Relational approach to order-of-magnitude reasoning

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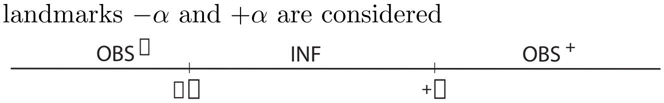
Abstract. This work concentrates on the automated deduction of logics of order-of-magnitude reasoning. Specifically, a translation of the multimodal logic of qualitative order-of-magnitude reasoning into relational logics is provided; then, a sound and complete Rasiowa-Sikorski proof system is presented for the relational version of the language.

1 Introduction

Qualitative order-of-magnitude reasoning has received considerable attention in the recent years; however, the analogous development of a logical approach has received little attention. Various multimodal approaches have been promulgated, for example, for qualitative spatial and temporal reasoning but, as far as we know, the only logic approaches to order-of-magnitude reasoning (OMR) are [1–3].

These first approaches to the logics of qualitative order-of-magnitude reasoning are based on a system with two landmarks, which is both simple enough to keep under control the complexity of the system and rich enough so as to permit the representation of a subset of the usual language of qualitative order-of-magnitude reasoning. The intuitive representation of our underlying frames is given below, in which two landmarks $-\alpha$ and $+\alpha$ are considered



In the picture, $-\alpha$ and $+\alpha$ represent respectively the greatest negative observable and the least positive observable, partitioning the real line in classes of positive observable $\textsc{Obs}^+$, negative observable $\textsc{Obs}^-$ and non-observable (also called infinitesimal) numbers $\textsc{Inf}$. This choice makes sense, in particular, when considering physical metric spaces in which we always have a smallest unit which can be measured; however, it is not possible to identify a least or greatest non-observable number.

In this paper the paradigm 'formulas are relations' formulated in [10] is applied to the modal logic for order-of-magnitude reasoning of [3]. A relational formalisation of logics is based on an observation that a standard relational structure (a Boolean algebra with a monoid) constitutes a common core of a great variety of nonclassical logics. Exhibiting this common core on all the three levels of syntax, semantics and deduction, enables us to create a general framework for representation, investigation and implementation of nonclassical logics. Relational formalization of nonclassical logics is performed on the following methodological levels:

Syntax: With the formal language of a logic L there is associated a language of relational terms.

Semantics and model theory: With logic L there is associated a class of relational models for L and in these models the formulas from L are interpreted as relations.

Proof theory: With logic L there is associated a relational logic $Re(L)$ for L such that its proof system provides a deduction method for L.

In relational representation of logical systems we articulate explicitly information about both their syntax and semantics. Generally speaking, formulas are represented as terms over some appropriate algebras of relations. Each of the propositional connectives becomes a relational operation and in this way an original syntactic form of formulas is coded. Semantic information about a formula which normally is included in a satisfiability condition for that formula, consists of the two basic parts: first, we say which states satisfy the subformulas of the given formula, and second, how those states are related to each other via an accessibility relation. Those two ingredients of semantic information are of course interrelated and unseparable. In relational representation of formulas the terms representing accessibility relations are included explicitly in the respective relational terms corresponding to the formulas. They become the arguments of the relational operations in a term in the same way as the other of its subterms, obtained from subformulas of the given formula. In this way semantic information is provided explicitly on the same level as syntactic information. Thus the relational term corresponding to a formula encodes both syntactic and semantic information about the formula.

In the paper we develop a relational logic $Re(OM)$ based on algebras of relations generated by some relations specific to the frames of OM-logics. We define a translantion from the language of OM-logics to the language of $Re(OM)$. Next, we construct a deduction system for $Re(OM)$ in the Rasiowa-Sikorski style [11]. The Rasiowa-Sikorski systems are dual to the Tableaux systems, as shown in [7, 12]. The system includes the rules of the classical relational logics and the rules specific to the relations from the frames of OM-logics. We present the basic steps of the proof of completeness theorem for this system The modular structure of the system enables us to use the existing implementation of relational proof systems [5] and to include to it the specific rules of $Re(OM)$ logic.

The structure of the paper is the following: The syntax and semantics of the language OM is given in Section 2, then a relational language for order-of-magnitude reasoning, $Re(OM)$, is presented in Section 3. Next, in Section 4 a translation function is given, which transforms a multimodal formula in OM into a relational formula in $Re(OM)$. Then, Section 5 introduces the relational proof system for the logic $Re(OM)$, together with proofs of some axioms of the proof system MQ^N of [3]. The next two sections are devoted to the soundness and completeness of the relational proof system. Finally, some conclusions are presented, together with prospects of future work.

2 The modal language *OM*

In our syntax we consider three types of modal connectives, each one associated to certain order relation: $\overrightarrow{\Box}$ and $\overleftarrow{\Box}$ to deal with the usual ordering $<$, the connectives $\overrightarrow{\blacksquare}$ and $\overleftarrow{\blacksquare}$ to deal with a second ordering $\sqsubset$ and the connectives $\overrightarrow{\boxed{n}}$ and $\overleftarrow{\boxed{n}}$ to deal with a third order relation $\prec$ (the specific conditions required on $\sqsubset$ and $\prec$ will be stated later).
The intuitive meanings of each modal connective is as follows:

$\overrightarrow{\Box}A$ means A *is true for all numbers which are greater than the current one.*
$\overrightarrow{\blacksquare}A$ is read A *is true for all numbers which are greater than and comparable with the current one.*
$\overleftarrow{\Box}A$ means A *is true for all numbers which are less than the current one.*
$\overleftarrow{\blacksquare}A$ means A *is true for all numbers which are less than and comparable with the current one.*
$\overrightarrow{\boxed{n}}A$ means A *is true for all numbers which are from which the current one is negligible.*
$\overleftarrow{\boxed{n}}A$ means A *is true for all numbers which are negligible from the current one.*

The intuitive description of the meaning of the negligibility-related modalities deserves some explaining comments. Depending on the particular context in which we are using the concept of negligibility, several possible definitions can arise. We have chosen to use an intrinsically directional notion of negligibility, in that negligible numbers are always to the left. There are other approaches in which the negligibility relation is bi-directional, so a point x can be negligible wrt points smaller than x and also wrt points greater than x, for instance, in [4, 13] it is the absolute value of an element that is considered before considering the negligibility relation, whereas in [1] yet another definition of bidirectional neglibility is presented.

The syntax of our initial language for qualitative reasoning with comparability and negligibility is introduced below:
The alphabet of the language OM is defined by using:

- A stock of atoms or propositional variables, $\mathcal{V}$.
- The classical connectives $\neg, \wedge, \vee$ and $\rightarrow$ and the constants $\top$ and $\bot$.
- The unary modal connectives $\overrightarrow{\Box}$, $\overleftarrow{\Box}$, $\overrightarrow{\blacksquare}$, $\overleftarrow{\blacksquare}$, $\overrightarrow{\boxed{n}}$ and $\overleftarrow{\boxed{n}}$.
- The constants α^+ and α^-.
- The auxiliary symbols: (,).

Formulas are generated from $\mathcal{V} \cup \{\alpha^+, \alpha^-, \top, \bot\}$ by the construction rules of classical propositional logic adding the following rule: If A is a formula, then so are $\overrightarrow{\Box}A$, $\overleftarrow{\Box}A$, $\overrightarrow{\blacksquare}A$, $\overleftarrow{\blacksquare}A$, $\overrightarrow{\boxed{n}}A$ and $\overleftarrow{\boxed{n}}A$.

The *mirror image* of A is the result of replacing in A each occurrence of $\overrightarrow{\Box}$, $\overleftarrow{\Box}$, $\overrightarrow{\blacksquare}$, $\overleftarrow{\blacksquare}$, $\overrightarrow{\boxed{n}}$, $\overleftarrow{\boxed{n}}$, α^+, α^- by $\overleftarrow{\Box}$, $\overrightarrow{\Box}$, $\overleftarrow{\blacksquare}$, $\overrightarrow{\blacksquare}$, $\overleftarrow{\boxed{n}}$, $\overrightarrow{\boxed{n}}$, α^-, α^+, respectively. We shall use the symbols $\overrightarrow{\Diamond}$, $\overleftarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$, $\overleftarrow{\blacklozenge}$, $\overrightarrow{\Diamond_n}$ and $\overleftarrow{\Diamond_n}$ as abbreviations respectively of $\neg\overrightarrow{\Box}\neg$, $\neg\overleftarrow{\Box}\neg$, $\neg\overrightarrow{\blacksquare}\neg$, $\neg\overleftarrow{\blacksquare}\neg$, $\neg\overrightarrow{\boxed{n}}\neg$ and $\neg\overleftarrow{\boxed{n}}\neg$.

The intended meaning of our language is based on a multi-modal approach, therefore the semantics is given by using the concept of frame.

Definition 1. *A* multimodal qualitative frame *for OM (or, simply, a* frame*) is a tuple $\Sigma = (\mathbb{S}, +\alpha, -\alpha, <, \prec)$, where*

1. $(\mathbb{S}, <)$ *is a linearly ordered set.*
2. $+\alpha$ *and* $-\alpha$ *are designated points in* $\mathbb{S}$ *(called* frame constants*) and allow to form the sets* $\textsc{Obs}^+$*,* $\textsc{Inf}$*, and* $\textsc{Obs}^-$ *that are defined as follows:*

$$\begin{aligned} \textsc{Obs}^- &= \{x \in \mathbb{S} \mid x \leq -\alpha\}; \\ \textsc{Inf} &= \{x \in \mathbb{S} \mid -\alpha < x < +\alpha\}; \\ \textsc{Obs}^+ &= \{x \in \mathbb{S} \mid +\alpha \leq x\} \end{aligned}$$

3. $\prec$ *is a restriction of* $<$*, i.e.* $\prec \subseteq <$*, and satisfies:*
   (i) *If* $x \prec y < z$*, then* $x \prec z$
   (ii) *If* $x < y \prec z$*, then* $x \prec z$
   (iii) *If* $x \prec y$*, then either* $x \notin \textsc{Inf}$ *or* $y \notin \textsc{Inf}$

We will use $x \sqsubset y$ *as an abbreviation of "*$x < y$ *and* $x, y \in \textsc{Eq}$*, where* $\textsc{Eq} \in \{\textsc{Inf}, \textsc{Obs}^+, \textsc{Obs}^-\}$*".*

It is worth noticing that as a consequence of items (i) and (ii) we have the transitivity of $\prec$; on the other hand, item (iii) states that two non-observable elements cannot be compared by the negligibility relation.

Definition 2. *Let Σ be a multimodal qualitative frame, a* multimodal qualitative model on Σ *is an ordered pair* $\mathcal{M} = (\Sigma, h)$*, where h is a* meaning function *(or,* interpretation*)* $h\colon \mathcal{V} \longrightarrow 2^{\mathbb{S}}$*.*

Any interpretation can be uniquely extended to the set of all formulas in OM (also denoted by h) by means of the usual conditions for the classical boolean connectives and the constants $\top$ and $\bot$, and the following conditions for the modal operators and frame constants:

$$
\begin{aligned}
h(\overrightarrow{\Box} A) &= \{x \in \mathbb{S} \mid y \in h(A) \text{ for all } y \text{ such that } x < y\} \\
h(\overrightarrow{\blacksquare} A) &= \{x \in \mathbb{S} \mid y \in h(A) \text{ for all } y \text{ such that } x \sqsubset y\} \\
h(\overrightarrow{\boxed{\scriptstyle\mathrm{n}}} A) &= \{x \in \mathbb{S} \mid y \in h(A) \text{ for all } y \text{ such that } x \prec y\} \\
h(\overleftarrow{\Box} A) &= \{x \in \mathbb{S} \mid y \in h(A) \text{ for all } y \text{ such that } y < x\} \\
h(\overleftarrow{\blacksquare} A) &= \{x \in \mathbb{S} \mid y \in h(A) \text{ for all } y \text{ such that } y \sqsubset x\} \\
h(\overleftarrow{\boxed{\scriptstyle\mathrm{n}}} A) &= \{x \in \mathbb{S} \mid y \in h(A) \text{ for all } y \text{ such that } y \prec x\} \\
h(\alpha^+) &= \{+\alpha\} \\
h(\alpha^-) &= \{-\alpha\}
\end{aligned}
$$

The concepts of truth and validity are defined in a straightforward manner.

3 The relational language *Re*(*OM*)

Syntax of *Re*(*OM*)

The alphabet of the language $Re(OM)$ consists of the disjoint sets listed below:

- A (nonempty) set $\mathbb{OV} = \{x, y, z, \dots\}$ of object variables.
- A set $\mathbb{OC} = \{\alpha^-, \alpha^+\}$ of object constants.
- A (nonempty) set $\mathbb{RV} = \{P, Q, R, \dots\}$ of binary relation variables.
- A set $\mathbb{RC} = \{1, 1', \aleph^-, \aleph^+, <, \sqsubset, \prec\}$ of relation constants.
- A set $\mathbb{OP} = \{-, \cup, \cap, ;, {}^{-1}\}$ of relational operation symbols.

Definition 3.

- *The set of* relation terms $\mathbb{RT}$ *is the smallest set of expressions that includes all the relational variables and relational constants and is closed with respect to the operation symbols from* $\mathbb{OP}$.
- *The set* $\mathbb{FR}$ *of* formulas, *consists of expressions of the form* xRy *where* x, y *denote individual (or object) variables or constants and* R *is a relational term built from the relational variables and the relational operators.*

The defined relations $>, \leq$ and $\geq$ will be used hereafter in order to simplify some relational formulas. The definition of these relations is given as follows:

$$
> := <^{-1} \qquad \leq := < \cup\, 1' \qquad \geq := <^{-1} \cup\, 1'
$$

Semantics of *Re*(*OM*)

A model for $Re(OM)$ is a pair $M = (W, m)$ where $W = W' \cup \{-\alpha, +\alpha\}$ for a nonempty set W', and m is the meaning function $m \colon \mathbb{RV} \to \wp(W \times W)$ such that:

1. Assigns elements of W to object constants as follows:
   (a) $m(\alpha^-) = -\alpha$
   (b) $m(\alpha^+) = +\alpha$
2. Assigns binary relations on W to relation constants as follows:
   For relation constants we should have:
   (a) $m(1) = W \times W$
   (b) $m(1') = \{(w, w) \mid w \in W\}$
   (c) $m(\aleph^-) = \{-\alpha\} \times W$
   (d) $m(\aleph^+) = \{+\alpha\} \times W$
   (e) $m(<)$ is an irreflexive, transitive and linear relation in W satisfying that $(-\alpha, +\alpha) \in m(<)$.
   Notice that the linearity of $m(<)$ allows to divide W into the classes $\textsc{Obs}^-$, $\textsc{Obs}^+$ and $\textsc{Inf}$, defined as in the previous section.
   (f) $m(\sqsubset) = m(<) \cap \big((\textsc{Obs}^- \times \textsc{Obs}^-) \cup (\textsc{Inf} \times \textsc{Inf}) \cup (\textsc{Obs}^+ \times \textsc{Obs}^+)\big)$
   Notice that, as a consequence of this requirement, $m(\sqsubset)$ turns out to inherit irreflexivity, left and right linearity and transitivity from $m(<)$.
   (g) $m(\prec)$ is a restriction of $m(<)$, i.e. $m(\prec) \subseteq m(<)$, which satisfies the following frame conditions:

$$\forall x, \forall y \text{ if } (x, y) \in m(\prec) \text{ and } (y, z) \in m(<), \text{ then } (x, z) \in m(\prec) \quad \textit{(fc-i)}$$
$$\forall x, \forall y \text{ if } (x, y) \in m(<) \text{ and } (y, z) \in m(\prec), \text{ then } (x, z) \in m(\prec) \quad \textit{(fc-ii)}$$
$$\forall x, \forall y \text{ if } x \in \textsc{Inf} \text{ and } (x, y) \in m(\prec), \text{ then } (+\alpha, y) \in m(< \cup 1') \quad \textit{(fc-iii)}$$
$$\forall x, \forall y \text{ if } x \in \textsc{Inf} \text{ and } (y, x) \in m(\prec), \text{ then } (y, -\alpha) \in m(< \cup 1') \quad \textit{(fc-iv)}$$

   Notice that these conditions mimic the requirements (3.i)–(3.iii) in the definition of frame for OM. The required conditions ensure that $m(\prec)$ is irreflexive and transitive.
3. Assigns binary relations on W to relation variables.
4. Assigns operations on binary relations to the relational operation symbols in $\mathbb{OP}$.
5. Extends homomorphically to the set of formulas in the usual manner:
   (a) $m(R \cup S) = m(R) \cup m(S)$ (union of relations)
   (b) $m(R \cap S) = m(R) \cap m(S)$ (intersection of relations)
   (c) $m(R; S) = m(R); m(S)$ (composition of relations)
   (d) $m(-R) = -m(R)$ (opposite relation)
   (e) $m(R^{-1}) = m(R)^{-1}$ (inverse relation)

We list below a set of frame conditions which are entailed by the requirements on the function m and will be used later:

$$\forall x \forall y, (x,y) \in m(\aleph^-) \text{ if and only if } (x,-\alpha) \in m(1') \quad \text{(fc-1)}$$

$$\forall x \forall y, (x,y) \in m(\aleph^+) \text{ if and only if } (x,+\alpha) \in m(1') \quad \text{(fc-2)}$$

$$\forall x, \text{ if } (x,-\alpha) \in m(1') \text{ then } (x,+\alpha) \in m(<) \quad \text{(fc-3)}$$

$$\forall x, \forall y \text{ if } (x,-\alpha) \in m(1') \text{ then } (x,y) \notin m(\sqsubset) \quad \text{(fc-4)}$$

$$\forall x, \forall y \text{ if } (y,+\alpha) \in m(1') \text{ then } (x,y) \notin m(\sqsubset) \quad \text{(fc-5)}$$

$$\forall x \forall y, \text{ if } x \in \textsc{Inf} \text{ and } (x,y) \in m(\sqsubset), \text{ then } (-\alpha,y) \in m(<) \quad \text{(fc-6)}$$

$$\forall x \forall y, \text{ if } x \in \textsc{Inf} \text{ and } (x,y) \in m(\sqsubset), \text{ then } (y,+\alpha) \in m(<) \quad \text{(fc-7)}$$

$$\forall x \forall y, \text{ if } (x,-\alpha) \in m(<) \text{ and } (x,y) \in m(\sqsubset), \text{ then } (y,-\alpha) \in m(< \cup 1') \quad \text{(fc-8)}$$

$$\forall x \forall y, \text{ if } (x,y) \in m(<) \text{ and } (y,-\alpha) \in m(< \cup 1'), \text{ then } (x,y) \in m(\sqsubset) \quad \text{(fc-9)}$$

$$\forall x \forall y, \text{ if } (x,y) \in m(<) \text{ and } (+\alpha,x) \in m(< \cup 1'), \text{ then } (x,y) \in m(\sqsubset) \quad \text{(fc-10)}$$

$$\forall x \forall y, \text{ if } (x,y) \in m(<) \text{ and } x \in \textsc{Inf} \text{ and } y \in \textsc{Inf}, \text{ then } (x,y) \in m(\sqsubset) \quad \text{(fc-11)}$$

$$\forall x, \forall y \text{ if } (x,y) \in m(\sqsubset), \text{ then } (x,y) \in m(<) \quad \text{(fc-12)}$$

Furthermore, it can be proved that the fulfillment of all the frame conditions, plus the requirements of $<$ being irreflexive, linear and transitive entail the properties from 2.c to 2.f in the definition of model. This fact will be used later during the proof of completeness.
Finally, the notions of satisfiability and validity in the relational logic are introduced as follows:

Definition 4.

- *A* valuation *in a model M is a function $v \colon \mathbb{OS} \to W$ such that $v(c) = m(c)$ for all constant symbols.*[4] *We say that v* satisfies *a relational formula xRy if $(v(x), v(y)) \in m(R)$.*
- *A relational formula xRy is* true *in M if every valuation in M satisfies xRy. Moreover, if xRy is true in every model, we say that xRy is* valid *in the relational logic.*

4 Translation from *OM* to *Re*(*OM*)

A translation function t from the language of OM to the language of $Re(OM)$ is introduced in this section.
The translation function $t \colon \Pi \to \mathbb{RV}$ from the set of propositional variables to the set of relational variables is defined for propositional connectives as follows:

$$t(p) := P;1 \qquad t(\neg A) := -t(A)$$
$$t(A \vee B) := t(A) \cup t(B) \qquad t(A \wedge B) := t(A) \cap t(B)$$
$$t(A \to B) := -t(A) \cup t(B)$$

[4] Notice the use of $\mathbb{OS}$ to denote the union of $\mathbb{OV}$ and $\mathbb{OC}$.

For the modal connectives, the translation is based on the general schema, which translates a modality based on a relation R as follows:

$$t(\langle R\rangle A) := R; t(A) \qquad t([R]A) := -(R; -t(A))$$

Specifically, in our case we have the following for the future connectives (for the past connectives the translation is similar):

- $t(\overrightarrow{\Diamond} A) := <; t(A)$
- $t(\overrightarrow{\Box} A) := -(<; -t(A))$
- $t(\overrightarrow{\blacklozenge} A) := \sqsubset; t(A)$
- $t(\overrightarrow{\blacksquare} A) := -(\sqsubset; -t(A))$
- $t(\overrightarrow{\Diamond_{n}} A) := \prec; t(A)$
- $t(\overrightarrow{\boxed{n}} A) := -(\prec; -t(A))$

Finally, the α-constants are translated, as expected, into the $\aleph$-relational constants:

$$t(\alpha^-) = \aleph^- \qquad t(\alpha^+) = \aleph^+$$

Proposition 1. *In relational logic Re(OM) we can verify both validity and entailment of logic OM, namely*

1. *A formula A of logic OM is valid iff a formula $x\, t(A)\, y$ of the corresponding logic Re(OM) is valid, where x, y are any object variables such that $x \neq y$,*
2. *$A_1, \ldots, A_n \models A$ in OM iff $x\, \big(1; -(t(A_1) \cap \cdots \cap t(A_n)); 1 \cup t(A)\big)\, y$ is valid in Re(OM).*

5 Relational proof systems for modal *Re*(*OM*)

Relational proofs have the form of finitely branching trees whose nodes are finite sets of formulas. Given a relational formula xAy, where A may be a compound relational expression, we successively apply decomposition or specific rules. In this way we form a tree whose root consists of xAy and each node (except the root) is obtained by an application of a rule to its predecessor node. We stop applying rules to formulas in a node after obtaining an axiomatic set, or when none of the rules is applicable to the formulas in this node. Such a tree is referred to as a proof tree for the formula xAy. A branch of a proof tree is said to be closed whenever it contains a node with an axiomatic set of formulas. A tree is closed iff all of its branches are closed.

5.1 Rules for the calculus of binary relations with equality

In the present section we, first, recall the deduction rules for the classical relational logic [9], that is the logic whose formulas xAy are built from the terms A generated by relation variables and constants 1 and $1'$ with the standard relational operations of union, intersection, complement, composition and converse. Second, we define the specific rules that characterise the specific relations of *Re*(*OM*). The rules apply to finite sets of relational formulas. As usual, we omit the set brackets when presenting the rules. The rules that refer to relational operations are decomposition rules. They enable us to decompose a formula in a set into some simpler formulas. As a result of decomposition we usually obtain finitely many new sets of formulas. The rules that encode properties of relational or object constants are referred to as specific rules. They enable us to modify a set to which they are applied, they have a status of structural rules. The role of axioms is played by what is called axiomatic sets. A variable is said to be restricted in a rule whenever it does not appear in any formula of the upper set in that rule.

A rule is said to be correct in $Re(OM)$ whenever the following holds: the upper set of formulas in the rule is valid iff all the lower sets are valid, where the validity of a finite set of formulas is understood as a validity of the (metalevel) disjunction of its elements. It follows that the branching in a rule is interpreted as conjunction.

As usual, we present the rules in a form of schemes. A scheme of the form A/B, where A and B are finite sets of formulas represents a family of rules $\Gamma \cup A / \Gamma \cup B$ for any finite set Γ of formulas, and similarly for the branching rules.

We recall here the standard rules for the calculus of binary relations. Note that the comma is interpreted disjunctively, whereas the vertical bar is interpreted conjunctively.

Firstly, we consider the rules for $\cup$:

$$\frac{x(R \cup S)y}{xRy, xSy}\ (\cup) \qquad \frac{x{-}(R \cup S)y}{x{-}Ry \mid x{-}Sy}\ (-\cup)$$

Rules for $\cap$

$$\frac{x(R \cap S)y}{xRy \mid xSy}\ (\cap) \qquad \frac{x{-}(R \cap S)y}{x{-}Ry, x{-}Sy}\ (-\cap)$$

Rules for double complement and inverse relation

$$\frac{x{-}{-}Ry}{xRy}\ ({-}{-}) \qquad \frac{xR^{-1}y}{yRx}\ ({}^{-1}) \qquad \frac{x{-}R^{-1}y}{y{-}Rx}\ ({-}^{-1})$$

Now, we state the rules for the composition

$$\frac{x(R;S)y}{xRz, x(R;S)y \mid zSy, x(R;S)y}\ z \text{ any variable} \quad (;)$$

$$\frac{x{-}(R;S)y}{x{-}Rz, z{-}Sy}\ z \text{ new variable} \quad (-;)$$

Finally, the rules for equality are introduced, where z is any variable

$$\frac{xRy}{xRz, xRy \mid y1'z, xRy}\ (1'\text{-}1) \qquad \frac{xRy}{x1'z, xRy \mid zRy, xRy}\ (1'\text{-}2)$$

5.2 Specific Rules for $Re(OM)$

Here we introduce the rules for handling the specific object constants and relation symbols $<, \sqsubset$ and $\prec$ of the language $Re(OM)$.
The rules below interpret adequately the behaviour of the relation constants $\aleph^-$ and $\aleph^+$:

$$\frac{x\aleph^- y}{x1'\alpha^-, x\aleph^- y}\ (\mathbf{c1a}) \qquad \frac{x{-}\aleph^- y}{x{-}1'\alpha^-, x{-}\aleph^- y}\ (\mathbf{c1b})$$

$$\frac{x\aleph^+ y}{x1'\alpha^+, x\aleph^+ y}\ (\mathbf{c2a}) \qquad \frac{x{-}\aleph^+ y}{x{-}1'\alpha^+, x{-}\aleph^+ y}\ (\mathbf{c2b})$$

The following rule expresses that α^- precedes α^+

$$\frac{x < \alpha^+}{x1'\alpha^-, x < \alpha^+}\ (\mathbf{c3})$$

The remaining rules are stated below. The numbering of the rules reflects their relationship with the corresponding frame conditions:

$$\frac{x{-}\sqsubset y}{x1'\alpha^-, x{-}\sqsubset y}\ (\mathbf{c4}) \qquad \frac{x{-}\sqsubset y}{y1'\alpha^+, x{-}\sqsubset y}\ (\mathbf{c5})$$

$$\frac{x \le \alpha^-, \alpha^+ \le x, x{-}\sqsubset y}{x \le \alpha^-, \alpha^+ \le x, x{-}\sqsubset y, y \le \alpha^-}\ (\mathbf{c6}) \qquad \frac{x \le \alpha^-, \alpha^+ \le x, x{-}\sqsubset y}{x \le \alpha^-, \alpha^+ \le x, x{-}\sqsubset y, \alpha^+ \le y}\ (\mathbf{c7})$$

$$\frac{\alpha^- \le x, x{-}\sqsubset y}{\alpha^- \le x, x{-}\sqsubset y, \alpha^- < y}\ (\mathbf{c8})$$

$$\frac{x{-}< y, \alpha^- < y}{x{-}< y, \alpha^- < y, x{-}\sqsubset y}\ (\mathbf{c9}) \qquad \frac{x{-}< y, x < \alpha^+}{x{-}< y, x < \alpha^+, x{-}\sqsubset y}\ (\mathbf{c10})$$

$$\frac{x \le \alpha^-, \alpha^+ \le x, y \le \alpha^-, \alpha^+ \le y, x \sqsubset y}{x \le \alpha^-, \alpha^+ \le x, y \le \alpha^-, \alpha^+ \le y, x \sqsubset y, x < y}\ (\mathbf{c11}) \qquad \frac{x{-}\sqsubset y}{x{-}\sqsubset y, x{-}{<}y}\ (\mathbf{c12})$$

We include below the rules for irreflexivity and linearity properties of the relation constant $<$.

$$\frac{}{x < x}\ (\mathbf{Iref}) \qquad \frac{}{y{-}{<}x \mid x{-}{<}y \mid x{-}1'y}\ (\mathbf{Lin})$$

The transitivity for the three relation constants is stated by the rule below, where $R \in \{<, \sqsubset, \prec\}$

$$\frac{xRy}{xRy, xRz, \mid xRy, zRy}\ (\mathbf{Tran})$$

The following cut-like rule will be needed later in the proof of completeness

$$\frac{}{x \sqsubset y \mid x -\!\sqsubset y} \ (\textbf{cut-} \sqsubset)$$

Finally, the following rules for $\prec$ reflect the frame conditions for negligibility:

$$\frac{x < y}{x \prec y, x < y} \ (\textbf{n-0})$$

$$\frac{x \prec z}{x \prec y, x \prec z \mid y < z, x \prec z} \ y \text{ any var} \quad (\textbf{n-i})$$

$$\frac{x \prec z}{x < y, x \prec z \mid y \prec z, x \prec z} \ y \text{ any var} \quad (\textbf{n-ii})$$

$$\frac{\alpha^+ \leq y}{\alpha^- < x, \alpha^+ \leq y \mid x < \alpha^+, \alpha^+ \leq y \mid x \prec y, \alpha^+ \leq y} \ (\textbf{n-iii})$$

$$\frac{y \leq \alpha^-}{\alpha^- < x, y \leq \alpha^- \mid x < \alpha^+, y \leq \alpha^- \mid y \prec x, y \leq \alpha^-} \ (\textbf{n-iv})$$

Axiomatic sets

An axiomatic set is any finite set of formulas which includes a subset of either of the following forms for a relational term R and x, y are any object variables. We have to introduce schemas of axiomatic sets for the universal relation, the identity relation and linearity, together with others which allow us to adequately interpret the constant relation symbols $\aleph$, together with the symbols $\pm\alpha$.
The axiomatic sets of $Re(OM)$ state valid formulas of the system, the following postulate the behaviour of the universal relation 1 and the equality relation $1'$, the *tertium non datur*, and the conditions related to the constant symbols α^- and α^+ are expressed by

$$\{x1y\} \qquad \{x1'x\} \qquad \{x{-}Ry, xRy\} \qquad \{\alpha^- < \alpha^+\}$$

where $x, y \in \mathbb{OS}$ and $R \in \mathbb{RT}$.

5.3 Proving some axioms of MQ^N

In this section we show the relational proof system at work, and prove some of the axioms of the system MQ^N of qualitative order-of-magnitude reasoning presented in [3].

Example 1. Axiom (c4): $\alpha^- \to \overrightarrow{\blacksquare} A$
The translated version of the axiom in the relational language is

$$-\aleph^- \cup -(\sqsubset; -(A;1))$$

We consider $x(-\aleph^- \cup -(\sqsubset; -(A;1)))y$, apply the rule $(\cup)$, and then, the following tree is generated:

$$\dfrac{\boxed{x -\aleph^- y}, x - (\sqsubset; -(A;1))y}{x -\aleph^- y, x -1'\alpha^-, \boxed{x - (\sqsubset; -(A;1))y}}\ (c1b)$$

$$\dfrac{x -\aleph^- y, x -1'\alpha^-, \boxed{x - (\sqsubset; -(A;1))y}}{x -\aleph^- y, x -1'\alpha^-, x- \sqsubset z, \boxed{--z(A;1)y}}\ (-;)\text{ any } z$$

$$\dfrac{x -\aleph^- y, x -1'\alpha^-, x- \sqsubset z, \boxed{--z(A;1)y}}{x -\aleph^- y, x -1'\alpha^-, x- \sqsubset z, \boxed{z(A;1)y}}\ (--)$$

$$\dfrac{x -\aleph^- y, x -1'\alpha^-, x- \sqsubset z, \boxed{z(A;1)y}}{\Gamma, zAw \mid \Gamma, \boxed{w1y}}\ (;)\ w \text{ new}$$

where $\Gamma = x -\aleph^- y, x -1'\alpha^-, x- \sqsubset z$.
The right branch closes because of $w1y$, whereas rule $(c4)$ applies to the left branch against $x -1'\alpha^-$, obtaining

$$\boxed{x-1'\alpha^-}, \boxed{x1'\alpha^-}, x- \sqsubset z, zAw$$

which closes.

Example 2. Axiom (c1): $\overleftarrow{\Diamond}\alpha^- \vee \alpha^- \vee \overrightarrow{\Diamond}\alpha^-$

$$\dfrac{\boxed{x(>;\aleph^-)y}, x\aleph^- y, x(<;\aleph^-)y}{x < \alpha^-, x\aleph^- y, x(>;\aleph^-)y, x(<;\aleph^-)y \mid \alpha^-\aleph^- y, x\aleph^- y, x(>;\aleph^-)y, x(<;\aleph^-)y}\ (;)[z/\alpha^-]$$

Note that the right branch closes, since it contains an axiomatic set for $\aleph^-$. On the other hand, the left branch continues as follows, where we use Γ to denote the pair of formulas $x(>;\aleph^-)y, x(<;\aleph^-)y$

$$\dfrac{x < \alpha^-, x\aleph^- y, x(>;\aleph^-)y, \boxed{x(<;\aleph^-)y}}{x < \alpha^-, \boxed{x\aleph^- y}, x > \alpha^-, \Gamma \mid x < \alpha^-, x\aleph^- y, \boxed{\alpha^-\aleph^- y}, \Gamma}\ (;)[z/\alpha^-]$$

the left branch closes after applying (c1a) and linearity, whereas the right branch closes because of the axiomatic set for $\aleph$.

6 Soundness of the relational proof system

Recall that a rule is said to be correct if the validity of the upper set entails the validity of the lower set and vice versa.
The frame conditions stated in Section 3 are used here in order to prove soundness of the relational proof system: we will show the equivalence between the correctness of the specific rules of *Re*(*OM*) and the validity of the corresponding frame conditions. As a result, since all the frame conditions hold in every model of *Re*(*OM*), we get that the specific rules of *Re*(*OM*) are all correct.

Proposition 2.

1. *For $k \in \{1, 2\}$, rules $(ck\,a)$ and $(ck\,b)$ are correct for a deduction system of $Re(OM)$ iff in every model of $Re(OM)$ condition (fc-k) is satisfied.*
2. *For $k \in \{3, \ldots, 12\}$, rule $(c\,k)$ is correct for a deduction system of $Re(OM)$ iff in every model of $Re(OM)$ condition (fc-k) is satisfied.*
3. *For $j \in \{i, ii, iii, iv\}$, rule $(n\,j)$ is correct for a deduction system of $Re(OM)$ iff in every model of $Re(OM)$ condition (fc-j) is satisfied.*

Proof. 1. Let us prove the case of (fc-2), since the other is similar:
Assume that the rules are correct and, and let us prove the two implications which form the frame condition. We proceed by contradiction and consider that the frame condition

$$\forall x \forall y, (x, y) \in m(\aleph^+) \text{ if and only if } (x, +\alpha) \in m(1') \qquad \text{(fc-2)}$$

does not hold.
Reasoning by cases, on the one hand, suppose that for some objects a, b we have $(a, +\alpha) \in m(1')$ and $(a, b) \notin m(\aleph^+)$. Consider the following instance of rule $(c2a)$, in which we add the context $\Gamma = x{-}1'\alpha^+$ to both the upper and lower sets:

$$\frac{x\aleph^+y, x{-}1'\alpha^+}{x\aleph^+y, x1'\alpha^+, x{-}1'\alpha^+}$$

The lower set is valid, so since the rule is correct, the upper set must be valid, that is, the formula $\forall x \forall y (x\aleph^+y \vee x{-}1'\alpha^+)$ is valid in first order logic. But the valuation v such that $v(x) = a$ and $v(y) = b$ is a counterexample.
On the other hand, suppose conversely that for some objects a, b we have $(a, b) \in m(\aleph^+)$ and $(a, +\alpha) \notin m(1')$. Consider the following instance of rule $(c2b)$, in which we add the context $\Gamma = x1'\alpha^+$ to both the upper and lower sets:

$$\frac{x1'\alpha^+, x{-}\aleph^+y}{x1'\alpha^+, x{-}1'\alpha^+, x{-}\aleph^+y}$$

The lower set is valid, so since the rule is correct, the upper set must be valid, that is, the formula $\forall x \forall y (x1'\alpha^+ \vee x{-}\aleph^+y)$ is valid in first order logic. But the valuation v such that $v(x) = a$ and $v(y) = b$ is a counterexample.
Reciprocally, assume the validity of the frame condition (fc-1^+) and let us prove that both rules (c1^+a) and (c1^+b) are correct. Clearly, validity of the upper set of the rules implies validity of the lower set. Now, assuming validity of the lower set, validity of the upper set follows easily from the frame condition.

2. For $k = 3$.
Assume that the rule is correct and suppose that (fc-3) does not hold, i.e., for some object a we have $(a, -\alpha) \in 1'$ and $(a, +\alpha) \notin <$. Consider the following instance of rule $(c3)$

$$\frac{x < \alpha^+, x{-}1'\alpha^-}{x1'\alpha^-, x < \alpha^+, x{-}1'\alpha^-}$$

Clearly, the lower set is valid, so since the rule is correct, the upper set must be valid. This means that the formula $\forall x(x < \alpha^+ \vee x{-}1'\alpha^-)$ is valid in first order logic. But the valuation v such that $v(x) = a$ does not satisfy that formula, a contradiction.
Reciprocally, assume (fc-3). Validity of the upper set of the rule implies validity of the lower set. Assuming validity of the lower set, validity of the upper set follows from the frame condition.
The proof for the rest of the cases is similar, we just introduce the context to be used when considering the instance for the corresponding rule.
For $k = 4$, assume $\Gamma = x(-1')\alpha^-$.
For $k = 5$, assume $\Gamma = y(-1')\alpha^+$.
For $k = 6$, assume $\Gamma = \alpha^- < y$.
For $k = 7$, assume $\Gamma = y < \alpha^+$.
For $k = 8$, assume $\Gamma = y \leq \alpha^-$.
For $k = 9, 10, 11$, assume $\Gamma = x \sqsubset y$.
For $k = 12$, assume $\Gamma = x < y$.

3. For $j = 0$, the context $\Gamma = x{-}{\prec}\, y$ proves that the rule (n-0) is correct if and only if $\prec$ is a restriction of $<$.
For $j = i$, take the context $x{-}{\prec}\, y, y{-}{<}\, z$.
For $j = ii$, consider $\Gamma = x{-}{<}\, y, y{-}{\prec}\, z$
For $j = iii$, assume $\Gamma = x \leq \alpha^-, \alpha^+ < x, x{-}{\prec}\, y$.
For $j = iv$, assume $\Gamma = x \leq \alpha^-, \alpha^+ < x, y{-}{\prec}\, x$.

□

The rest of the rules are the standard ones for defining properties related of order relations and the equality. As a result, we have the following proposition:

Proposition 3.

1. *All the rules of the deduction system for $Re(OM)$ are correct.*
2. *All the axiomatic sets are valid.*

The soundness theorem follows from the correcteness of the rules and from validity of the axiomatic sets of the system.

Proposition 4 (Soundness). *If there is a closed proof tree for a formula xAy, then xAy is valid.*

7 Completeness of the relational proof system

A completeness proof for dual tableaux systems involves a notion of a complete proof tree. Intuitively, a proof tree is complete if all the rules that can be applied to its nodes have been applied. A non-closed branch b of a proof tree is complete whenever it satisfies some appropriate completion conditions. The conditions say that given a rule applicable to a node of b, there is a node on b which contains a set of formulas resulting from an application of that rule.

Completion conditions. A non-closed branch b of a proof tree is said to be complete whenever for all $x, y \in \mathbb{OS}$ it satisfies the completion conditions on Table 1.
It is known that any proof tree can be extended to a complete proof tree. A complete and non-closed branch is said to be open.
Let b be an open branch of a proof tree. We define a branch structure $M^b = (W^b, m^b)$:

$$W^b = \mathbb{OV} \cup \mathbb{OC}$$
$$m^b(R) = \{(x,y) \in W^b \times W^b \mid xRy \notin b\} \text{ for } R \in \mathbb{RV} \cup \mathbb{RC}$$
$$m^b(\alpha^+) = \alpha^+, \qquad m^b(\alpha^-) = \alpha^-$$

and m^b extends homomorphically to all the relation terms.
Let $v^b \colon \mathbb{OV} \to W^b \setminus \mathbb{OC}$ be an identity valuation, i.e., $v^b(x) = x$ for every object variable x.
Throughout the paper we shall often write R^b for $m^b(R)$.

Note that, as in the case of first order logic with equality, the relation $1'^b$ can only be proved to be an equivalence relation.

Lemma 1. *The relation $1'^b$ is an equivalence relation.*

Proof. $1'^b$ is reflexive: We have $x1'x \notin b$ (otherwise b would be closed) which means, by definition of m^b, that $(x,x) \in 1'^b$.
$1'^b$ is symmetric: In order to reach a contradiction, consider $x, y \in W^b$ such that $(x,y) \in 1'^b$ but $(y,x) \notin 1'^b$, then by definition of m^b we have both $x1'y \notin b$ and $y1'x \in b$. Now from the completion condition (cpl $1'$-1), we have either $y1'y \in b$ or $x1'y \in b$. Since b is open, we obtain $x1'y \in b$, a contradiction.
$1'^b$ is transitive: Consider $x, y, z \in W^b$ such that $(x,y) \in 1'^b, (y,z) \in 1'^b$ and $(x,z) \notin 1'^b$, which means, by definition of m^b, that $x1'y \notin b$, $y1'z \notin b$ and $x1'z \in b$. Given $x1'z \in b$, from the completion condition (cpl $Tran$) we have either $x1'y \in b$ or $y1'z \in b$ and we reach a contradiction in both cases. □

In order to obtain the expected behaviour of $1'^b$ as the equality relation, we consider a quotient model $[M^b]_{1'^b} = \big([W^b]_{1'^b}, n\big)$ where:

- $[W^b]_{1'^b}$ is the set of equivalence classes of W^b wrt $1'^b$.
- $n(R) = \big\{([x]_{1'^b}, [y]_{1'^b}) \mid (x,y) \in R^b\big\}$ for $R \in \mathbb{RT}$.
- Valuation u in $[M^b]_{1'^b}$ is such that $u(x) = [x]_{1'^b}$.

Now, we have the following proposition:

Proposition 5.

1. *For every formula xAy, $[M^b]_{1'^b}, u \models xAy$ iff $M^b, v^b \models xAy$.*
2. *$[M^b]_{1'^b}$ is a model of $Re(OM)$.*

(cpl $\cup$**)** If $x(R \cup S)y \in b$, then both $xRy \in b$ and $xSy \in b$

(cpl $-\cup$**)** If $x-(R \cup S)y \in b$, then either $x - Ry \in b$ or $x - Sy \in b$

(cpl $\cap$**)** If $x(R \cap S)y \in b$, then either $xRy \in b$ or $xSy \in b$

(cpl $-\cap$**)** If $x-(R \cap S)y \in b$, then both $x - Ry \in b$ and $x - Sy \in b$

(cpl $--$**)** If $x - - Ry \in b$, then $xRy \in b$

(cpl $^{-1}$**)** If $xR^{-1}y \in b$, then $yRx \in b$

(cpl $-^{-1}$**)** If $x - R^{-1}y \in b$, then $y - Rx \in b$

(cpl ;**)** If $x(R;S)y \in b$, then for every $z \in \mathbb{OS}$, either $xRz \in b$ or $zSy \in b$

(cpl $-$;**)** If $x-(R;S) \in b$, then for some $z \in \mathbb{OS}$ both $x - Rz \in b$ and $z - Sy \in b$

(cpl $1'$**-1)** If $xRy \in b$, then for every $z \in \mathbb{OS}$ either $xRz \in b$ or $y1'z \in b$

(cpl $1'$**-2)** If $xRy \in b$, then for every $z \in \mathbb{OS}$ either $x1'z \in b$ or $zRy \in b$

(cpl $c1a$**)** If $x\aleph^{-}y \in b$ then $x1'\alpha^{-} \in b$

(cpl $c1b$**)** If $x{-}\aleph^{-}y \in b$, then $x{-}1'\alpha^{-} \in b$

(cpl $c2a$**)** If $x\aleph^{+}y \in b$ then $x1'\alpha^{+} \in b$

(cpl $c2b$**)** If $x{-}\aleph^{+}y \in b$, then $x{-}1'\alpha^{+} \in b$

(cpl $c3$**)** If $x < \alpha^{+} \in b$ then $x1'\alpha^{-} \in b$

(cpl $c4$**)** If $x - \sqsubset y \in b$ then $x1'\alpha^{-} \in b$

(cpl $c5$**)** If $x - \sqsubset y \in b$ then $y1'\alpha^{+} \in b$

(cpl $c6$**)** If $x \le \alpha^{-} \in b$, $\alpha^{+} \le x \in b$ and $x - \sqsubset y \in b$, then $y \le \alpha^{-} \in b$

(cpl $c7$**)** If $x \le \alpha^{-} \in b$, $\alpha^{+} \le x \in b$ and $x - \sqsubset y \in b$, then $\alpha^{+} \le y \in b$

(cpl $c8$**)** If $\alpha^{-} \le x \in b$ and $x - \sqsubset y \in b$, then $\alpha^{-} < y \in b$

(cpl $c9$**)** If both $x - < y \in b$ and $\alpha^{-} < y \in b$, then $x - \sqsubset y \in b$

(cpl $c10$**)** If both $x - < y \in b$ and $x < \alpha^{+} \in b$, then $x - \sqsubset y \in b$

(cpl $c11$**)** If $x \le \alpha^{-} \in b, \alpha^{+} \le x \in b, y \le \alpha^{-} \in b, \alpha^{+} \le y \in b$ and $x \sqsubset y \in b$ then $x < y \in b$,

(cpl $c12$**)** If $x - \sqsubset y \in b$, then $x - < y \in b$

(cpl cut-$\sqsubset$**)** Either $x \sqsubset y \in b$ or $x - \sqsubset y \in b$

(cpl *n-0*) If $x < y \in b$, then $x \prec y \in b$

(cpl *n-i*) If $x \prec z \in b$, then for every $y \in \mathbb{OS}$ either $x \prec y \in b$ or $y < z \in b$

(cpl *n-ii*) If $x \prec z \in b$, then for every $y \in \mathbb{OS}$ either $x < y \in b$ or $y \prec z \in b$

(cpl *n-iii*) If $\alpha^{+} \le y \in b$, then $\alpha^{-} < x \in b$ or $x < \alpha^{+} \in b$ or $x \prec y \in b$

(cpl *n-iv*) If $y \le \alpha^{-} \in b$, then $\alpha^{-} < x \in b$ or $x < \alpha^{+} \in b$ or $y \prec x \in b$

(cpl $Iref$**)** $x < x \in b$

(cpl $Tran$**)** If $xRy \in b$, then for every $z \in \mathbb{OS}$, either $xRz \in b$ or $zRy \in b$ (where $R \in \{<, \sqsubset, \prec\}$).

(cpl Lin**)** Either $x - < y \in b$ or $x - 1'y \in b$ or $y - < x \in b$

Table 1. Completion conditions.

Proof.
1. This condition is easily verified using the corresponding definitions.
2. We only give the proofs for some conditions on the model; the proofs of the remaining conditions are similar.

(a) $n(1) = [W^b]_{1'^b} \times [W^b]_{1'^b}$
Since b is open, $x1y \notin b$ for all $x, y \in \mathbb{OS}$. So, by definition of m^b, we get $(x, y) \in m^b(1)$; note that this means that $M^b, v^b \models x1y$. Now, by the item 1 above, we have $[M^b]_{1'^b}, u \models x1y$. Hence $([x]_{1'^b}, [y]_{1'^b}) \in n(1)$.
(c) $n(\aleph^-) = \{[\alpha^-]_{1'^b}\} \times [W^b]_{1'^b}$
We have that

$$\begin{aligned} ([x]_{1'^b}, [y]_{1'^b}) \in n(\aleph^-) &\text{ if and only if } [M^b]_{1'^b}, u \models x\aleph^- y \\ &\text{ if and only if } M^b, v^b \models x\aleph^- y \text{ (by item 1 above)} \\ &\text{ if and only if } (x, y) \in m^b(\aleph^-) \\ &\text{ if and only if } x\aleph^- y \notin b. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} [x]_{1'^b} \neq [\alpha^-]_{1'^b} &\text{ if and only if } ([x]_{1'^b}, [\alpha^-]_{1'^b}) \notin n(1) \\ &\text{ if and only if } [M^b]_{1'^b}, u \not\models x1'\alpha^- \text{ (by item 1 above)} \\ &\text{ if and only if } M^b, v^b \not\models x1'\alpha^- \\ &\text{ if and only if } x1'\alpha^- \in b. \end{aligned}$$

If either $n(\aleph^-) \subset \{[\alpha^-]_{1'^b}\} \times [W^b]_{1'^b}$ or $n(\aleph^-) \supset \{[\alpha^-]_{1'^b}\} \times [W^b]_{1'^b}$would not hold, then completion conditions (cpl c1a) and (cpl c1b) would generate a contradiction.

In the proofs of the remaining conditions we shall abuse the notation and the symbols of quotient classes will not be written, and moreover, we shall write A^b instead of $n(A)$, and W^b instead of $[W^b]_{1'^b}$.

fc-3 Let us show that $\forall x \in W^b,$ if $(x, \alpha^-) \in 1'^b$ then $(x, \alpha^+) \in <^b$.
Assume that $(x, \alpha^-) \in 1'^b$ and suppose that $(x, \alpha^+) \notin <^b$. By definition of m^b we get $x1'\alpha^- \notin b$ and $x < \alpha^+ \in b$. From the completion condition (cpl c2) we get $x1'\alpha^- \in b$. Hence $(x, \alpha^-) \notin 1'^b$, a contradiction.
fc-4 We show that $\forall x, y \in W^b$ if $(x, \alpha^-) \in 1'^b,$ then $(x, y) \notin \sqsubset^b$.
Assume that $(x, \alpha^-) \in 1'^b$ and $(x, y) \in \sqsubset^b$. By definition of m^b, we have $x1'\alpha^- \notin b$ and $x \sqsubset y \notin b$. From the completion condition (clp $c4$) we get $x- \sqsubset y \notin b$, and by the completion condition (cpl cut-$\sqsubset$) we have $x \sqsubset y \in b$, so by definition of m^b we obtain $(x, y) \notin \sqsubset^b$, a contradiction.
Condition *fc*-5 can be proved in a similar way as condition above.
fc-6 $\forall x, y \in W^b$ if $x \in \textsc{Inf}^b$ and $(x, y) \in \sqsubset^b,$ then $(\alpha^-, y) \in <^b$.
Assume that $(\alpha^-, x) \in <^b, (x, \alpha^+) \in <^b$ (that is, $x \in \textsc{Inf}^b$) and $(x, y) \in \sqsubset^b$. Suppose also that $(\alpha^-, y) \notin <^b$. By definition of m^b, we get $\alpha^- < x \notin b$, $x < \alpha^+ \notin b, x \sqsubset y \notin b$ and $\alpha^- < y \in b$. Now we have $y \leq \alpha^- \notin b$

(otherwise b should be closed). From the completion condition (clp $c6$) we obtain $x \leq \alpha^- \notin b$ or $\alpha^+ \leq x \notin b$ or $x- \sqsubset y \notin b$. From the completion condition (cpl cut-$\sqsubset$) we get $x \sqsubset y \notin b$ and, by definition of m^b, we have that $(x, \alpha^-) \in \leq^b$ or $(\alpha^+, x) \in \leq^b$ or $(x, y) \notin \sqsubset^b$. In any case we easily reach a contradiction.

Condition *fc*-7 can be proved in a similar way as condition above.

fc-8 $\forall x, y \in W^b,$ if $(x, \alpha^-) \in <^b$ and $(x, y) \in \sqsubset^b,$ then $(y, \alpha^-) \in \leq^b$.
Assume that $(x, \alpha^-) \in <^b$ and $(x, y) \in \sqsubset^b$. Suppose also that $(y, \alpha^-) \notin \leq^b$. By definition of m^b, we get $\alpha^- < x \notin b, x < \alpha^+ \notin b, x \sqsubset y \notin b$ and $y \leq \alpha^- \in b$. Now we have $\alpha^- < y \notin b$ (otherwise b should be closed). From the completion condition (clp $c8$) we get $\alpha^- \leq x \notin b$ or $x- \sqsubset y \notin b$. Hence by completion condition (cpl cut-$\sqsubset$) and definition of m^b, we have that $(\alpha^-, x) \in \leq^b$ or $(x, y) \notin \sqsubset^b$. In any case we get a contradiction with the hypothesis.

fc-9 $\forall x, y \in W^b,$ if $(x, y) \in <^b$ and $(y, \alpha^-) \in \leq^b,$ then $(x, y) \in \sqsubset^b$.
Suppose that $(x, y) \in <^b$ and $(y, \alpha^-) \in \leq^b$. In order to reach a contradiction assume also that $(x, y) \notin \sqsubset^b$. Then $x \sqsubset y \in b$, i.e., $x- \sqsubset y \notin b$ (otherwise b should be closed). From the completion condition (clp $c9$) we get $x- < y \notin b$ or $\alpha^- < y \notin b$. Thus, by definition of m^b, we have $(x, y) \notin <^b$ or $(\alpha^-, y) \in <^b$. In any case we get a contradiction.

Condition *fc*-10 can be proved in a similar way as condition above.

fc-11 $\forall x, y \in W^b,$ if $(x, y) \in <^b, x \in \textsc{Inf}^b$ and $y \in \textsc{Inf}^b,$ then $(x, y) \in \sqsubset^b$.
Assume $(x, y) \in <^b, (\alpha^-, x) \in <^b, (x, \alpha^+) \in <^b, (\alpha^-, y) \in <^b$ and $(y, \alpha^+) \in <^b$. We have to prove that $(x, y) \in \sqsubset^b$. Then $x < y \notin b, \alpha^- < x \notin b, x < \alpha^+ \notin b, \alpha^- < y \notin b$ and $y < \alpha^+ \notin b$. If it were $(x, y) \notin \sqsubset^b$, then $x \sqsubset y \in b$. From the completion condition (clp $c8$) we get $x < y \in b$, a contradiction.

fc-12 $\forall x, y \in W^b,$ if $(x, y) \in \sqsubset^b,$ then $(x, y) \in <^b$.
Assume that $(x, y) \in \sqsubset^b$ and $(x, y) \notin <^b$. Then by definition of m^b, we get $x \sqsubset y \notin b$ and $x < y \in b$. From the completion condition (cpl cut-$\sqsubset$), we obtain $x - \sqsubset y \in b$. Now by the completion condition (cpl $c12$) we get $x - < y \in b$. As a result b is closed, a contradiction.

fc-0 $\forall x, y \in W^b,$ if $(x, y) \in \prec^b,$ then $(x, y) \in <^b$.
Assume that $(x, y) \in \prec^b$. Thus by definition of m^b we have $x \prec y \notin b$. If it were $(x, y) \notin <^b$ also, then by definition of m^b again, we obtain $x < y \in b$. Then by the completion condition (cpl *n-0*), we get $x \prec y \in b$ too, a contradiction.

fc-i $\forall x, y \in W^b,$ if both $(x, y) \in \prec^b$ and $(y, z) \in <^b,$ then $(x, z) \in \prec^b$.
Assume that $(x, y) \in \prec^b$ and $(y, z) \in <^b$. Then $x \prec y \notin b$ and $y < z \notin b$. If it were $(x, z) \notin \prec^b$ also, then $x \prec z \in b$. By the completion condition (clp *n-i*) we obtain either $x \prec y \notin b$ or $y < z \notin b$. In both cases we get a contradiction.

fc-ii $\forall x, y \in W^b$,if both $(x, y) \in <^b$ and $(y, z) \in \prec^b$, then $(x, z) \in \prec^b$. The proof is similar to the item above by using (clp *n-ii*) instead of (clp *n-i*).

fc-iii $\forall x, y \in W^b,$ if $x \in \textsc{Inf}^b$ and $(x, y) \in \prec^b,$ then $(\alpha^+, y) \in \leq^b$.
Assume that $(\alpha^-, x) \in <^b, (x, \alpha^+) \in <^b$ and $(x, y) \in \prec^b$ and also $(\alpha^+, y) \notin \leq^b$. Then, by definition of m^b, we obtain $\alpha^- < x \notin b, x < \alpha^+ \notin b, x \prec y \notin b$ and $\alpha^+ \leq y \in b$. Given $\alpha^+ \leq y \in b$, by the completion condition (cpl *n-iii*)

we have that $\alpha^- < x \in b$ or $x < \alpha^+ \in b$ or $x \prec y \in b$, which lead us a contradiction in any case.

fc-iv The proof is similar to that of the item above. □

Proposition 6. *For every open branch b of a proof tree and for every formula xAy the following holds:* $M^b, v^b \models xAy$ *implies* $xAy \notin b$.

The standard proof is by induction on the structure of term A.
Now the completeness theorem follows.

Theorem 1 (Completeness). *If a formula is valid, then there is a closed proof tree for it.*

Proof. Assume that a formula xAy is valid. Suppose all the proof trees for xAy are not closed. Take any of those trees. We may assume that it is complete. Let b be one of its open branches (which exists by the König's lemma). Since $xAy \in b$, by the previous proposition we know that v^b does not satisfy xAy in M^b. Therefore also the valuation u in the quotient model $M^b/1'^b$ does not satisfy xAy, a contradiction. □

8 Conclusions

A relational deduction system for the logic OM of order of magnitude reasoning has been presented. OM is a propositional logic with modal operators determined by three accessibility relations related to each other and their converses. We defined a translation from the language of OM to a target relational language such that both accessibility relations from the frames of OM and the formulas of OM became the relational terms. All the frame conditions on the accessibility relations were postulated as the axioms in the target language.
Two groups of deduction rules have been defined: those that characterize the relational operators of the target language which corresponded to the propositional operators of OM, and those that reflect the axioms imposed on the accesibility relations.
We proved completeness of our proof system adjusting a standard method to the specific features of OM. The key steps of the proof include a development of the completion conditions associated with every rule which enable us to express the notion of a complete proof tree (or equivalently a saturated proof tree) and a development of a branch structure determined by a branch of a proof tree which must then be proved to be a model of the target relational language.
An implementation of translation procedures from the languages of nonclassical logics to relational languages is presented in [6]. Another recent implementation of the core rules of relational proof systems is described in [5]; further work is planned on relational proof systems for variants of OM logic and on implementation of the specific rules for OM within the latter system.

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