec_i \Gamma$ and $\Gamma \sim_T \Phi_{\Sigma_1^{\Im}}(t_i)$. Thus, as in (II.2.a), we have three subcases which can be discussed by applying the same reasoning.

(II.2.c): If there exists a flow $T_{w''}$, with $w'' \neq w$ and $w'' \neq w'$, and there exists $t_{w''} \in T_{w''}$ such that $\Phi_{\Sigma_1^{\Im}}(t_{w''}) \searrow \Phi_{\Sigma_1^{\Im}}(t_w)$ and $\Phi_{\Sigma_1^{\Im}}(t_{w''}) \searrow \Phi_{\Sigma_1^{\Im}}(t_{w'})$, once again by Diamond Theorem, there exists an *mc*-set Γ such that $\Phi_{\Sigma_1^{\Im}}(t_w) \prec_i \Gamma$ and $\Gamma \sim_T \Phi_{\Sigma_1^{\Im}}(t_i)$ and the reasoning applied in (II.2.a) is also applicable to the present case. This ends the proof of the lemma.

□

We study in the following section the only remaining case of incompleteness in our approach.

[4] However, if $\Gamma = \Phi_{\Sigma_1^{\Im}}(t_i)$ and injectivity has to be preserved, it is not necessary, unlike in (II.1.b), to consider subcases (A$_1$) and (A$_2$), since we have two different functions.

4 Incompleteness of the system $\mathcal{S}^{\Im}$-$Surj$

The incompleteness of this system can be obtained by giving a formula X valid in the class $\mathbb{K}_{surj}$ of all ind-functional frames where every function is surjective, which is not a theorem of $\mathcal{S}^{\Im}$-$Surj$. To do this, we will prove the following:

1. Consider $i \in \Im$. Let X be the formula $<i> (PP\top \vee FF\top) \rightarrow (PFF\top \vee FF\top)$. Then, X is valid in the class $\mathbb{K}_{surj}$. In fact, $\mathbb{K}_{surj}$ is the class of all ind-functional frames for $\mathcal{S}^{\Im}$-$Surj$, that is, the class of all ind-functional frames in which every theorem of $\mathcal{S}^{\Im}$-$Surj$ is valid.
2. There exists a model $\mathcal{M}$ such that:
   (a) Every theorem of $\mathcal{S}^{\Im}$-$Surj$ is valid in $\mathcal{M}$.
   (b) There exists an instance of X which is not valid in $\mathcal{M}$.

Now, a reasonable question arises: What happens if we add the previous formula X to our system $\mathcal{S}^{\Im}$-$Surj$? The answer is that the resulting system is also incomplete. In fact, we prove in this section that non only the system $\mathcal{S}^{\Im}$-$Surj$ is incomplete, but also that an increasing sequence of systems, obtained by extending $\mathcal{S}^{\Im}$-$Surj$ in a natural way are also incomplete. To begin with, we give the following definition.

Definition 4.1 If $\gamma \in \{P, F\}$ and n is a positive integer, we denote by γ^n the string formed by the repetition times of γ. We will call $(\Theta_n)^{\Im}$ the set of formulas $\{<i> (P^n\top \vee F^n\top) \rightarrow (PF^n\top \vee F^n\top) \mid i \in \Im\}$.

Notice that the role of formulas of the form $<i> (P^n\top \vee F^n\top) \rightarrow (PF^n\top \vee F^n\top)$ is to establish a lower bound of the number of elements of the linear order under consideration. Concretely, the condition expressed by this kind of formulas is a necessary condition for surjectivity: if there are at least n points in the set of images, then there are at least n points in the set where the domain is defined. In particular, the formula X given at the beginning of this section is an element of $(\Theta_2)^{\Im}$. We can introduce now the following sequence of systems:

Definition 4.2 Let n be a positive integer, then the system $\mathcal{S}^{\Im}$-$Surj$-Θ-(n) is defined as follows:

$$\begin{cases} \text{for } n = 1: & \mathcal{S}^{\Im}\text{-}Surj \\ \text{for } n > 1: & \mathcal{S}^{\Im}\text{-}Surj\text{-}\Theta\text{-}(n\text{-}1) \cup (\Theta_n)^{\Im} \end{cases}$$

The following Propositions are needed in order to obtain the result of incompleteness of this sequence of systems. The first one is a direct consequence of the above comment about the role of formulas in $(\Theta_n)^{\Im}$.

Proposition 4.3 *For each positive integer n, all formulas in $(\Theta_n)^{\Im}$ are valid in the class of ind-functional frames $\mathbb{K}_{surj} = \{\Sigma^{\Im} = (W, \mathcal{T}, \mathcal{F}) \mid \mathcal{F}$ is a class of surjective functions$\}$.*

Proposition 4.4 *For every positive integer n, $\mathbb{K}_{surj}$ is the class of all ind-functional frames for $\mathcal{S}^{\Im}$-$Surj$-Θ-(n).*

Proof. $\mathcal{S}^{\Im}$-$Surj$ is sound with respect to $\mathbb{K}_{surj}$ and, by Definition 4.2, we have that $\mathcal{S}^{\Im}$-$Surj$-Θ-(n) is a system obtained as an extension of $\mathcal{S}^{\Im}$-$Surj$ by adding the formulas $(\Theta_k)^{\Im}$ for $k = 2, \ldots, n$, which are all valid in $\mathbb{K}_{surj}$, by Proposition 4.3. Thus $\mathbb{K}_{surj}$ is a class of ind-functional frames for $\mathcal{S}^{\Im}$-$Surj$-Θ-(n). Moreover, the definability proved in Theorem 2.5 lead us finally to conclude that $\mathbb{K}_{surj}$ is the class of *all* ind-functional frames for $\mathcal{S}^{\Im}$-$Surj$-Θ-(n).

□

Proposition 4.5 *Let n be a positive integer. The formulas $(\Theta_{n+1})^{\Im}$ are not theorems of $\mathcal{S}^{\Im}$-$Surj$-Θ-(n).*

Proof. We fix $i \in \Im$ and let $\Psi = (W, \mathcal{T}, \mathcal{F})$ be a tuple such that:

- $W = \{w, i\}$;
- $\mathcal{T} = \{(T_w, <_w), (T_i, R_i)\}$, where $T_w = \{1_w, ..., (n+1)_w\}$; $<_w$ is a restriction of the usual strict linear order relation on the set of natural numbers, and $T_i = \{1_i\}, R_i = T_i \times T_i$;
- $\mathcal{F} = \{f_{wi}\}$ where $f_{w0} : T_w \rightarrow T_i$ with $f_{wi}(1_w) = \cdots = f_{wi}((n+1)_w) = 1_i$.

Notice that Ψ is not an ind-functional frame as given in Definition 2.1, because R_i is not a strict linear order.
We now give a model $\mathcal{M}$, based on Ψ, such that:

1. Every theorem of $\mathcal{S}^{\mathfrak{I}}\text{-}Surj\text{-}\Theta\text{-}(n)$ is valid in $\mathcal{M}$.
2. $<i>(P^{n+1}\top \vee F^{n+1}\top) \rightarrow (PF^{n+1}\top \vee F^{n+1}\top)$ is not valid in $\mathcal{M}$.

For our purpose, it suffices to consider any arbitrary model $\mathcal{M} = (\Psi, h)$.

Let us prove (1): It is easy to see that the axiom schemata of $\mathcal{S}^{\mathfrak{I}}\text{-}Surj$ are valid in $\mathcal{M}$ and that the rules preserve validity. We omit details. We focus our attention in the formulas ok kind $(\Theta_k)^{\mathfrak{I}}$ for $k < n+1$. Consider any such k and $i \in \mathfrak{I}$. So we deal with $<i>(P^k\top \vee F^k\top) \rightarrow (PF^k\top \vee F^k\top)$. Take $m_w \in T_w$ such that $1 \leq m \leq n+1$. If $m = 1$, then $m_w \in h(F^k\top)$ because $<_w$ is a strict linear order relation on T_w and there are k points to the right of 1_w in T_w. If $m > 1$, then $m_w \in h(PF^k\top)$ since $1_w <_w m_w$. In any case, $m_w \in h(PF^k\top \vee F^k\top)$ and the proof of validity is finished.

Let us prove now (2): The formula $P^{n+1}\top \vee F^{n+1}\top$ is true at 1_i in the model $\mathcal{M}$ under consideration, because R_i is reflexive. Thus, $<i>(P^{n+1}\top \vee F^{n+1}\top)$ is true at 1_w. However, since $(T_w, <_w)$ is a strict linear order with *only* $(n+1)$ elements, the formula $PF^{n+1}\top \vee F^{n+1}\top$ is false at 1_w and, therefore, $<i>(P^{n+1}\top \vee F^{n+1}\top) \rightarrow (PF^{n+1}\top \vee F^{n+1}\top)$ is not valid in $\mathcal{M}$. □

The previous Propositions 4.3, 4.4 and 4.5 allow us to state our desired result of incompleteness.

Theorem 4.6 *For each positive integer n, the system $\mathcal{S}^{\mathfrak{I}}\text{-}Surj\text{-}\Theta\text{-}(n)$ is incomplete.*

5 Conclusions and Future Work

Following the tradition of using non-standard logics to study different mathematical theories, we have proposed a combination of modal and temporal logics in order to obtain axiomatic systems dealing with several properties of functions. We have proved the soundness and completeness of the indexed systems introduced in previous works which deal with injective and with increasing accessibility functions. Similar proofs can be given for the rest of the properties considered except for surjectivity. In fact, we have shown not only that the system obtained with the formulas which define this property is incomplete, but also that there exist a sequence of natural extensions of this system that remains incomplete. As a future work, we are studying the completeness of the system obtained by extending $\mathcal{S}^{\mathfrak{I}}\text{-}Surj$ by adding *all* the formulas considered in the previous sequence. Another possibility would be to consider double indexes in order to represent the domain and image of each accessibility function. Other future works are related to the extension of this approach to more properties of accessibility functions, and the consideration of non-deterministic operators, in the line of [1, 11].

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6 APPENDIX: Completeness for $\mathcal{S}^{\mathfrak{I}}$-$Inc$

As said above, we consider this case due to its peculiarity. We present here only the results which are substantially different from the presented ones the case of injective functions.

Lemma 6.1 *Every formula of the following set is a theorem of $\mathcal{S}^{\mathfrak{I}}$-$Inc$:*

$$\{(<i>\top \wedge F<i>A) \rightarrow <i>(A \vee FA),\ (<i>\top \wedge P<i>A) \rightarrow <i>(PA \vee A) \mid i \in \mathfrak{I}\}$$

Proposition 6.2 *If $\Gamma_1 \prec_T \Gamma_2, \Gamma_1 \prec_i \Gamma_3$ and $\Gamma_2 \prec_i \Gamma_4$, then we have that $\Gamma_3 \prec_T \Gamma_4$ or $\Gamma_3 = \Gamma_4$.*

Proposition 6.3 *If $\Gamma_1 \prec_T \Gamma_2$, $\Gamma_1 \prec_i \Gamma_3$ (resp., $\Gamma_2 \prec_i \Gamma_3$) and $<i>\top \in \Gamma_2$ (resp., $<i>\top \in \Gamma_1$), then there exists $\Gamma_4 \in \mathcal{MC}$ such that $\Gamma_2 \prec_i \Gamma_4$ (resp., $\Gamma_1 \prec_i \Gamma_4$) and either $\Gamma_3 \prec_T \Gamma_4$ (resp., $\Gamma_4 \prec_T \Gamma_3$) or $\Gamma_3 = \Gamma_4$.*

In order to preserve the *increasing property* for mc-sets, we have to make a more subtle reasoning than the given one in the injective case, because of the apparition of equality in the definition of this property. To begin with, we present the following result.

Proposition 6.4 *If $\Gamma_1 \prec_T \Gamma \prec_T \Gamma_2, <i>\top \in \Gamma, \Gamma_1 \prec_i \Gamma_3, \Gamma_2 \prec_i \Gamma_3$ and $\Gamma_3 \not\prec_T \Gamma_3$, then we have $\Gamma \prec_i \Gamma_3$.*

Notice that the hypothesis $\Gamma_3 \not\prec_T \Gamma_3$ is the key in the proof of the Proposition above. As a consequence of this, we have to impose that the traces in our construction hold the following property:

$(\mathbf{P}_{inc})$ *For all $w \in W$ and $i \in \mathfrak{I}$ such that $\stackrel{w\ i}{\longrightarrow} \in \mathcal{F}_w$ and every coordinates $t_w, t'_w \in Dom(\stackrel{w\ i}{\longrightarrow})$ such that $t_w \neq t'_w$, we have that: If $\stackrel{w\ i}{\longrightarrow}(t_w) = \stackrel{w\ i}{\longrightarrow}(t'_w)$, then $\Phi_{\Sigma_k^{\mathfrak{I}}}(\stackrel{w\ i}{\longrightarrow}(t_w)) \not\prec_T \Phi_{\Sigma_k^{\mathfrak{I}}}(\stackrel{w\ i}{\longrightarrow}(t'_w))$*

We can now prove the corresponding Exhausting Lemma for our system.

Theorem 6.5 *Let $\Phi_{\Sigma_1^{\Im}}$ be a coherent trace of a finite, increasing, admissible and rooted ind-functional frame $\Sigma_1^{\Im}$ which verifies the property $(\mathbf{P}_{inc})$, and let (α) be a conditional for $\Phi_{\Sigma_1^{\Im}}$ which is active. Then there is an extension of $\Sigma_1^{\Im}$, called $\Sigma_2^{\Im}$, which is finite, increasing, admissible, rooted and also verifies the property $(\mathbf{P}_{inc})$. Moreover, there is a coherent trace $\Phi_{\Sigma_2^{\Im}}$ of $\Sigma_2^{\Im}$, extension of $\Phi_{\Sigma_1^{\Im}}$, such that (α) is a conditional for $\Phi_{\Sigma_2^{\Im}}$ which is exhausted.*

Proof. Assume that $i \in \Im$ and consider an active ind-possibilistic conditional for $\Phi_{\Sigma^{\Im}}$, that is, suppose $<i> A \in \Phi_{\Sigma_1^{\Im}}(t_w)$, but there is no $t_i = \xrightarrow{w\,i} (t_w)$ such that $A \in \Phi_{\Sigma_1^{\Im}}(t_w)$. We only consider the case $i \in W_1$ and $\xrightarrow{w\,i}$ is defined in $\mathcal{F}_1$, the rest of the cases can be reasoned similarly to Theorem 3.16. We have two possibilities:

(a) there are only elements of $(\leftarrow, t_w)$ with image in T_i.

(b) there are only elements of $(t_w, \rightarrow)$ with image in T_i.

(c) there are elements of $(\leftarrow, t_w)$ and of $(t_w, \rightarrow)$ with image in T_i.

If we have the first possibility of (a) ((b) is reasoned similarly), let t'_w be the maximum of the elements of $(\leftarrow, t_w)$ with image in T_i and denote $t'_i = \xrightarrow{w\,i} (t'_w)$. Then, by using Proposition 6.3, there exists $\Gamma \in \mathcal{MC}$ such that $\Phi_{\Sigma_1^{\Im}}(t_w) \prec_i \Gamma$. Now, we have the following subcases: (a.1)$\Phi_{\Sigma_1^{\Im}}(t'_i) \prec_T \Gamma$; (a.2)$\Phi_{\Sigma_1^{\Im}}(t'_i) = \Gamma$.

If (a.1) holds, then $\Sigma_2^{\Im}$ extends $\Sigma_1^{\Im}$ by considering the number s of successors of t'_i as in Theorem 3.16. On the other hand, if (a.2) holds but we are not in case (a.1), that is $\Phi_{\Sigma_1^{\Im}}(t'_i) = \Gamma$ but $\Phi_{\Sigma_1^{\Im}}(t'_i) \not\prec_T \Gamma$ [5], we extend our ind-functional frame with the condition that the image of t_w is t'_i.

Finally, let us suppose that possibility (c) holds. Let t'_w be the maximum of the elements of $(\leftarrow, t_w)$ with image in T_i and let t''_w be the minimum of the elements of $(t_w, \rightarrow)$ with image in T_i. Let us denote $t'_i = \xrightarrow{w\,i} (t'_w)$ and $t''_i = \xrightarrow{w\,i} (t''_w)$. It is clear that we have any of the following possibilities: (c.1) $t'_i <_i t''_i$ (c.2) $t'_i = t''_i$.

If we have (c.1), by Proposition 6.3, there exists $\Gamma \in \mathcal{MC}$ such that $\Phi_{\Sigma_n^{\Im}}(t_w) \prec_i \Gamma$ and, we have any of the following subcases: (i) $\Phi_{\Sigma_1^{\Im}}(t'_i) \prec_T \Gamma$; (ii) $\Phi_{\Sigma_1^{\Im}}(t'_i) = \Gamma$.

If (i) holds, by Proposition 6.2, we have also that either $\Gamma \prec_T \Phi_{\Sigma_1^{\Im}}(t''_i)$ or $\Gamma = \Phi_{\Sigma_1^{\Im}}(t''_i)$. If the first possibility holds, we have to extend $\Sigma_1^{\Im}$ with a new element which belongs to (t'_i, t''_i) which has to be associated to Γ, as in Theorem 3.16. Otherwise, that is, if $\Gamma \not\prec_T \Phi_{\Sigma_1^{\Im}}(t''_i)$ and, as a consequence, $\Gamma = \Phi_{\Sigma_1^{\Im}}(t''_i)$, we extend $\xrightarrow{w\,i}$ such that the image of t_w is t''_i. On the other hand, if we have (ii) $\Phi_{\Sigma_1^{\Im}}(t'_i) \not\prec_T \Gamma$, we extend $\xrightarrow{w\,i}$ such that the image of t_w is t'_i. Finally, if (c.2) holds, by hypothesis property $(\mathbf{P}_{inc})$ holds [6] for $\Phi_{\Sigma_1^{\Im}}$, that is, $\Phi_{\Sigma_n^{\Im}}(t'_i) \not\prec_T \Phi_{\Sigma_1^{\Im}}(t''_i)$, hence we can use Proposition 6.4. As a consequence of this, we can extend $\xrightarrow{w\,i}$ such that the image of t_w is t'_i, and we reason as in previous cases.

□

As a consequence, we have the Completeness Theorem for $\mathcal{S}^{\Im}$-Inc:

Theorem 6.6 *If a formula $A \in L^{\Im}$ is valid in the class of ind-functional frames*

$$\{(W, \mathcal{T}, \mathcal{F}) \mid \mathcal{F} \text{ is a class of increasing functions}\}$$

then A is a theorem of $\mathcal{S}^{\Im}$-Inc.

[5] This assumption is very important, in order to preserve the property $(\mathbf{P}_{inc})$.
[6] This is the key step where we need this property.