Matemáticas III

GIE

Abstract

Selección de ejercicios de examen de diversas universidades americanas y europeas.

I. Multivariable calculus

1. Consider the function
$$F(x,y,z) = z\sqrt{x^2+y} + 2\frac{y}{z}.$$
The point P_0 : $(1,3,2)$ lies on the surface $F(x,y,z) = 7$. Find the equation of the tangent plane to the surface $F = 7$ at P_0. If starting at P_0 a small change were to be made in only *one* of the variables, which one would produce the largest change (in absolute value) in F?

2. For each of the following functions, determine whether or not it is differentiable at $(0,0)$. Justify your answers.
$$f(x,y) = \begin{cases} x^2y^2(x^2+y^2)^{-1} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$
$$g(x,y) = \begin{cases} x^2(x^2+y^2)^{-1} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

3. For the following function from $\mathbb{R}^2 \to \mathbb{R}$, give with proof the set of points at which it is differentiable:
$$g(x,y) = (x^2+y^2)\sin\frac{1}{x^2+y^2},$$
if at least one of x, y is not 0, and $g(0,0) = 0$.

4. Show that the map $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by
$$f(x,y,z) = (x-y-z, x^2+y^2+z^2, xyz)$$
is differentiable everywhere and find its differential map. Stating accurately any theorem that you require, show that f has a differentiable local inverse at a point (x,y,z) if and only if
$$(x+y)(x+z)(y-z) \neq 0.$$

5. Consider the function
$$f(x,y) = x^2+y^2-\frac{1}{2}x^4-2x^2y^2-\frac{1}{2}y^4.$$
Find the critical points of $f(x,y)$. Determine the type of each critical point and sketch contours of constant $f(x,y)$ and plot $z = f(x,y)$.

6. Let $f(x,y) = x+4y+\dfrac{2}{xy}$. Find the critical points of $f(x,y)$. Use the second-derivative test to test the critical points found.

7. A rectangular box is placed in the first octant with one corner at the origin and the three adjacent faces in the coordinate planes. The opposite point $P : (x,y,z)$ is constrained to lie on the paraboloid $x^2+y^2+z = 1$. Which P gives the box of gratest volume? One could use Lagrange multipliers to maximize the volume $V = xyz$ with the constraint $x^2+y^2+z = 1$.

II. PDE

1. Let a be a positive constant. By considering the lines $y = a(x-x_0)$ for constant x_0, or otherwise, show that any solution of the equation
$$\frac{\partial f}{\partial x} + a\frac{\partial f}{\partial y} = 0$$
is of the form $f(x,y) = F(y-ax)$ for some function F.

2. Let a be a positive constant. Solve the equation
$$\frac{\partial f}{\partial x} + a\frac{\partial f}{\partial y} = f^2$$
subject to $f(0,y) = g(y)$ for a given function g.

3. Check that $u(x,t) = \dfrac{1}{(1+4t)^{1/2}}\exp\left(-\dfrac{x^2}{1+4t}\right)$ is the solution of the Heat problem:
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ u = \exp(-x^2) \text{ at } t = 0,$$
with $-\infty < x < \infty$ and $t > 0$.