# Matemáticas II

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**Abstract**

*Selección de ejercicios de examen de diversas universidades americanas y europeas.*

## I. Ordinary differential equations

1. Solve $\frac{dz}{dt} = z^2$ subject to $z(0) = z_0$. For which $z_0$ is the solution finite for all $t \in \mathbb{R}$?

2. Show that the solution to
$$y'' - 4y' + 3y = te^{-3t},$$
subject to the conditions $y(0) = 0$ and $y'(0) = 1$, is given by
$$y(t) = \frac{37}{72}e^{3t} - \frac{17}{32}e^{t} + \left(\frac{5}{288} + \frac{1}{24}t\right)e^{-3t}$$
when $t \geq 0$.

3. Find the solution of the differential equation
$$y'' - y' - 6y = 0$$
that is bounded as $x \to \infty$ and satisfies $y = 1$ when $x = 0$.

4. Find the general solution of the equation
$$\frac{dy}{dx} - 2y = e^{\lambda x}, \tag{1}$$
where $\lambda$ is a constant not equal to 2. By subtracting from the particular integral an appropriate multiple of the complementary function, obtain the limit as $\lambda \to 2$ of the general solution of (1) and confirm that it yields the general solution for $\lambda = 2$. Solve equation (1) with $\lambda = 2$ and $y(1) = 2$.

5. Legendre's differential equation may be written
$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0,\ y(1) = 1.$$
This equation has a solution $P_n(x)$ that is a polynomial of degree $n$. Find $P_0$, $P_1$ and $P_2$ explicitly. Find the solution when $n = 0$.

6. Solve the equation
$$\ddot{y} - \dot{y} - 2y = 3e^{2t} + 3e^{-t} + 3 + 6t$$
subject to the conditions $y = \dot{y} = 0$ at $t = 0$.

7. Use the transformation $z = \ln x$ to solve
$$\ddot{z} = -\dot{z}^2 - 1 - e^{-z}$$
subject to the conditions $z = 0$ and $\dot{z} = V$ at $t = 0$, where $V$ is a positive constant. Show that then $\dot{z} = 0$
$$z = \ln\left(\sqrt{V^2+4} - 1\right).$$

## II. Systems of ordinary differential equations

1. (*a*) By considering eigenvectors, find the general solution of the equations
$$\frac{dx}{dt} = 2x + 5y,\ \frac{dy}{dt} = -x - 2y, \tag{2}$$
and show that it can be written in the form
$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} 5\cos t \\ -2\cos t - \sin t \end{pmatrix} + \beta \begin{pmatrix} 5\sin t \\ \cos t - 2\sin t \end{pmatrix},$$
where $\alpha$ and $\beta$ are constants.

   (*b*) For any square matrix $M$, $\exp(M)$ is defined by
$$\exp(M) = \sum_{n=0}^{\infty} \frac{M^n}{n!}.$$
Show that if $M$ has constant elements, the vector equation $\frac{dx}{dt} = Mx$ has a solution $x = \exp(Mt)x_0$, where $x_0$ is a constant vector. Hence solve (2) and show that your solution is consistent with the result of part (*a*).