An Approach to Fuzzy Modal Logic of Time Intervals

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Abstract. Temporal reasoning based on intervals is nowadays ubiquitous in artificial intelligence, and the most representative interval temporal logic, called HS, was introduced by Halpern and Shoham in the eighties. There has been a great effort in the past in studying the expressive power and computational properties of the satisfiability problem for HS and its fragments, but only recently HS has been proposed as a suitable formalism for artificial intelligence applications. Such applications highlighted some of the intrinsic limits of HS: sometimes, when dealing with real-life data one is not able to express temporal relations and propositional labels in a definite, crisp way. In this paper, following the seminal ideas of Fitting and Zadeh, among others, we present a fuzzy generalization of HS that partially solves such problems of expressive power, and we prove that, as in the crisp case, its satisfiability problem is generally undecidable.

1 Introduction

Temporal reasoning based on intervals has been deeply studied in the past years. Starting with Allen's interval algebra for existential reasoning about sets of events and their relative positions [2], later sharpened in several studies concerning fragments of the interval algebra with better computational properties [27], the focus has moved progressively toward the logical level, with the introduction of Halpern and Shoham's modal logic for temporal intervals [25], also called HS, and with the systematic study of classical problems of fragments of HS, that include sub-logics in which the underlying temporal structure is constrained [34], the set of modal operators is restricted [1, 10], the semantics is softened to a reflexive one [33], the nesting of modal operators is reduced [11], or the propositional power of the languages is limited [12]; a common denominator to all such proposals is the *crisp* semantics of the languages. However, when HS is used to describe real data, as suggested in [8, 32], the need to generalize its syntax and semantics emerges in order to improve its ability of describing and working with concrete situations. For example, while from the point of view of a logician it is perfectly acceptable that a combination of symptoms such as *a period of fever, followed by a period of headache* is described by a crisp formula in which the two events (*fever* and *headache*) have precisely one point in common (the ending point of the first one, which is equal to the beginning point of the second one), from the point of view of a physician such a description may be too restrictive. In particular, *followed by* may be represented by the Allen's relation *meets* but also, to a certain extent, by the Allen's relation *overlaps* (provided that the overlapping period is not too long), and by the relation *later* (provided that the distance between the two events is not too long).

Propositional *many-valued* (or *fuzzy*) logics (from the early work of Łukasiewicz, Post, and Tarski) extend Boolean propositional logic by allowing more than two truth values [23]. Fuzzy modal logics, at least in the sense we will think of them, were introduced by Fitting [19] and have enjoyed sustained attention in recent years [7, 14, 22, 40]. Fitting, in particular, gives a very general approach to fuzzy modal logic in which not only propositions, but also accessibility relations are not just true or false, but may take different truth values. On similar basis we introduce here a fuzzy version of HS. We start by defining the concept of fuzzy linear ordering, following Zadeh [41], Bodenhofer [6], Kundu [28], and Ovchinnikov [37]. Then we build, on a fuzzy linear ordering, the classical infrastructure of Allen's relations in terms of the fuzzy version of equality and linear ordering relations (unlike, and more generally than, [18], in which they are defined on a underlying crisp linear ordering as functions that depend on the distances between points), and, finally, we give a fuzzy semantics to formulæ of HS; the resulting logic is called *Fuzzy HS* (*FHS*, for short). Following Fitting, our approach is parameterized by a Heyting algebra (which generalizes the Boolean algebra of truth values), so that both the propositional values and Allen's relations are relativized to it. As a result, FHS is a very general logic, whose semantics can be customized by modulating the properties of the algebra on which it is based. We also study to what extent FHS is less expressive than HS in terms of frame properties definability and inter-definability of modal operators, and we prove that its satisfiability problem is still undecidable even under our very general assumptions.

This work is organized as follows. In Section 2 we revise the current literature in fuzzy modal and temporal logic. In Section 3 we recall the basic elements of HS. Then, in Section 4 we study the fuzzy generalization of HS, before concluding.

2 Related Work

Fitting introduces in a systematic way *fuzzy* modal logics in [19], and, since then, fuzzy modal logics have been studied by several authors (see, e.g. [7, 14, 15, 22, 40]). Fitting's approach, in particular, consists of defining a generalization of a Kripke frame relativized to an algebra $\mathcal{A}$, so that, given two worlds v, w, the value vRw for an accessibility relation R is a value in $\mathcal{A}$, instead of a Boolean value. As in classical fuzzy propositional logics, propositional truth values are relativized to $\mathcal{A}$ as well. In this paper, we follow a similar approach, which can be considered more general than the one presented in [14], as we use a generic complete Heyting Algebra, instead of a particular one.

Temporal logics have been studied from a fuzzy point of view to

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some extent. For instance, in [5] a model for the representation and handling of fuzzy temporal references has been presented, and the concepts of date, time extent, and interval, according to the formalism of possibility theory have been introduced. In [35], the formalism of Fuzzy Linear Temporal Logic (FLTL) is defined as a generalization of propositional linear temporal logic with fuzzy temporal events and fuzzy temporal states; this logic was extended also to its branching version. Unlike our approach, FLTL is based on an absolute and linear time model with a continuous time domain, where fuzzy temporal events are defined as fuzzy numbers. Other attempts to work with non-Boolean temporal logics include, among others, a study on model checking fuzzy CTL formulæ [16]. Concerning the fuzzyfication of Allen's interval algebra, in [18] the authors define fuzzy Allen's relations from a fuzzy partition made by three possible fuzzy relations between dates (*approximately equal*, *clearly smaller*, and *clearly greater*). Unlike our fuzzy approach, these are specific functions which take their values in the real interval $[0, 1]$. On the other hand, in [39], it is shown how temporal reasoning about fuzzy time intervals can be reduced to reasoning about linear constraints, without using any version of fuzzy Allen's relations. To the best of our knowledge, the only previous attempt to define a fuzzy version of HS is [26], in which a fuzzy extension of HS, suitable for representation of preferences, has been given, and preference logic operators are interpreted with a *degree* α belonging to a finite subset of the real interval $[0, 1]$. More recently, it has been presented in [38] a fuzzy logic whose sentences are Boolean combinations of propositional variables and Allen's relations between temporal intervals. However, they are considered in their classical (crisp) definition and embedded in the propositional language, and the fuzziness is restricted only to the interpretation of the formulæ, which take values, again, in $[0, 1]$ in $\mathbb{R}$.

3 The Modal Logic of Time Intervals

Syntax and semantics. Let $\mathbb{D} = \langle D, \leq\rangle$ be a linearly ordered set, in which we assume that the equality is defined in the standard way (that is, $\forall x, y((x = y) \Leftrightarrow (x \leq y) \,\&\, (y \leq x)))$, and where we use the shortcut $x < y$ for $x \leq y \,\&\, x \neq y$. An *interval* over $\mathbb{D}$ is an ordered pair $[x, y]$, where $x, y \in D$ and $x < y$. While in the original approach to interval temporal logic intervals with coincident endpoints were included in the semantics, in the recent literature they tend to be excluded except, for instance, in [4] where a two-sorted approach has been studied. If we exclude the identity relation, there are 12 different relations between two intervals in a linear order, often called *Allen's relations* [2]: the six relations R_A (adjacent to, or meets), R_L (later than), R_B (begins), R_E (ends), R_D (during), and R_O (overlaps), depicted in Figure 1, together with their inverses $R_{\overline{X}} = (R_X)^{-1}$, for each $X \in \{A, L, B, E, D, O\}$. We interpret interval structures as Kripke structures, with Allen's relations playing the role of the accessibility relations. Thus, we associate a universal modality $[X]$ and an existential modality $\langle X\rangle$ with each Allen's relation R_X. For each $X \in \{A, L, B, E, D, O\}$, the *inverse* of the modalities $[X]$ and $\langle X\rangle$ are the modalities $[\overline{X}]$ and $\langle\overline{X}\rangle$, corresponding to the inverse relation $R_{\overline{X}}$ of R_X. Halpern and Shoham's logic, denoted HS [24], is a multi-modal logic with formulæ built from a finite, non-empty set $\mathcal{AP}$ of atomic propositions (also referred to as propositional letters), the classical propositional connectives, and a modal operator for each Allen's relation, as follows:

$$\varphi ::= \bot \mid p \mid \neg\psi \mid \psi \vee \xi \mid \langle X\rangle\psi.$$

In the above grammar, $p \in \mathcal{AP}$ and $X \in \{A, L, B, E, D, O, \overline{A}, \overline{L}, \overline{B}, \overline{E}, \overline{D}, \overline{O}\}$. The other propositional connectives and constants (e.g., $\rightarrow$, and $\top$), as well as the dual modalities (e.g., $[A]\varphi \equiv \neg\langle A\rangle\neg\varphi$), can be defined in the standard way. Given a formula of HS, its *inverse* formula is obtained by substituting every operator $\langle X\rangle$ with its inverse one $\langle\overline{X}\rangle$, and the other way around, for $X \in \{A, L, B, E, D, O\}$, while its *symmetric* is obtained by substituting every operator $\langle X\rangle$ with its inverse one $\langle\overline{X}\rangle$, and the other way around, for $X \in \{A, L, O\}$, and every $\langle B\rangle$ (resp., $\langle\overline{B}\rangle$) with $\langle E\rangle$ (resp., $\langle\overline{E}\rangle$), and the other way around.

The semantics of HS is given in terms of *interval models* of the type:

$$M = \langle\mathbb{I}(\mathbb{D}), V\rangle,$$

where $\mathbb{D}$ is a linear order, $\mathbb{I}(\mathbb{D})$ is the set of all intervals over $\mathbb{D}$, and V is a *valuation function* $V : \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{D})}$, which assigns to each atomic proposition $p \in \mathcal{AP}$ the set of intervals $V(p)$ on which p holds. The *truth* of a formula φ on a given interval $[x, y]$ in an interval model M is defined by structural induction on formulæ as follows:

$$\begin{aligned}
&M, [x, y] \Vdash p \text{ if } [x, y] \in V(p), \text{ for } p \in \mathcal{AP};\\
&M, [x, y] \Vdash \neg\psi \text{ if } M, [x, y] \not\Vdash \psi;\\
&M, [x, y] \Vdash \psi \vee \xi \text{ if } M, [x, y] \Vdash \psi \text{ or } M, [x, y] \Vdash \xi;\\
&M, [x, y] \Vdash \langle X\rangle\psi \text{ if } M, [z, t] \Vdash \psi \text{ for a } [z, t] \text{ s.t. } [x, y]R_X[z, t],\\
&\qquad\qquad \text{for } X \in \{A, L, B, E, D, O, \overline{A}, \overline{L}, \overline{B}, \overline{E}, \overline{D}, \overline{O}\}.
\end{aligned}$$

In the recent literature, several computational problems related to the logic HS have been studied: *(i)* the satisfiability problem, analyzed for the full logic in the original work by Halpern and Shoham [24], in which the authors prove that it is undecidable when the logic is interpreted in virtually all interesting classes of linearly ordered sets, and for various fragments (with different computational behaviours) in, among others, [1, 10, 11, 12, 30, 33, 34]; *(ii)* the model checking problem, in [29, 31, 32], and, more recently *(iii)* different knowledge extraction problems, in [8, 13]. Some fragments of HS present interesting properties. In the current literature, a syntactical fragment of HS characterized by encompassing only a subset of modal operators is denoted by the name of its modalities, so that the fragment with the modalities $\langle X_1\rangle, \ldots \langle X_n\rangle$ only is denoted by $\mathrm{X_1 \ldots X_n}$; examples of interesting fragments are AB and the fragment O. The inter-definability among different operators of HS has been studied at large, with the aim to determine which are, under given conditions, the expressively different fragments of HS. As a consequence, by joining the expressive power results in [24] and the inter-definability result that can be found in [1], one may obtain a rather complete picture of interesting validities of HS in the case of a general linear order.

Modelling temporal knowledge. In many different application fields temporal information can be described by using intervals instead of points. Moreover, temporal databases store the period of validity of certain tuples as an interval by design, although it has been argued that not all such information is truly interval-based; think, for example, in the case of storing the current salary of an employee during a certain period: that information is really point-based (the employee has been receiving such a salary every month of the stored period), and the interval is merely a convenient description. On the

HS	Allen's relations
$\langle A \rangle$	$[x, y] R_A [x', y'] \Leftrightarrow y = x'$
$\langle L \rangle$	$[x, y] R_L [x', y'] \Leftrightarrow y < x'$
$\langle B \rangle$	$[x, y] R_B [x', y'] \Leftrightarrow x = x', y' < y$
$\langle E \rangle$	$[x, y] R_E [x', y'] \Leftrightarrow y = y', x < x'$
$\langle D \rangle$	$[x, y] R_D [x', y'] \Leftrightarrow x < x', y' < y$
$\langle O \rangle$	$[x, y] R_O [x', y'] \Leftrightarrow x < x' < y < y'$

Graphical representation

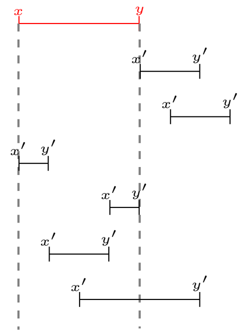

Figure 1. Allen's interval relations and HS modalities.

other side, in other domains, intervals are used as primitive, as we can see in the following examples.

In the *medical* domain (see, e.g., [8, 32]), patients can be described by timelines that collect all relevant pieces of information about tests, results, symptoms, and hospitalizations that occurred during the entire observation period, and each of these can be thought of as intervals; in this case, for instance, we can model the fact that during a certain period a patient suffered *high fever* using an interval, so that in most (but not necessary all) observation moments during that interval his/her body temperature was, in fact, high. By abstracting the relevant knowledge in this way, we can model non-trivial relationships between events, such as, for instance, the fact that *high fever* occurs *during* the administration of a certain *therapy*. As another example, in the *natural language processing domain*, a *context* can be seen as an interval during a conversation in which a particular topic is being discussed, and, because of its nature, it cannot be forced to be uninterrupted sequences of instants in which that particular topic is being discussed, but, instead, it is naturally represented by abstracted intervals (see, e.g., [36]); for instance, during a conversation between a seller and a potential buyer, the *price* of the object to be sold is a possible context, as well as the *known advantages* of a particular product over other similar ones. Extracting and elaborating contexts during a conversation is a typical problem in *chatbot* design; the ability of automatically identify a context, and therefore describing in which temporal relations contexts are related to each other is essential for the elaboration of an answer, or to decide that human intervention is necessary. As a third example, consider the *smart home* environment [3, 17, 21]. In a typical case, sensors are attached to people as well as being strategically placed at several points within a smart home. Personal sensors describe position, status, and several other parameters of the person that wears them, while sensors in the rooms take into account presence, absence, and possible activities of the subjects. This information is clearly interval-based: *the subject is sleeping (while) on the couch*, or *while in the kitchen, the subject started cooking and then went to the living room*, are possible examples of interesting natural language statements that we want to describe.

4 A Fuzzy Generalization of the Modal Logic of Time Intervals

Syntax and semantics. Following Fitting, a formula of a fuzzy modal logic is evaluated in a Heyting Algebra. A *Heyting Algebra* is a structure $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$, where $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice with (non-empty) domain A. Recall that a bounded distributive lattice is a set with internal operations $\wedge$ (*meet*[5]) and $\vee$ (*join*), both commutative, associative, and connected by the absorption law, in which a partial order can be defined:

$$\alpha \preceq \beta \Leftrightarrow \alpha \wedge \beta = \alpha \Leftrightarrow \alpha \vee \beta = \beta.$$

The symbols 0 and 1 denote, respectively, least and the greatest elements of $\mathcal{A}$. In other words, a Heyting algebra is a bounded distributive lattice in which the *relative pseudo-complement* of α w.r.t. β, defined as:

$$\bigvee \{\gamma \mid \alpha \wedge \gamma \preceq \beta\},$$

and denoted by $\alpha \rightarrow \beta$ (it is also called *Heyting implication*), exists for every α and β [20]. A Heyting algebra is said to be *complete* if for every subset $S \subseteq A$, both its least upper bound $\bigvee S$ and its greatest lower bound $\bigwedge S$ exist, and it is said to be a *chain* if $\preceq$ is total. In the following we restrict our attention to complete Heyting chains. Typical realizations of Heyting algebras include the two-element Boolean algebra, the closed interval $[0, 1]$ in $\mathbb{R}$, and any finite linear chain.

There are several possible definitions of *fuzzy* linear orders. For example Zadeh [41], defines a similarity relation in a set, imposing that it is reflexive, symmetric, and transitive, as well as a notion of fuzzy ordering, with a form of antisymmetry and fuzzy versions of totality. Similarly, Bodenhofer [6] advocates for the use of similarity-based fuzzy orderings, in which the linearity is in a strong form; the same notion is also used in [28]. On the other hand, in [37] Ovchinnikov proposes a notion of fuzzy ordering with a non-strict ordering relation. A common denominator to all such proposals is the definition of a very weak fuzzy version of the transitivity property, which allows one to obtain very general definitions. Among these previous works, the proposal that is most similar to ours is Zadeh's, which we modify to take into account both the fuzzy linear order and the fuzzy similarity in the same structure. Thus, assuming that $\mathcal{A}$ is a complete Heyting algebra with domain A, as discussed above, we start with a domain D enriched with two functions:

$$\tilde{<}, \tilde{=} : D \times D \rightarrow A,$$

and we say that the structure $\tilde{\mathbb{D}} = \langle D, \tilde{<}, \tilde{=} \rangle$ is a *fuzzy strictly linearly ordered set* if it holds, for every x, y, and z:

[5] This is the classical nomenclature in lattice theory, and it should not be confused with Allen's relation *meets*, used in this paper.

1. $\widetilde{=}(x, y) = 1 \Leftrightarrow x = y$ (*reflexivity of* $\widetilde{=}$);
2. $\widetilde{=}(x, y) = \widetilde{=}(y, x)$ (*symmetry of* $\widetilde{=}$);
3. $\widetilde{<}(x, x) = 0$ (*irreflexivity of* $\widetilde{<}$);
4. $\widetilde{<}(x, z) \succeq \widetilde{<}(x, y) \wedge \widetilde{<}(y, z)$ (*transitivity of* $\widetilde{<}$);
5. $\widetilde{<}(x, y) \succ 0 \,\&\, \widetilde{<}(y, z) \succ 0 \Rightarrow \widetilde{<}(x, z) \succ 0$ (*transfer of* $\widetilde{<}$);
6. $\widetilde{<}(x, y) = 0 \,\&\, \widetilde{<}(y, x) = 0 \Rightarrow \widetilde{=}(y, x) = 1$ (*weak totality*);
7. $\widetilde{=}(x, y) \succ 0 \Rightarrow \widetilde{<}(x, y) \prec 1$ (*non-contradiction of* $\widetilde{<}$ *over* $\widetilde{=}$),

where we have used, as before, a classical meta-language with quantification, logical implication ($\Rightarrow$), logical equivalence ($\Leftrightarrow$), and logical conjunction &. Observe that the conditions 4 and 5 are independent from each other, and that, as expected, the function $\widetilde{<}$ is asymmetric: if we had that both $\widetilde{<}(x, y)$ and $\widetilde{<}(y, x)$ were positive, then by transitivity $\widetilde{<}(x, x)$ would be also positive, which is in contradiction with the irreflexivity of $\widetilde{<}$ itself. Observe also that by irreflexivity of $\widetilde{<}$ and reflexivity of $\widetilde{=}$, one obtains that $\widetilde{<}(x, y) \succ 0$ implies that $\widetilde{=}(x, y) \prec 1$, that is, that $\widetilde{=}$ does not contradict $\widetilde{<}$. Moreover, it is worth to point out that 4 is the standard transitivity used in fuzzy orderings; a stronger version of it, such as, for example, imposing that $\widetilde{<}(x, z)$ is greater or equal to the *join* of $\widetilde{<}(x, y)$ and $\widetilde{<}(y, z)$, would lead, in fact, to a system that can only be realized in a crisp ordering.

Under such premises we say that, given a set of propositional letters $\mathcal{AP}$ and a complete Heyting algebra $\mathcal{A}$, a well-formed *fuzzy interval temporal logic* (FHS, for short) formula is obtained by the following grammar:

$$\varphi ::= \alpha \mid p \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \langle X \rangle \varphi \mid [X]\varphi,$$

where $\alpha \in A$, $p \in \mathcal{AP}$, and, as in the crisp case, $X \in \{A, L, B, E, D, O, \overline{A}, \overline{L}, \overline{B}, \overline{E}, \overline{D}, \overline{O}\}$. We use $\neg\varphi$ to denote the formula $\varphi \rightarrow 0$.

Given a fuzzy strictly linearly ordered set, we can now define the set of fuzzy strict intervals in $\widetilde{\mathbb{D}}$:

$$\mathbb{I}(\widetilde{\mathbb{D}}) = \{[x, y] \mid \widetilde{<}(x, y) \succ 0\}.$$

Generalizing classical Boolean evaluation, propositional letters are directly evaluated in the underlying algebra, by defining a valuation function $\widetilde{V} : \mathcal{AP} \times \mathbb{I}(\widetilde{\mathbb{D}}) \rightarrow A$ that generalizes the crisp function V. Apart from the fuzzyfication of valuations, following Fitting, we need to define how accessibility relations behave in the fuzzy context. Unlike classical modal logic, however, in interval temporal logic accessibility relations are not primitive, but they are defined over the underlying linear order. The natural definition of fuzzy Allen's relations, therefore, is obtained by generalizing the original, crisp definition, and substituting every $=$ with $\widetilde{=}$ and every $<$ with $\widetilde{<}$:

$$\begin{aligned}
&\widetilde{R}_A([x, y], [z, t]) = \widetilde{=}(y, z);\\
&\widetilde{R}_L([x, y], [z, t]) = \widetilde{<}(y, z);\\
&\widetilde{R}_B([x, y], [z, t]) = \widetilde{=}(x, z) \wedge \widetilde{<}(t, y);\\
&\widetilde{R}_E([x, y], [z, t]) = \widetilde{<}(x, z) \wedge \widetilde{=}(y, t);\\
&\widetilde{R}_D([x, y], [z, t]) = \widetilde{<}(x, z) \wedge \widetilde{<}(t, y);\\
&\widetilde{R}_O([x, y], [z, t]) = \widetilde{<}(x, z) \wedge \widetilde{<}(z, y) \wedge \widetilde{<}(y, t).
\end{aligned}$$

Now, we say that an $\mathcal{A}$*-valued interval model* (or *fuzzy interval model*) is a tuple of the type:

$$\widetilde{M} = \langle \mathbb{I}(\widetilde{\mathbb{D}}), \widetilde{V} \rangle,$$

where $\widetilde{\mathbb{D}}$ is a fuzzy strictly linearly ordered set that respects the properties 1-7, and $\widetilde{V}$ is a fuzzy valuation function. We interpret an FHS formula in a fuzzy interval model $\widetilde{M}$ and an interval $[x, y]$ by extending the valuation $\widetilde{V}$ of propositional letters as follows, where $X \in \{A, L, B, E, D, O, \overline{A}, \overline{L}, \overline{B}, \overline{E}, \overline{D}, \overline{O}\}$ and $[z, t]$ varies in $\mathbb{I}(\widetilde{\mathbb{D}})$:

$$\begin{aligned}
&\widetilde{V}(\alpha, [x, y]) = \alpha;\\
&\widetilde{V}(\varphi \wedge \psi, [x, y]) = \widetilde{V}(\varphi, [x, y]) \wedge \widetilde{V}(\psi, [x, y]);\\
&\widetilde{V}(\varphi \vee \psi, [x, y]) = \widetilde{V}(\varphi, [x, y]) \vee \widetilde{V}(\psi, [x, y]);\\
&\widetilde{V}(\varphi \rightarrow \psi, [x, y]) = \widetilde{V}(\varphi, [x, y]) \rightarrow \widetilde{V}(\psi, [x, y]);\\
&\widetilde{V}(\langle X \rangle \varphi, [x, y]) = \bigvee_{[z,t]} \{\widetilde{R}_X([x, y], [z, t]) \wedge \widetilde{V}(\varphi, [z, t])\};\\
&\widetilde{V}([X]\varphi, [x, y]) = \bigwedge_{[z,t]} \{\widetilde{R}_X([x, y], [z, t]) \rightarrow \widetilde{V}(\varphi, [z, t])\}.
\end{aligned}$$

We say that a formula of FHS φ is α*-satisfied* at an interval $[x, y]$ in a fuzzy interval model $\widetilde{M}$ if $\widetilde{V}(\varphi, [x, y]) \succeq \alpha$. The formula φ is α*-satisfiable* if and only if there exists a fuzzy interval model and an interval in that model where it is α-satisfied. A formula is *satisfiable* if it is α-satisfiable for some $\alpha \in \mathcal{A}$, $\alpha \neq 0$. A formula is α*-valid* if it is α-satisfied at every interval in every model, and *valid* if it is 1-valid. Observe that since a Heyting algebra, in general, does not encompass classical negation, and since our definition of satisfiability is graded, instead of absolute, then the usual duality of satisfiability and validity does not hold anymore.

Modelling uncertain temporal knowledge. Let us consider the three situations described in Section 3, and let us assume, to fix the ideas, $[0, 1]$ in $\mathbb{R}$ as underlying algebra.

Considering, as before, a timeline that describes a patient, using FHS we can assert that on a certain interval $[x, y]$ *high fever* has value 0.5, because the patient's temperature is high, but not high enough to be sure, and that in some sub-interval $[z, t]$ *high fever* has value 0.9, because his/her temperature is sufficiently high. Suppose, now that, *therapy* has value 1 at some interval $[u, v]$ that is vaguely before $[z, t]$ and vaguely overlapping $[x, y]$: this could be modelled by suitably choosing the values of $\widetilde{=}$ and $\widetilde{<}$ over pairs in $\{x, y, z, t, u, v\}$, and by asserting that *high fever occurs (fuzzily) after therapy*. Taking again into consideration the example of modelling a phone conversation between a seller and a potential customer, using FHS we can describe the inherently imprecise relationship between contexts, as well as the fact that labelling a context is naturally vague. A context can be recognized with keywords, but in many situations such an operation cannot be accurately described as Boolean. In our example, the *price* context may emerge because of words that refer to currency, numbers, or cost; similarly, the context *advantages* may emerge because of words that include comparisons or quality adjectives, among others. One can easily design a function that assigns a value to a context in a certain interval, and such value can be represented in FHS in a natural way: we can then say that *price* has value, for example, 0.5 on a certain interval $[x, y]$ because two out of four possible keywords have been detected during that interval. Finally, re-considering the case of smart home designing, the crisp solution to the inherent vagueness of the information, such as *being in the kitchen*, would be to take arbitrary decisions in the form of threshold, e.g., if the kitchen sensor is on at least 75% of the instants during the interval $[x, y]$, then $[x, y]$ is labelled with *in kitchen*. Using FSH, now we can simply give *in kitchen* the value 0.75. In a similar way, we can model the vagueness in temporal relationships between events, by giving a non-crisp value to the relation *after*, for example to assert that *the subject started cooking and (after) went to the living room*.

Transfer of validities. As we have mentioned, the logic HS has been studied in depth, and it is interesting to understand to which extent its fuzzy version retains its original expressive power and characteristics. As we shall see, FHS is somewhat weaker than HS in terms of expressive power, but some familiar properties are preserved; of those, some depend on the general (Fitting-like) approach to the fuzzyfication of a modal logic, while some other are related to our particular choices on the properties 1-7. From [1, 24], we know that:

- HS is a modal normal temporal logic, that is, that the following formulæ, and their inverse ones, are valid for every $X \in \{A, L, B, E, D, O\}$:
  - the K axioms: $[X](p \to q) \to ([X]p \to [X]q)$;
  - the temporal axioms: $p \to [X]\langle \overline{X} \rangle p$;
- the relations R_L, R_D, R_B, R_E, and their inverse ones, are all transitive, that is, the following formulæ, and their inverses, are valid:
  - $\langle D\rangle\langle D\rangle p \to \langle D\rangle p$;
  - $\langle B\rangle\langle B\rangle p \to \langle B\rangle p$;
  - $\langle L\rangle\langle L\rangle p \to \langle L\rangle p$;
  - $\langle E\rangle\langle E\rangle p \to \langle E\rangle p$;
- the following inter-definabilities hold, that is, the following formulas are valid:
  - $\langle L\rangle p \leftrightarrow \langle A\rangle\langle A\rangle p$, its inverse, and its symmetric version;
  - $\langle L\rangle p \leftrightarrow \langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p$ and its symmetric version;
  - $\langle D\rangle p \leftrightarrow \langle B\rangle\langle E\rangle p$, its inverse, and its symmetric version;
  - $\langle D\rangle p \leftrightarrow \langle E\rangle\langle B\rangle p$, its inverse, and its symmetric version;
  - $\langle O\rangle p \leftrightarrow \langle E\rangle\langle \overline{B}\rangle p$ and its symmetric version.

In the first group of properties, FHS retains the full power of HS, as the following theorem proves.

Theorem 1 *FHS is a normal temporal logic, that is, the following formulæ, and their inverse ones, are valid in FHS for every* $X \in \{A, L, B, E, D, O\}$*:*

- *the K axiom;*
- $p \to [X]\langle \overline{X}\rangle p$.

Proof. The fact that K holds for every relation is a consequence of a more general result proven in [7].

Now, we prove that the temporal axiom still holds for every modal operator, using the fact that, in a Heyting algebra, $\alpha \to \beta = 1$ if and only if $\alpha \preceq \beta$. Therefore, we prove that for every $X \in \{A, L, B, E, D, O, \overline{A}, \overline{L}, \overline{B}, \overline{E}, \overline{D}, \overline{O}\}$, it holds that $\widetilde{V}([X]\langle \overline{X}\rangle p, [x, y]) \succeq \widetilde{V}(p, [x, y])$ in every fuzzy model and interval (we stipulate that $\langle \overline{\overline{X}}\rangle$ is equal to $\langle X\rangle$). Indeed:

$$\begin{aligned}
&\widetilde{V}([X]\langle \overline{X}\rangle p, [x, y]) = \\
&= \bigwedge_{[z,t]} \{\widetilde{R}_X([x, y], [z, t]) \to \bigvee_{[u,v]} \{\widetilde{R}_{\overline{X}}([z, t], [u, v]) \wedge \widetilde{V}(p, [u, v])\}\} \\
&\succeq \bigwedge_{[z,t]} \{\widetilde{R}_X([x, y], [z, t]) \to (\widetilde{R}_{\overline{X}}([z, t], [x, y]) \wedge \widetilde{V}(p, [x, y]))\} \\
&= \bigwedge_{[z,t]} \{\widetilde{R}_X([x, y], [z, t]) \to (\widetilde{R}_X([x, y], [z, t]) \wedge \widetilde{V}(p, [x, y]))\} \\
&\succeq \widetilde{V}(p, [x, y]).
\end{aligned}$$

Observe, in particular, that the last inequality follows from the fact that, in a Heyting algebra, $\alpha \preceq (\beta \to (\beta \wedge \alpha))$ (by the definition of $\to$ itself). ■

In terms of transitivity of operators, FHS turns out to be sensibly weaker than HS, as only $\langle D\rangle$ and its inverse can be proven to be transitive.

Theorem 2 *The following formula and its inverse one are valid in FHS:*

- $\langle D\rangle\langle D\rangle p \to \langle D\rangle p$.

Conversely, the following formulæ, and their inverse ones, are not valid in FHS:

- $\langle B\rangle\langle B\rangle p \to \langle B\rangle p$;
- $\langle L\rangle\langle L\rangle p \to \langle L\rangle p$;
- $\langle E\rangle\langle E\rangle p \to \langle E\rangle p$.

Proof. Let us start by proving that $\langle D\rangle\langle D\rangle p \to \langle D\rangle p$ is valid in FHS. First, observe that, given any fuzzy strictly linearly ordered set $\widetilde{\mathbb{D}} = \langle D, \widetilde{<}, \widetilde{=}\rangle$ and any three intervals $[x, y], [z, t], [u, v] \in \mathbb{I}(\widetilde{\mathbb{D}})$, it is the case that $\widetilde{R}_D([x, y], [z, t]) \wedge \widetilde{R}_D([z, t], [u, v]) = \widetilde{<}(x, z) \wedge \widetilde{<}(t, y) \wedge \widetilde{<}(z, u) \wedge \widetilde{<}(v, t) \preceq \widetilde{<}(x, u) \wedge \widetilde{<}(v, y) = \widetilde{R}_D([x, y], [u, v])$, where the inequality holds by the transitivity of $\widetilde{<}$ (property 4 above). Thus, we have that:

$$\begin{aligned}
&\widetilde{V}(\langle D\rangle\langle D\rangle p, [x, y]) = \\
&= \bigvee_{[z,t]} \{\widetilde{R}_D([x, y], [z, t]) \wedge \bigvee_{[u,v]} \{\widetilde{R}_D([z, t], [u, v]) \wedge \widetilde{V}(p, [u, v])\}\} \\
&= \bigvee_{[z,t]} \bigvee_{[u,v]} \{\widetilde{R}_D([x, y], [z, t]) \wedge \widetilde{R}_D([z, t], [u, v]) \wedge \widetilde{V}(p, [u, v])\} \\
&\preceq \bigvee_{[u,v]} \{\widetilde{R}_D([x, y], [u, v]) \wedge \widetilde{V}(p, [u, v])\} \\
&= \widetilde{V}(\langle D\rangle p, [x, y]).
\end{aligned}$$

Observe, in particular, that the second equality holds because Heyting algebras are join infinite distributive.

Now, let us prove that $\langle B\rangle\langle B\rangle p \to \langle B\rangle p$ is not valid in FHS. Consider a model $\widetilde{M} = \langle \mathbb{I}(\widetilde{\mathbb{D}}), \widetilde{V}\rangle$ with domain $D = \{0, 1, 2, 3, 4, 5\}$, and let $\mathcal{A}$ be the three elements Heyting chain $\{0 \prec \frac{1}{2} \prec 1\}$ with $\widetilde{=}(x, y) = \max\{0, 1 - \frac{1}{2}|x - y|\}$ and $\widetilde{<}(x, y) = \min\{1, \max\{\frac{1}{2}(y - x), 0\}\}$. It is easy to check that both $\widetilde{=}$ and $\widetilde{<}$, so defined, satisfy the required conditions. Further, consider the valuation $\widetilde{V}$ with $\widetilde{V}(p, [2, 3]) = \frac{1}{2}$ and with $\widetilde{V}(p, [x, y]) = 0$ for every interval $[x, y] \neq [2, 3]$. It is easy to see that the only significant intervals to be checked in order to obtain $\widetilde{V}(\langle B\rangle\langle B\rangle p, [0, 5])$ are $[0, 5]$, $[1, 4]$ and $[2, 3]$, hence:

$$
\begin{aligned}
&\widetilde{V}(\langle B\rangle\langle B\rangle p,[0,5]) = \\
&= \bigvee_{[z,t]}\{\widetilde{R}_B([0,5],[z,t]) \wedge \bigvee_{[u,v]}\{\widetilde{R}_B([z,t],[u,v]) \wedge \widetilde{V}(p,[u,v])\}\} \\
&= \widetilde{R}_B([0,5],[1,4]) \wedge \widetilde{R}_B([1,4],[2,3]) \wedge \widetilde{V}(p,[2,3]) \\
&= \frac{1}{2} \wedge \frac{1}{2} \wedge \frac{1}{2} = \frac{1}{2}.
\end{aligned}
$$

On the other hand:

$$
\begin{aligned}
&\widetilde{V}(\langle B\rangle p,[0,5]) = \\
&\quad = \bigvee_{[z,t]}\{\widetilde{R}_B([0,5],[z,t]) \wedge \widetilde{V}(p,[z,t])\} \\
&\quad = \widetilde{R}_B([0,5],[2,3]) \wedge \widetilde{V}(p,[2,3]) = (0 \wedge 1) \wedge \frac{1}{2} = 0.
\end{aligned}
$$

Since $\widetilde{V}(\langle B\rangle\langle B\rangle p,[0,5]) \succ \widetilde{V}(\langle B\rangle p,[0,5])$, we have the result. ■

Finally, all inter-definability of operators still hold in FHS, but in only one direction; unfortunately, this means that, unlike HS, no operator can be defined in terms of the others, which, in turn, means that all fragments of FHS are expressively different, and they may present different properties.

Theorem 3 *The following formulæ are valid in FHS:*

- $\langle L\rangle p \rightarrow \langle A\rangle\langle A\rangle p$;
- $\langle D\rangle p \rightarrow \langle B\rangle\langle E\rangle p$;
- $\langle D\rangle p \rightarrow \langle E\rangle\langle B\rangle p$;
- $\langle O\rangle p \rightarrow \langle E\rangle\langle \overline{B}\rangle p$.

Conversely, their right-to-left versions are not valid in FHS; moreover the following formula and its right-to-left version are not valid in FHS either:

- $\langle L\rangle p \rightarrow \langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p$.

Proof. Let us focus first on proving that $\langle L\rangle\varphi \rightarrow \langle A\rangle\langle A\rangle\varphi$ is valid in FHS. We begin by noticing that if $\widetilde{R}_L([x,y],[z,t]) = \widetilde{<}(y,z) \succ 0$, then $[y,z] \in \mathbb{I}(\widetilde{\mathbb{D}})$, and, so, $\widetilde{R}_A([x,y],[y,z]) = \widetilde{=}(y,y) = 1$ and $\widetilde{R}_A([y,z],[z,t]) = \widetilde{=}(z,z) = 1$. Therefore:

$$
\begin{aligned}
&\widetilde{V}(\langle L\rangle\phi,[x,y]) = \\
&= \bigvee_{[z,t]}\{\widetilde{R}_L([x,y],[z,t]) \wedge \widetilde{V}(\phi,[z,t])\} \\
&\preceq \bigvee_{[z,t]}\{\widetilde{R}_A([x,y],[y,z]) \wedge \widetilde{R}_A([y,z],[z,t]) \wedge \widetilde{V}(\phi,[z,t])\} \\
&\preceq \bigvee_{[z,t]}\bigvee_{[u,v]}\{\widetilde{R}_A([x,y],[u,v]) \wedge \widetilde{R}_A([u,v],[z,t]) \wedge \widetilde{V}(\phi,[z,t])\} \\
&= \bigvee_{[u,v]}\{\widetilde{R}_A([x,y],[u,v]) \wedge \bigvee_{[z,t]}\{\widetilde{R}_A([u,v],[z,t]) \wedge \widetilde{V}(\phi,[z,t])\}\} \\
&= \widetilde{V}(\langle A\rangle\langle A\rangle\phi,[x,y]).
\end{aligned}
$$

The remaining valid formulæ can be treated in a similar way.

Now, in order to prove that $\langle A\rangle\langle A\rangle p \rightarrow \langle L\rangle p$ is not valid, take a model $\widetilde{M} = \langle \mathbb{I}(\widetilde{\mathbb{D}}), \widetilde{V}\rangle$ with domain $D = \{0,1,2,3,4\}$ and let $\mathcal{A}$ be the three elements Heyting chain $\{0 \prec \frac{1}{2} \prec 1\}$ with $\widetilde{=}(x,y) = \max\{0, 1-\frac{1}{2}|x-y|\}$ and $\widetilde{<}(x,y) = \min\{1, \max\{\frac{1}{2}(y-x), 0\}\}$. Then, consider a valuation $\widetilde{V}$ such that $\widetilde{V}(p,[2,4]) = 1$ and $\widetilde{V}(p,[x,y]) = 0$ for every $[x,y] \neq [2,4]$. Then, the only interval $[x,y]$ such that $\widetilde{R}_L([0,2],[x,y]) \succ 0$ is $[3,4]$, and, in particular, $\widetilde{R}_L([0,2],[3,4]) = \widetilde{<}(2,3) = \frac{1}{2}$. So $\widetilde{V}(\langle L\rangle p,[0,2]) = \widetilde{R}_L([0,2],[3,4]) \wedge \widetilde{V}(p,[3,4]) = \frac{1}{2} \wedge 0 = 0$. On the other hand, $\widetilde{V}(\langle A\rangle\langle A\rangle p,[0,2]) \succeq \widetilde{R}_A([0,2],[1,2]) \wedge (\widetilde{R}_A([1,2],[2,4]) \wedge \widetilde{V}(p,[2,4])) = \frac{1}{2} \wedge 1 \wedge 1 = \frac{1}{2}$. Therefore, $\widetilde{V}(\langle A\rangle\langle A\rangle\varphi \rightarrow \langle L\rangle\varphi,[0,2]) = \frac{1}{2} \rightarrow 0 = 0$, and so $\langle A\rangle\langle A\rangle\varphi \rightarrow \langle L\rangle\varphi$ is not valid. The other right-to-left implications can be shown not to be valid using similar arguments.

To conclude, let us prove that neither $\langle L\rangle p \rightarrow \langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p$ nor $\langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p \rightarrow \langle L\rangle p$ is valid, starting with the former. Consider a model $\widetilde{M} = \langle \mathbb{I}(\widetilde{\mathbb{D}}), \widetilde{V}\rangle$ with domain $D = \{0,3,5,6\}$ and let, again, $\mathcal{A}$ be the three elements Heyting chain $\{0 \prec \frac{1}{2} \prec 1\}$ with $\widetilde{=}(x,y) = \max\{0, 1-\frac{1}{2}|x-y|\}$ and $\widetilde{<}(x,y) = \min\{1, \max\{\frac{1}{2}(y-x), 0\}\}$. Then, consider a valuation $\widetilde{V}$ such that $\widetilde{V}(p,[5,6]) = 1$ and $\widetilde{V}(p,[x,y]) = 0$ for every $[x,y] \neq [5,6]$. Then, the only interval $[x,y]$ such that $\widetilde{R}_L([0,3],[x,y]) \succ 0$ is $[5,6]$, and, in particular, $\widetilde{R}_L([0,3],[5,6]) = \widetilde{<}(3,5) = 1$. Then $\widetilde{V}(\langle L\rangle p,[0,3]) = \widetilde{R}_L([0,3],[5,6]) \wedge \widetilde{V}(p,[5,6]) = 1$. On the other hand, the only intervals $[z,t]$ such that $\widetilde{R}_{\overline{B}}([0,3],[z,t]) \succ 0$ are $[0,5]$ and $[0,6]$, so, for $[z,t]$ varying in $\{[0,5],[0,6]\}$, it holds:

$$
\begin{aligned}
&\widetilde{V}(\langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p,[0,3]) = \\
&= \bigvee_{[z,t]}\{\widetilde{R}_{\overline{B}}([0,3],[z,t]) \wedge \widetilde{V}([E]\langle \overline{B}\rangle\langle E\rangle p,[z,t])\}.
\end{aligned}
$$

The only intervals $[u,v]$ such that $\widetilde{R}_E([0,5],[x,y]) \succ 0$ are $[3,5]$, $[3,6]$, and $[5,6]$, so for $[u,v]$ varying in $\{[3,5],[3,6],[5,6]\}$, we have: we have:

$$
\begin{aligned}
&\widetilde{V}([E]\langle \overline{B}\rangle\langle E\rangle p,[0,5]) = \\
&= \bigwedge_{[u,v]}\{\widetilde{R}_E([0,5],[u,v]) \rightarrow \widetilde{V}(\langle \overline{B}\rangle\langle E\rangle p,[u,v])\} \\
&\preceq [\widetilde{R}_E([0,5],[3,6]) \rightarrow \widetilde{V}(\langle \overline{B}\rangle\langle E\rangle p,[3,6])] = \frac{1}{2} \rightarrow 0 = 0.
\end{aligned}
$$

Similarly, it can be proved that $\widetilde{V}([E]\langle \overline{B}\rangle\langle E\rangle p,[0,6]) = 0$; hence $\widetilde{V}(\langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p,[0,3]) = 0$. Therefore, $\widetilde{V}(\langle L\rangle p \rightarrow \langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p,[0,3]) = 1 \rightarrow 0 = 0$, proving that $\langle L\rangle p \rightarrow \langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p$ is not valid. In order to prove that $\langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p \rightarrow \langle L\rangle p$ is not valid either, take a model $\widetilde{M} = \langle \mathbb{I}(\widetilde{\mathbb{D}}), \widetilde{V}\rangle$ with domain $D = \{0, \frac{1}{2}, 1, 3\}$ and set $\mathcal{A}$ as the Heyting chain $\{0 \prec \frac{1}{4} \prec \frac{1}{2} \prec \frac{3}{4} \prec 1\}$. Define, as above, $\widetilde{=}(x,y) = \max\{0, 1-\frac{1}{2}|x-y|\}$ and $\widetilde{<}(x,y) = \min\{1, \max\{\frac{1}{2}(y-x), 0\}\}$. Consider now a valuation $\widetilde{V}$ such that $\widetilde{V}(p,[x,y]) = 0$ for every $[x,y] \in \mathbb{I}(\widetilde{\mathbb{D}})$. This implies that $\widetilde{V}(\langle L\rangle p,[0,\frac{1}{2}]) = 0$. On the other hand, $\widetilde{R}_{\overline{B}},([0,\frac{1}{2}],[1,3]) = \frac{1}{2}$ and, for all $[x,y] \in \mathbb{I}(\widetilde{\mathbb{D}})$ it happens that $\widetilde{R}_E([1,3],[x,y]) = 0$. This implies that $\widetilde{V}(\langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p,[0,\frac{1}{2}]) \neq 0$ and, as recalled above, $\widetilde{V}(\langle L\rangle p,[0,\frac{1}{2}]) = 0$, proving that $\langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p \rightarrow \langle L\rangle p$ is not valid. ■

Undecidability of the satisfiability problem. In the fuzzy case, the *satisfiability problem* is not uniquely defined. On the one hand, the fuzzy 1-satisfiability problem corresponds to the satisfiability problem in the crisp case; more in general, this holds for the fuzzy α-satisfiability problem *for a given* α, solving which immediately reduces to solving the 1-satisfiability problem. On the other hand, the α-satisfiability problem (which is the one corresponding to our definition in Section 4, and asks the question of whether a given formula is satisfiable at *any* degree at all) does not have an immediate crisp counterpart: this is the one studied in this section. The satisfiability problem for crisp HS has been studied in a comprehensive way [1, 10, 11, 12, 30, 33, 34]. Without syntactical or semantical restrictions, it is generally undecidable, regardless the properties of the underlying linear order and, in addition, it remains undecidable for most syntactical fragments of HS. As we have seen in the previous section, the taxonomy of expressively different syntactical fragments of FHS is very different from that of HS; however, the negative computational properties still transfer from included to including fragments. In other words, if any satisfiability problem is undecidable for a syntactical fragment $\mathrm{FX_1 \dots X_n}$ of FHS under a certain hypothesis, then so is the same problem for full FHS under the same hypothesis. Let us consider the particular case of the fragment FO (i.e., the fuzzy counterpart of the HS fragment O) of FHS, whose crisp counterpart has been studied in [9]. Observe that every formula of the fragment O, modulo writing 0 for $\perp$ and $\varphi \to 0$ for $\neg\varphi$, is, in fact, a formula of FO. Now, we want to show that a formula φ of the fragment O of HS is satisfiable if and only if its fuzzy counterpart is α-satisfiable in FHS.

Theorem 4 *Let φ be a formula of the fragment* O *of (crisp) HS. Then, φ is satisfiable in the class of all linear orders if and only its fuzzy counterpart is α-satisfiable in the class of all fuzzy linear orders for some $\alpha \succ 0$, $\alpha \in \mathcal{A}$.*

Proof. The left-to-right direction follows easily, by observing that any interval model $M = \langle \mathbb{I}(\mathbb{D}), V\rangle$ can be seen as a fuzzy interval model $\widetilde{M} = \langle \mathbb{I}(\widetilde{\mathbb{D}}), \widetilde{V}\rangle$ with $\widetilde{\mathbb{D}} = \langle D, \widetilde{<}, \widetilde{=}\rangle$, where $\widetilde{<}(x, y) = 1$ if $x < y$ and $\widetilde{<}(x, y) = 0$, otherwise, $\widetilde{=}(x, y) = 1$ if $x = y$ and $\widetilde{=}(x, y) = 0$, otherwise, and $\widetilde{V}(p, [x, y]) = 1$ if $[x, y] \in V(p)$ and $\widetilde{V}(p, [x, y]) = 0$, otherwise.

Conversely, suppose that, for some fuzzy model $\widetilde{M} = \langle \mathbb{I}(\widetilde{\mathbb{D}}), \widetilde{V}\rangle$ with $\widetilde{\mathbb{D}} = \langle D, \widetilde{<}, \widetilde{=}\rangle$ based on a Heyting chain $\mathcal{A}$, and some $[x, y] \in \mathbb{I}(\widetilde{\mathbb{D}})$, we have $\widetilde{V}(\varphi, [x, y]) = \alpha \succ 0$. Consider the interval model $M = \langle \mathbb{I}(\mathbb{D}), V\rangle$ obtained from $\widetilde{M}$ by setting $\mathbb{D} = \langle D, <\rangle$, where $x < y$ if and only if $\widetilde{<}(x, y) \neq 0$, and $V(p) = \{[x, y] \in \mathbb{I}(\widetilde{\mathbb{D}}) \mid \widetilde{V}(p, [x, y]) \neq 0\}$. It is easy to check (using the irreflexivity, transitivity, transfer and weak totality of $\widetilde{<}$, as well as and the reflexivity of $\widetilde{=}$) that $\mathbb{D} = \langle D, <\rangle$, so defined, is, in fact, a linear order. Moreover, $\mathbb{I}(\mathbb{D}) = \mathbb{I}(\widetilde{\mathbb{D}})$. Observe, now, that for all $[x, y], [z, t] \in \mathbb{I}(\mathbb{D})$ it is the case that $[x, y]R_O[z, t]$ if and only if $\widetilde{R}_O([x, y], [z, t]) \neq 0$. To see this, suppose that $[x, y]R_O[z, t]$. Then $x < z < y < t$ and so, by definition of $<$, we have $\widetilde{<}(x, z), \widetilde{<}(z, y), \widetilde{<}(y, t) \neq 0$ and, since $\mathcal{A}$ is a chain, $\widetilde{R}_O([x, y], [z, t]) = \widetilde{<}(x, z) \wedge \widetilde{<}(z, y) \wedge \widetilde{<}(y, t) \neq 0$. Conversely, suppose that $\widetilde{R}_O([x, y], [z, t]) \neq 0$, i.e., $\widetilde{<}(x, z) \wedge \widetilde{<}(z, y) \wedge \widetilde{<}(y, t) \neq 0$. This implies that $\widetilde{<}(x, z), \widetilde{<}(z, y), \widetilde{<}(y, t) \neq 0$. So, by definition, we must have $x < z < y < t$, and, hence, that $[x, y]R_O[z, t]$. Now, we can prove that, for every O-formula φ, and every interval $[x, y] \in \mathbb{I}(\mathbb{D})$, it is the case that $[x, y] \in V(\varphi)$ if and only if $\widetilde{V}(\varphi, [x, y]) \neq 0$. To see this, we proceed by structural induction on φ. The base case for propositional variables is immediate by the definition of V, and the case for $\perp$ is trivial. The inductive cases for φ of the form $\theta \wedge \psi$ or $\theta \vee \psi$ are straightforward. Suppose, now, that φ is of the form $\theta \to \psi$, and suppose that $\widetilde{V}(\theta \to \psi, [x, y]) = 0$. Then, $\widetilde{V}(\theta, [x, y]) \to \widetilde{V}(\psi, [x, y]) = 0$, and hence $\widetilde{V}(\theta, [x, y]) \neq 0$ while $\widetilde{V}(\psi, [x, y]) = 0$ (because we are assuming that $\mathcal{A}$ is a chain). By the inductive hypothesis, $[x, y] \in V(\theta)$ and $[x, y] \notin V(\psi)$, so $[x, y] \notin V(\theta \to \psi)$. Conversely, suppose that $\widetilde{V}(\theta \to \psi, [x, y]) = (\widetilde{V}(\theta, [x, y]) \to \widetilde{V}(\psi, [x, y])) \neq 0$. Then either $\widetilde{V}(\theta, [x, y]) = 0 = \widetilde{V}(\psi, [x, y])$, or $\widetilde{V}(\psi, [x, y]) \neq 0$. In both cases, the inductive hypothesis yields $[x, y] \in V(\theta \to \psi)$. Now, consider the case in which φ is of the form $\langle O\rangle\psi$, and suppose that $[x, y] \in V(\langle O\rangle\psi)$. So, there is an interval $[z, t] \in \mathbb{I}(\mathbb{D})$ such that $[x, y]R_O[z, t]$ and $[z, t] \in V(\psi)$. As we have proved before, we have that $\widetilde{R}_O([x, y], [z, t]) \neq 0$ and, by the inductive hypothesis, $\widetilde{V}(\psi, [z, t]) \neq 0$. Therefore $\widetilde{V}(\langle O\rangle\psi, [x, y]) = \bigvee\{\widetilde{R}_O([x, y], [z, t]) \wedge \widetilde{V}(\varphi, [z, t]) \mid [z, t] \in \mathbb{I}(\widetilde{\mathbb{D}})\} \neq 0$. Conversely, suppose that $\widetilde{V}(\langle O\rangle\psi, [x, y]) = \bigvee\{\widetilde{R}_O([x, y], [z, t]) \wedge \widetilde{V}(\varphi, [z, t]) \mid [z, t] \in \mathbb{I}(\widetilde{\mathbb{D}})\} \neq 0$. Then, there must be an interval $[z, t] \in \mathbb{I}(\widetilde{\mathbb{D}})$ such that $\widetilde{R}_O([x, y], [z, t]) \neq 0$ and $\widetilde{V}(\varphi, [z, t]) \neq 0$. As we have seen above, this means that $[x, y]R_O[z, t]$ and, by the inductive hypothesis, $[z, t] \in V(\psi)$, so $[x, y] \in V(\langle O\rangle\psi)$. The case for ϕ of the form $[O]\psi$ can be proved using a similar argumentation. ∎

Corollary 1 *The satisfiability problem for FHS interpreted in the class of all fuzzy linear orders is undecidable.*

The satisfiability problem for HS and its fragments has been studied in particular classes of linearly ordered sets, such as the class of all finite linearly ordered sets, the class of all strongly discrete linearly ordered sets, and so on. For most of cases, the computational properties are unaffected by the additional conditions of the underlying linear order, at least for classes with at least one arbitrarily long linear order. The fragment O, among others, is undecidable in all such cases. The investigation of the satisfiability problem for (fragments of) FHS with respect to different linear orders would require first to identify a set of axioms characterizing the fuzzy counterpart of each of these classes of linear orders (along the line of the set of axioms 1–7 provided here for the class of all linear orders).

5 Conclusions

We proposed a fuzzy generalization of the interval logic HS, defining its syntax, semantics, and discussing its expressive power by comparing it to the one of crisp HS. We have studied the decidability status of its satisfiability problem, which turned out to be undecidable at least in the case in which the underlying algebra is a chain. While fuzzy modal logics were proposed more than twenty years ago, this is the first systematic definition of a fuzzy version of HS that follows the seminal, general approach indicated by Fitting. The result is a very powerful logic with the potential to be applicable in a number of contexts.

There are several open questions concerning FHS. First, we have proved that the satisfiability problem for FHS is undecidable in the class of all fuzzy linear orders when the underlying algebra is a chain, and asking the same question in the general case of Heyting algebras, and for other classes of linear orders would be very natural. Moreover, it would be important to understand whether the general undecidability of fragments of HS can be transferred to the fragments of FHS, and under which hypothesis. Furthermore, the need of FHS origins from modern applications of interval temporal logics in the context of information extraction. These include interval model checking, used to verify temporal datasets, temporal rule extraction, and temporal decision tree extraction; adapting these algorithms to the case of FHS is an open, and non-trivial, problem.

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