Matemáticas II

GIE

Abstract

Selección de ejercicios de examen de diversas universidades americanas y europeas.

I. One variable

1. (*a*) Find an integral formula for the arc length of the curve $y = 2\sqrt{x+1}$ for $0 \leq x \leq 1$. Do not evaluate.

   (*b*) Find an integral formula for the surface area of the curve in part (*a*) rotated around the x-asis. Simplify the integrand and evaluate the integral.

2. Estimate the following to two decimal places: $\sin(\pi + 1/100)$ and $\sqrt{101}$.

3. Find the equation of the line tangent to the graph of $x^2y^2 + y^3 = 2$ at the point $(1, 1)$ on the graph. Give the equation in the form $y = mx + n$.

4. Let $F(x) = \int_0^x e^{-t^2} dt$.

   (*a*) Find $F'(1)$ and $F''(1)$.

   (*b*) Express $\int_1^2 e^{-u^2/4} du$ in terms of values of $F(x)$.

5. Define what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be differentiable at a point $a \in \mathbb{R}$. If f is differentiable everywhere on $\mathbb{R}$, must f' be continuous everywhere? Justify your answer.

II. Series

1. For what values of p does $\sum_{n=1}^{\infty} \frac{n}{\sqrt{4+n^p}}$ converge? Indicate reasoning.

2. (*a*) State the Taylor's theorem with the ramainder in Lagrange's form.

   (*b*) Suppose that $e : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $e(0) = 1$ and $e'(x) = e(x)$ for all $x \in \mathbb{R}$. Use the result of (*a*) to prove that

   $$e(x) = \sum_{n \geq 0} \frac{x^n}{n!} \quad \text{for all} \quad x \in \mathbb{R}.$$

   No property of the exponential function may be assumed.

3. (*a*) Find by differentiating the function $f(x) = \sqrt{1+x}$ the first four non-zero terms of its Taylor series around $x = 0$.

   (*b*) Use the correct answer to (*a*) to calculate $\sqrt{1.2}$ to four decimal places.

4. Find the Taylor series for $\tan^{-1} x$ around $x = 0$ by using term-by-term differentiation or integration on the appropiate geometric series. Give enough terms to make the pattern clear.

5. For $s \in \mathbb{R}$, prove that $\sum_{n=1}^{\infty} n^{-s}$ converges if and only if $s > 1$.

6. Prove that the real series

   $$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$$

   $$g(x) = \sum_{n=0} (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

   have radius of convergence ∞.

7. State Taylor's theorem. Show that

   $$(1+x)^{1/2} = 1 + \sum_{n \geq 1} c_n x^n,$$

   for all $x \in (0, 1)$, where

   $$c_n = \frac{\frac{1}{2}(\frac{1}{2}-1)\dots(\frac{1}{2}-n+1)}{n!}.$$