Matemáticas I

GIE

Abstract

Selección de ejercicios de examen de diversas universidades americanas y europeas.

I. Vector spaces and linear transformations

1. Let e_1, e_2 be a basis of $\mathbb{R}^2$. For which values of λ do $\lambda e_1 + e_2$, $e_1 + \lambda e_2$ form a basis of $\mathbb{R}^2$?

2. (a). Consider the transformation $T : \mathbb{R}_n[x] \to \mathbb{R}$, given by $T(p) := \int_0^1 p(s)ds$. Show that T is a linear transformation.

   (b). For the linear transformation T from part (a), you are given the relation
   $$T(x^k) = \int_0^1 x^k dx = \frac{1}{k+1},\ k \geq 0.$$
   Pick a basis for the input space, a basis for the output space, and find the corresponding matrix that represents T.

3. Consider the vector space of polynomials of the form $p(x) = ax^3 + bx^2 + cx + d$, where a, b, c and d can be any real numbers. Are the following subspaces? Explain briefly in a way that we are sure you understand subspaces.

   (a). Those $p(x)$ for which $p(1) = 0$.

   (b). Those $p(x)$ for which $p(0) = 1$.

   (c). Those $p(x)$ for which $a + b = c + d$.

   (d). Those $p(x)$ for which $a^b + b^2 = c^2 + d^2$.

4. Suppose A is the 6×6 matrix
   $$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

   (a). What is the rank of A?

   (b). Give a basis for $\ker(A)$.

5. The vector space S consists of 2×2 matrices whose entries are linear functions of the symbol x. For example, $\begin{pmatrix} x & 2-x \\ 1+x & 4+10x \end{pmatrix}$ is one member of S, and the general form of a member of S is
   $$A = \begin{pmatrix} a+bx & e+fx \\ c+dx & g+hx \end{pmatrix}.$$
   Write down a basis for S.

II. Diagonalization and Jordan form

1. (a) Consider the matrix
   $$M = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}.$$
   Determine whether or not M is diagonalizable.

   (b) Prove that if A and B are similar matrices then A and B have the same eigenvalues with the same corresponding algebraic multiplicities. Is the inverse true? Give either a proof (if true) or a counterexample with a brief reason (if false).

   (c) State the Cayley-Hamilton theorem for a matrix A and prove it in the case when A is a 2×2 diagonalizable matrix.

2. Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

   (a). What are the eigenvalues of A? (Explain briefly.)

   (b). What is the rank of A?

   (c). What is the Jordan form of A? (Explain briefly.)

   (d). Compute in simplest form e^{tA}.