Matemáticas III

GIE

Abstract

Selección de ejercicios de examen de diversas universidades americanas y europeas.

I. Line integral

1. *a*) Show that $\mathbf{F} = (3x^2 - 6y^2)\vec{i} + (-12xy + 4y)\vec{j}$ is conservative. *b*) Find a potential function for $\mathbf{F}$. *c*). Let C be the curve $x = 1 + y^3(1-y)^3$, $0 \le y \le 1$. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

2. Let $\mathbf{F} = (2xy + z^3)\vec{i} + (x^2 + 2yz)\vec{j} + (y^2 + 3xz^2 - 1)\vec{k}$. *a*) Show that $\mathbf{F}$ is conservative. *b*) Using a systematic method, find a potential function $f(x, y, z)$ such that $\mathbf{F} = \nabla f$. Show your work, even if you can do it mentally.

3. For $x^2 + y^2 > 0$, let $\mathbf{F}$ be the vector field defined by

$$\mathbf{F} = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, z \right).$$

Show that $\nabla \times \mathbf{F} = 0$. Let C be the curve which is the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $z = x + 200$. Calculate $\oint_C \mathbf{F} \cdot d\mathbf{x}$.

II. Double integral and Green's theorem

1. *a*) Draw a picture of the region of integration of $\int_0^1 \int_x^{2x} dydx$. Exchange the order of integration to express the integral in part (*a*) in terms of integration in the order $dxdy$. Warning: your answer will hace two pieces.

2. *a*) Express the work done by the force field $\mathbf{F} = (5x + 3y)\vec{i} + (1 + \cos y)\vec{j}$ on a particle moving counterclockwise once around the unit circle centered at the origin in the form $\int_a^b f(t)dt$. Do not evaluate the integral; don't even simplify $f(t)$. *b*) Evaluate the line integral using Green's theorem.

III. Surface integral and Stokes' theorem

1. Let $\mathbf{F} = -2xz\vec{i} + y^2\vec{k}$. *a*) Calculate $\operatorname{curl} \mathbf{F}$. *b*) Show that $\iint_R \operatorname{curl} \mathbf{F} \cdot \mathbf{n} dS = 0$ for any finite portion R of the unit sphere $x^2 + y^2 + z^2 = 1$. (take the normal vector $\mathbf{n}$ pointing outward. *c*) Show that $\oint_C \mathbf{F} \cdot dr = 0$ for any simple closed curve C on the unit sphere $x^2 + y^2 + z^2 = 1$.

IV. Triple integral and Gauss's theorem

1. Let S be the surface formed by the part of the paraboloid $z = 1 - x^2 - y^2$ lying above the xy-plane, and let $\mathbf{F} = x\vec{i} + y\vec{j} + 2(1 - z)\vec{k}$. Calculate the flux of $\mathbf{F}$ across S, taking the upward direction as the one for which the flux is positive. Do this in two ways: *a*) by direct calculation of $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$; *b*) by computing the flux of $\mathbf{F}$ across a simpler surface and using the divergence theorem.

2. State (without proof) the divergence theorem for a vector field $\mathbf{F}$ with continuous first-order partial derivatives throughout a volume V enclosed by a bounded oriented piecewise-smooth non-self-intersecting surface S. By calculating the relevant volume and surface integrals explicitly, verify the divergence theorem for the vector field $\mathbf{F} = (x^3 + 2xy^2, y^3 + 2yz^2, z^3 + 2zx^2)$, defined within a sphere of radius R centred at the origin.