Matemáticas I

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Abstract

Selección de ejercicios de examen de diversas universidades americanas y europeas.

I. Bilinear and quadratic forms

1. Find a linear change of coordinates such that the quadratic form

$$2x^2 + 8xy - 6xz + y^2 - 4yz + 2z^2$$

takes the form

$$\alpha x^2 + \beta y^2 + \gamma z^2,$$

for real numbers α, β and γ.

2. Let q denote a quadratic form on a real vector space V. Define the rank and signature of q. Find the rank and signature of the following quadratic forms.

(a) $q(x, y, z) = x^2 + y^2 + z^2 - 2xz - 2yz$.

(b) $q(x, y, z) = xy - xz$.

(c) $q(x, y, z) = xy - 2z^2$.

3. Let V denote the vector space of all real polynomials of degree at most 2. Show that

$$(f, g) = \int_{-1}^{1} f(x)g(x)dx$$

defines an inner product on V. Find an orthonormal basis of V.

4. Precisely one of the four matrices specified below is not orthogonal. Which is it? Give a brief justification.

$$\frac{1}{\sqrt{6}}\begin{pmatrix} 1 & -\sqrt{3} & \sqrt{2} \\ 1 & \sqrt{3} & \sqrt{2} \\ -2 & 0 & \sqrt{2} \end{pmatrix}, \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & -2 & -1 \\ 2 & 1 & 2 \end{pmatrix},$$

$$\frac{1}{\sqrt{6}}\begin{pmatrix} 1 & -2 & 1 \\ -\sqrt{6} & 0 & \sqrt{6} \\ 1 & 1 & 1 \end{pmatrix}, \frac{1}{9}\begin{pmatrix} 7 & -4 & -4 \\ -4 & 1 & -8 \\ -4 & -8 & 1 \end{pmatrix}.$$

Given that the four matrices represent transformations of $\mathbb{R}^3$ corresponding (in no particular order) to a rotation, a reflection, a combination of a rotation and a reflexion, and none of these, identify each matrix. Explain your reasoning.

5. Let R be a real orthogonal 3×3 matrix with a real eigenvalue λ corresponding to some real eigenvector. Show that $\lambda = \pm 1$ and interpret this result geometrically. Each of the matrices

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, N = \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix},$$

$$P = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

has and eigenvalue $\lambda = 1$. Confirm this by finding as many independent eigenvectors as possible with this eigenvalue, for each matrix in turn. Show that one of the matrices above represents a rotation, and find the axis and angle of rotation. State, with brief explanations, whether the matrices M, N, P are diagonalisable (i) over the real numbers; (ii) over the complex numbers.

II. Affine space

1. Consider the quadratic surface Q in $\mathbb{R}^3$ defined by

$$2x^2 - 4xy + 5y^2 - z^2 + 6\sqrt{5}y = 0.$$

Find the position of the origin $\tilde{O}$ and orthonormal coordinate basis vectors $\tilde{e}_1$, $\tilde{e}_2$ and $\tilde{e}_3$, for a coordinate system $(\tilde{x}, \tilde{y}, \tilde{z})$ in which Q takes the form

$$\alpha\tilde{x}^2 + \beta\tilde{y}^2 + \gamma\tilde{z}^2 = 1.$$

Also determine the values of α, β and γ, and describe the surface geometrically.

2. Show that the quadric $\mathcal{Q}$ in $\mathbb{R}^3$ defined by

$$3x^2 + 2xy + 2y^2 + 2xz + 2z^2 = 1$$

is an ellipsoid.