Multi-adjoint t-concept lattices

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Abstract

The t-concept lattice is introduced as a set of triples associated to graded tabular information interpreted in a non-commutative fuzzy logic. Following the general techniques of formal concept analysis, and based on the works by Georgescu and Popescu, given a non-commutative conjunctor it is possible to provide generalizations of the mappings for the intension and the extension in two different ways, and this generates a pair of concept lattices. In this paper, we show that the information common to both concept lattices can be seen as a sublattice of the Cartesian product of both concept lattices. The multi-adjoint framework can be applied to this general t-concept lattice, and its usefulness is illustrated by a working example.

Key words: concept lattice, multi-adjoint lattice, implication triple, Galois connection.

1 Introduction

Since its introduction by Wille in the 1980's, formal concept analysis has become an important and appealing research topic both from the theoretical perspective [20, 32, 35] and from the applied one. Regarding applications, we can find papers ranging from ontology merging [12, 30], to applications to the Semantic Web by using the notion of concept similarity [13], and from processing of medical records in the clinical domain [16] to the development of recommender systems [10].

Soon after the introduction of "classical" formal concept analysis, a number of different approaches towards its generalization were introduced and, nowadays, there are works which extend the theory by using ideas from fuzzy set theory [3, 23, 24] or fuzzy logic reasoning [2, 5, 11] or from rough set theory [22, 33, 36] or some integrated approaches such as fuzzy and rough [34], or rough and domain theory [21].

This paper is related to fuzzy extensions of formal concept analysis, for which a number of different approaches have been presented. To the best of our knowledge, the first one was given in [8], although its authors did not advance much beyond the basic definitions, probably due to the fact that they did not use residuated implications. This type of mappings, together with complete residuated lattices as structures for the truth degrees, was used in [3,31], and a representation theorem was proved directly in a fuzzy framework in [4], setting the basis of most of the subsequent direct proofs.

In recent years, there has been an increased interest in studying formal concept analysis based on non-commutative conjunctors. This approach is justified by the fact that conjunctions learnt usually do not satisfy commutativity.

Non-commutative logic and similarity were used to develop new kinds of concept lattices in [15]. The same approach was developed in an asymmetric way in [19], where the so-called generalised concept lattices were introduced. More recently, we can find even further generalisations, such as the variable threshold concept lattices [37] and multi-adjoint concept lattices [27]. Nowadays, a number of different versions have been introduced and it is not surprising to discover relationships between them (see for instance [6, 18, 26]).

Multi-adjoint concept lattices were introduced [25, 27] as a new general approach in which the philosophy of the multi-adjoint paradigm [17,29] is applied to formal concept analysis. With the idea of providing a general framework in which the different approaches stated above could be conveniently accommodated, the authors worked in a general non-commutative environment; and this naturally led to the consideration of adjoint triples, also called implication triples [1] or bi-residuated structures [28] as the main building blocks of a multi-adjoint concept lattice.

Following the general techniques of formal concept analysis and based on the initial work [15], given a non-commutative conjunctor, it is possible to provide generalizations of the mappings for the intension and the extension in two different ways, generating a pair of concept lattices. In this paper, continuing the study of the multi-adjoint concept lattices, we show that the common information to both concept lattices can be seen as a sublattice of the Cartesian product of both concept lattices. In some sense, this common information may be thought of as "neutral" information with regard to the

non-commutativity of the conjunctor. The multi-adjoint framework is applied to this general t-concept lattice, providing an enriched version of the approach by Georgescu and Popescu; finally, its usefulness is illustrated by a working example.

The structure of the paper is as follows: in Section 2 we recall the definition of the multi-adjoint concept lattices and, in particular, the mappings α and β required in the definition of lattice representing a multi-adjoint concept lattice. Then, in Section 3, we prove some new results concerning α and β. The concept lattice of t-concepts is introduced in Section 4, and its representation theorem is stated and proved in Section 5. A detailed example follows in order to show the flexibility and expressive power of the use of t-concepts.

2 Multi-adjoint concept lattices

To make this paper self-contained, we will recall some definitions and results from [27] which will be used hereafter.

The first definition introduces the basic building blocks of the multi-adjoint concept lattices, the *adjoint triples*, which are generalisations of the notion of adjoint pair under the hypothesis of having a non-commutative conjunctor.

The lack of commutativity of the conjunctor, directly provides two different ways of generalising the well-known adjoint property between a t-norm and its residuated implication, depending on which argument is fixed in the conjunction.

Definition 1 *Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets and $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$ be mappings,[1] then $(\&, \swarrow, \nwarrow)$ is an adjoint triple with respect to P_1, P_2, P_3 if:*

(1) $\&$ is order-preserving in both arguments.
(2) $\swarrow$ and $\nwarrow$ are order-preserving in the consequent and order-reversing in the antecedent.
(3) $x \leq_1 z \swarrow y \quad iff \quad x \& y \leq_3 z \quad iff \quad y \leq_2 z \nwarrow x, \quad where \ x \in P_1, \ y \in P_2 \ and \ z \in P_3.$

Example 2 *The usual pairs formed by a t-norm and its residuated implication can be seen as degenerate examples of adjoint triples. As a t-norm is commutative, we have that $\swarrow$ and $\nwarrow$ coincide, and are equal to the residuated implication.*

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[1] Note that the arrow symbol $\swarrow$ also appears in [14], but with a different meaning.

Note that in the domain and codomain of the considered conjunctor we have three (in principle) different sorts, thus providing a more flexible language to a potential user. Furthermore, notice that no boundary condition is required, in difference to the usual definition of multi-adjoint lattice [29] or implication triple [1]. Nevertheless, some boundary conditions follow from the definition, specifically, from the adjoint property (condition (3) above).

Lemma 3 (see [27]) *If* $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ *have bottom element and* $(\&, \swarrow, \nwarrow)$ *is an adjoint triple, then* $(P_1, \leq_1)$ *and* $(P_2, \leq_2)$ *have top element and for all* $x \in P_1, y \in P_2$ *and* $z \in P_3$ *the following properties hold:*

(1) $\bot_1 \& y = \bot_3, \qquad x \& \bot_2 = \bot_3.$
(2) $z \nwarrow \bot_1 = \top_2, \qquad z \swarrow \bot_2 = \top_1.$

Example 4 *Let us consider the following conjunctor and pair of implications defined by the regular partition of* $[0,1]$ *into four subintervals* $[0,1]_4 = \{0, 1/4, 1/2, 3/4, 1\}$.

&	0	1/4	1/2	3/4	1
0	0	0	0	0	0
1/4	0	0	0	0	1/4
1/2	0	0	0	0	1/4
3/4	0	0	1/4	1/2	3/4
1	0	1/4	1/2	3/4	1

$\swarrow$	0	1/4	1/2	3/4	1
0	1	3/4	1/2	1/2	0
1/4	1	1	3/4	1/2	1/2
1/2	1	1	1	3/4	1/2
3/4	1	1	1	1	3/4
1	1	1	1	1	1

$\nwarrow$	0	1/4	1/2	3/4	1
0	1	3/4	3/4	1/4	0
1/4	1	1	1	1/2	1/4
1/2	1	1	1	3/4	1/2
3/4	1	1	1	1	3/4
1	1	1	1	1	1

We have that the operator $\&$ *is not commutative because*

$$3/4 \& 1/2 = 1/4 \neq 0 = 1/2 \& 3/4$$

and, moreover $(\&, \swarrow, \nwarrow)$ *is an adjoint triple.*

In order to provide more flexibility into our language, we will allow the existence of several adjoint triples for a given triplet of posets. Notice, however, that since these triplets will be used as the underlying structures of our generalization of concept lattice, it is reasonable to require the lattice structure on some of the posets in the definition of adjoint triple.

Definition 5 *A* multi-adjoint frame $\mathcal{L}$ *is a tuple*

$$(L_1, L_2, P, \preceq_1, \preceq_2, \leq, \&_1, \swarrow^1, \nwarrow_1, \ldots, \&_n, \swarrow^n, \nwarrow_n)$$

where $(L_1, \preceq_1)$ *and* $(L_2, \preceq_2)$ *are complete lattices,* $(P, \leq)$ *is a poset and, for all* $i \in \{1, \ldots, n\}$, $(\&_i, \swarrow^i, \nwarrow_i)$ *is an adjoint triple with respect to* L_1, L_2, P.

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For short, a multi-adjoint frame will be denoted as $(L_1, L_2, P, \&_1, \ldots, \&_n)$.

Following the usual approach to formal concept analysis, given a frame, we define a *multi-adjoint context* as a tuple consisting of sets of objects and attributes and a fuzzy relation among them; in addition, the multi-adjoint approach also includes a function which assigns an adjoint triple to each object (or attribute). This feature is important in that it allows for defining subgroups of objects or attributes in terms of different degrees of preference, see [27]. Formally, the definition is the following:

Definition 6 *Given a multi-adjoint frame $(L_1, L_2, P, \&_1, \ldots, \&_n)$, a context is a tuple (A, B, R, σ) such that A and B are non-empty sets (usually interpreted as attributes and objects, respectively), R is a P-fuzzy relation $R\colon A \times B \to P$ and $\sigma\colon B \to \{1, \ldots, n\}$ is a mapping which associates any element in B with some particular adjoint triple in the frame.* [2]

Once we have fixed a multi-adjoint frame and a context for that frame, we can define the following mappings $\uparrow^\sigma\colon L_2^B \longrightarrow L_1^A$ and $\downarrow^\sigma\colon L_1^A \longrightarrow L_2^B$ which can be seen as generalisations of those given in [5, 19]:

$$g^{\uparrow_\sigma}(a) = \inf\{R(a, b) \swarrow^{\sigma(b)} g(b) \mid b \in B\} \tag{1}$$

$$f^{\downarrow^\sigma}(b) = \inf\{R(a, b) \searrow_{\sigma(b)} f(a) \mid a \in A\} \tag{2}$$

It is not difficult to show that these two arrows generate a Galois connection [27]. This concept is defined below:

Definition 7 *Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be posets, and $\downarrow\colon P_1 \to P_2$, $\uparrow\colon P_2 \to P_1$ mappings, the pair $(\uparrow, \downarrow)$ forms a* Galois connection *between P_1 and P_2 whenever the following conditions hold:*

(1) $\uparrow$ and $\downarrow$ are order-reversing.
(2) $x \leq_1 x^{\downarrow\uparrow}$ for all $x \in P_1$.
(3) $y \leq_2 y^{\uparrow\downarrow}$ for all $y \in P_2$.

Proposition 8 (see [27]) *Let $(L_1, L_2, P, \&_1, \ldots, \&_n)$ be a multi-adjoint frame and (A, B, R, σ) be a context, then the pair $(\uparrow_\sigma, \downarrow^\sigma)$ is a Galois connection between L_1^A and L_2^B.*

Now, a *multi-adjoint concept* is a pair $\langle g, f \rangle$ satisfying that $g \in L_2^B$, $f \in L_1^A$ and that $g^{\uparrow_\sigma} = f$ and $f^{\downarrow^\sigma} = g$; where $(\uparrow_\sigma, \downarrow^\sigma)$ is the Galois connection defined above.

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[2] A similar theory could be developed by considering a mapping $\tau\colon A \to \{1, \ldots, n\}$ which associates any element in A with some particular adjoint triple in the frame.

Definition 9 *The* multi-adjoint concept lattice *associated to a multi-adjoint frame* $(L_1, L_2, P, \&_1, \dots, \&_n)$ *and a context* (A, B, R, σ) *is the set*

$$\mathcal{M} = \{\langle g, f \rangle \mid g \in L_2^B, f \in L_1^A \text{ and } g^{\uparrow_\sigma} = f, f^{\downarrow^\sigma} = g\}$$

in which the ordering is defined by $\langle g_1, f_1 \rangle \preceq \langle g_2, f_2 \rangle$ *if and only if* $g_1 \preceq_2 g_2$ *(equivalently* $f_2 \preceq_1 f_1$*).*

The ordering defined above provides $\mathcal{M}$ with the structure of a complete lattice. This follows from Proposition 8 (the arrows $(^{\uparrow_\sigma}, ^{\downarrow^\sigma})$ forms a Galois connection) and the theorem below.

Theorem 10 (see [9]) *Let* $(L_1, \preceq_1)$*,* $(L_2, \preceq_2)$ *be complete lattices, let* $(^\uparrow, ^\downarrow)$ *be a Galois connection between* L_1, L_2 *and consider* $\mathcal{C} = \{\langle x, y \rangle \mid x^\uparrow = y, x = y^\downarrow; x \in L_1, y \in L_2\}$*; then* $(\mathcal{C}, \preceq)$ *is a complete lattice, where*

$$\bigwedge_{i \in I} \langle x_i, y_i \rangle = \langle \bigwedge_{i \in I} x_i, (\bigvee_{i \in I} y_i)^{\downarrow\uparrow} \rangle; \qquad \bigvee_{i \in I} \langle x_i, y_i \rangle = \langle (\bigvee_{i \in I} x_i)^{\uparrow\downarrow}, \bigwedge_{i \in I} y_i \rangle$$

and $\langle x_1, y_1 \rangle \preceq \langle x_2, y_2 \rangle$ *if and only if* $x_1 \preceq_1 x_2$*.*

Example 11 *Let us consider the set* $B = \{$*Mercury, Earth, Jupiter, Neptune*$\}$ *of objects, the attributes* $A = \{$*size, dist, temp*$\}$*, together with the fuzzy relation in Figure 1, which (roughly) assigns a weighted value corresponding to the degree in which every particular attribute is satisfied by an object.*

R	Mercury	Earth	Jupiter	Neptune
size	$1/4$	$1/2$	1	$3/4$
dist	$1/4$	$1/2$	$3/4$	1
temp	$1/4$	$1/2$	$3/4$	1

Fig. 1. Fuzzy relation between the objects and the attributes.

We will consider the frame $([0, 1]_4, [0, 1]_4, [0, 1]_4, \leq, \leq, \leq, \&)$*, and the context* (A, B, R, σ)*, where* $(\&, \swarrow, \searrow)$ *is the adjoint triple defined in Example 4 and* σ *is the constant mapping that applies the conjunctor* $\&$ *to each attribute.*[3]

In order to give an example of concept in this formal context, let us obtain an element of the corresponding concept lattice $\mathcal{M}$ *generated by the mapping* $g_0 \colon B \to [0, 1]_4$ *defined as* $g(b) = 0$*, for all* $b \in B$*. The concept obtained is* $\langle g_0^{\uparrow\downarrow}, g_0^\uparrow \rangle$ *which is shown in Figure 2, where the last row in each table is obtained by the infimum of the corresponding column (that is, by applying the definitions of* $g_0^\uparrow$ *and* $g_0^{\uparrow\downarrow}$*).*

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[3] A more complex example with a non-constant σ can be seen in the final section.

	size	dist	temp
Mercury	1	1	1
Earth	1	1	1
Jupiter	1	1	1
Neptune	1	1	1
$g_0^{\uparrow}$	1	1	1

	Mercury	Earth	Jupiter	Neptune
size	$1/4$	$1/2$	1	$3/4$
dist	$1/4$	$1/2$	$3/4$	1
temp	$1/4$	$1/2$	$3/4$	1
$g_0^{\uparrow\downarrow}$	$1/4$	$1/2$	$3/4$	$3/4$

Fig. 2. Concept $\langle g_0^{\uparrow\downarrow}, g_0^{\uparrow}\rangle$ obtained from g_0.

The concept obtained in Figure 2 can be interpreted as the order of the planets considering completely all attributes. We can summarize the representation of this concept as in Figure 3.

	g_0		$g_0^{\uparrow\downarrow}$		$g_0^{\uparrow}$
Mercury	0		$1/4$	size	1
Earth	0		$1/2$	dist	1
Jupiter	0		$3/4$	temp	1
Neptune	0		$3/4$		

Fig. 3. Representation of concept $\langle g_0^{\uparrow\downarrow}, g_0^{\uparrow}\rangle$ obtained from g_0.

Another possibility to build a concept is to consider mappings defined on the set of attributes. For instance, we can consider the mapping $f_s\colon A \to [0,1]_4$, where the attribute "size" is the only initially assumed (this means that f_s is defined as $f_s(size) = 1$, $f_s(dist) = 0$ and $f_s(temp) = 0$). After applying the mapping $^{\downarrow}$, in Figure 4, we obtain an order among the planets as if we consider mainly the attribute "size", and neglect the rest of attributes.

	f_s		$f_s^{\downarrow\uparrow}$		$f_s^{\downarrow}$
size	1		1	Mercury	$1/4$
dist	0		$3/4$	Earth	$1/2$
temp	0		$3/4$	Jupiter	1
				Neptune	$3/4$

Fig. 4. Concept $\langle f_s^{\downarrow}, f_s^{\downarrow\uparrow}\rangle$ obtained from f_s.

On the other hand, if we consider the mappings $f_d\colon A \to [0,1]_4$, $f_t\colon A \to [0,1]_4$, defined as $f_d(size) = 0$, $f_d(dist) = 1$, $f_d(temp) = 0$ and $f_t(size) = 0$, $f_t(dist) = 0$, $f_t(temp) = 1$, respectively, we obtain the same concept in both cases. This occurs because the attributes dist and temp are certainly correlated.

A possible consequence of this fact, although out of the scope of this paper, is that in order to obtain the concepts of $\mathcal{M}$ we could erase one of the attributes above because of the existence of redundant information.

	f_d	f_t		$f_d^{\downarrow\uparrow}$	$f_t^{\downarrow\uparrow}$		$f_d^{\downarrow}$	$f_t^{\downarrow}$
size	0	0		$^3/_4$	$^3/_4$	Mercury	$^1/_4$	$^1/_4$
dist	1	0		1	1	Earth	$^1/_2$	$^1/_2$
temp	0	1		1	1	Jupiter	$^3/_4$	$^3/_4$
						Neptune	1	1

Fig. 5. Concept $\langle f_d^{\downarrow}, f_d^{\downarrow\uparrow} \rangle$ obtained from f_d, and $\langle f_t^{\downarrow}, f_t^{\downarrow\uparrow} \rangle$ obtained from f_t.

From now on, we will fix a multi-adjoint frame $(L_1, L_2, P, \&_1, \ldots, \&_n)$ and context (A, B, R, σ). Moreover, to improve readability, we will write $(^{\uparrow}, ^{\downarrow})$ instead of $(^{\uparrow\sigma}, ^{\downarrow\sigma})$ and $\nearrow^b$, $\searrow_b$ instead of $\nearrow^{\sigma(b)}$, $\searrow_{\sigma(b)}$.

In the next section, we will present some new properties about the functions α and β involved in the representation (or fundamental) theorem for the multi-adjoint framework presented in [27]. In order to do this, we will recall some necessary definitions.

Definition 12 *Given a complete lattice L, a subset $K \subseteq L$ is* infimum-dense *(resp.* supremum-dense*) if and only if for all $x \in L$ there exists $K' \subseteq K$ such that $x = \inf(K')$ (resp. $x = \sup(K')$).*

The notion of multi-adjoint concept lattice represented by a complete lattice is presented in the definition below:

Definition 13 *A multi-adjoint concept lattice $(\mathcal{M}, \preceq)$ is represented by a complete lattice $(V, \sqsubseteq)$ if there exists a pair of mappings $\alpha\colon A \times L_1 \to V$ and $\beta\colon B \times L_2 \to V$ such that:*

1a) $\alpha[A \times L_1]$ is infimum-dense;
1b) $\beta[B \times L_2]$ is supremum-dense; and
2) For all $a \in A$, $b \in B$, $x \in L_1$, $y \in L_2$:

$$\beta(b, y) \sqsubseteq \alpha(a, x) \quad \text{if and only if} \quad x \&_b y \leq R(a, b)$$

From the definition of representability the following properties follow:

Proposition 14 (see [27]) *Given a complete lattice $(V, \sqsubseteq)$ which represents a multi-adjoint concept lattice $(\mathcal{M}, \preceq)$, and mappings $f \in L_1^A$ and $g \in L_2^B$, we have:*

(1) β is order-preserving in the second argument.

(2) α is order-reversing in the second argument.

(3) $g^{\uparrow}(a) = \sup\{x \in L_1 \mid v_g \sqsubseteq \alpha(a,x)\}$, where $v_g = \sup\{\beta(b,g(b)) \mid b \in B\}$.

(4) $f^{\downarrow}(b) = \sup\{y \in L_2 \mid \beta(b,y) \sqsubseteq v_f\}$, where $v_f = \inf\{\alpha(a,f(a)) \mid a \in A\}$.

(5) If $g_v(b) = \sup\{y \in L_2 \mid \beta(b,y) \sqsubseteq v\}$, then $\sup\{\beta(b,g_v(b)) \mid b \in B\} = v$.

(6) If $f_v(a) = \sup\{x \in L_1 \mid v \sqsubseteq \alpha(a,x)\}$, then $\sup\{\alpha(a,f_v(a)) \mid a \in A\} = v$.

Finally, the fundamental theorem for multi-adjoint concept lattices presented in [27] is the following.

Theorem 15 (see [27]) *A complete lattice $(V, \sqsubseteq)$ represents a multi-adjoint concept lattice $(\mathcal{M}, \preceq)$ if and only if $(V, \sqsubseteq)$ is isomorphic to $(\mathcal{M}, \preceq)$.*

3   New results about the mappings α and β

In this section, we introduce some new interesting properties about the mappings α and β which appear in the representation theorem. So, let us assume a complete lattice $(V, \sqsubseteq)$ which represents a multi-adjoint concept lattice $(\mathcal{M}, \preceq)$ and the mappings $\alpha \colon A \times L_1 \to V$, $\beta \colon B \times L_2 \to V$.

We will restate below the isomorphism constructed in fundamental theorem, based on both the α and β functions, since these expressions will be used later.

Proposition 16 (see [27]) *If a complete lattice $(V, \sqsubseteq)$ represents a multi-adjoint concept lattice $(\mathcal{M}, \preceq)$, then there exists an isomorphism $\varphi \colon \mathcal{M} \to V$ and two mappings $\beta \colon B \times L_2 \to V$, $\alpha \colon A \times L_1 \to V$, such that:*

$$\varphi(\langle g, f \rangle) = \sup\{\beta(b, g(b)) \mid b \in B\} = \inf\{\alpha(a, f(a)) \mid a \in A\}$$

for all concepts $\langle g, f \rangle \in \mathcal{M}$.

The following result introduces some continuity-related properties of α and β in their second argument.

Proposition 17 *The applications $\beta \colon B \times L_2 \to V$, $\alpha \colon A \times L_1 \to V$ satisfy that:*

(1) *For all indexed set $Y = \{y_i\}_{i \in I} \subseteq L_2$ and $b \in B$:*

$$\beta(b, \sup\{y_i \mid i \in I\}) = \sup\{\beta(b, y_i) \mid i \in I\}$$

(2) *For all indexed set $X = \{x_i\}_{i \in I} \subseteq L_1$ and $a \in A$:*

$$\alpha(a, \sup\{x_i \mid i \in I\}) = \inf\{\alpha(a, x_i) \mid i \in I\}$$

PROOF. 1. Let $b \in B$ and $Y = \{y_i\}_{i \in I} \subseteq L_2$, as $\alpha[A \times L_1]$ is infimum-dense and $\beta(b, \sup Y) \in V$, there exists an indexing set Λ such that $\beta(b, \sup Y) = \inf\{\alpha(a_j, x_j) \mid j \in \Lambda\}$; as a result $\beta(b, \sup Y) \sqsubseteq \alpha(a_j, x_j)$ for every $j \in \Lambda$. From Proposition 14(1), we obtain that $\beta(b, y_i) \sqsubseteq \alpha(a_j, x_j)$, for every $i \in I$ and $j \in \Lambda$, and hence $\sup\{\beta(b, y_i) \mid i \in I\} \sqsubseteq \alpha(a_j, x_j)$ for every $j \in \Lambda$, then

$$\sup\{\beta(b, y_i) \mid i \in I\} \sqsubseteq \inf\{\alpha(a_j, x_j) \mid j \in \Lambda\} = \beta(b, \sup Y)$$

For the other inequality, let us consider $\sup\{\beta(b, y_i) \mid i \in I\}$; as $\alpha[A \times L_1]$ is infimum-dense, there exists an indexing set Λ' such that $\sup\{\beta(b, y_i) \mid i \in I\} = \inf\{\alpha(a_j, x_j) \mid j \in \Lambda'\}$. Now, for all $i \in I$ and $j \in \Lambda'$ we obtain that $\beta(b, y_i) \sqsubseteq \alpha(a_j, x_j)$, therefore, from Definition 13(2), $x_j \mathbin{\&_b} y_i \leq R(a_j, b)$. Now, as $(\mathbin{\&_b}, \nearrow^b, \nwarrow_b)$ is an adjoint triple, we have the following chain of equivalent statements:

$$\begin{aligned} x_j \mathbin{\&_b} y_i &\leq R(a_j, b) && \text{for all } i \in I \\ y_i &\preceq_2 R(a_j, b) \nwarrow_b x_j && \text{for all } i \in I \\ \sup Y &\preceq_2 R(a_j, b) \nwarrow_b x_j \\ x_j \mathbin{\&_b} \sup Y &\leq R(a_j, b) \end{aligned}$$

so, $\beta(b, \sup Y) \sqsubseteq \alpha(a_j, x_j)$ for every $j \in \Lambda'$, and thus

$$\beta(b, \sup Y) \sqsubseteq \inf\{\alpha(a_j, x_j) \mid j \in \Lambda'\} = \sup\{\beta(b, y_i) \mid i \in I\}$$

2. The core of the proof is the same; it follows by, firstly, interchanging supremum-dense and infimum-dense and, secondly, the implications $\nwarrow$ and $\nearrow$. $\square$

We continue below by proving some boundary conditions fulfilled by α and β.

Proposition 18 *The applications $\alpha \colon A \times L_1 \to V$ and $\beta \colon B \times L_2 \to V$ are such that $\alpha(a, \perp_1) = \top_V$ and $\beta(b, \perp_2) = \perp_V$ for all $b \in B$ and $a \in A$.*

PROOF. Given $a \in A$, let us prove that $\alpha(a, \perp_1) = \top_V$. Firstly, recall that Lemma 3 implies that $\perp_1 \mathbin{\&_b} y \leq R(a, b)$ for all $b \in B$ and $y \in L_2$; now, from Definition 13(2) we obtain that $\beta(b, y) \sqsubseteq \alpha(a, \perp_1)$ for all $b \in B$ and $y \in L_2$, that is, $\alpha(a, \perp_1)$ is an upper bound of the set of elements $\beta(b, y)$ for all $b \in B$ and $y \in L_2$. Now, as β is supremum-dense, there is an indexing set Λ such that $\top_V = \sup\{\beta(b_i, y_i) \mid i \in \Lambda\}$, therefore, we have that: $\top_V \sqsubseteq \alpha(a, \perp_1)$. Hence, $\top_V = \alpha(a, \perp_1)$.

The other equality follows similarly. $\square$

From the propositions above, we obtain the following corollary which states

the behaviour of α and β regarding suprema of any set (including the empty set, contrariwise to Proposition 17).

Corollary 19 *The applications $\beta\colon B \times L_2 \to V$, $\alpha\colon A \times L_1 \to V$ satisfy that:*

(1) $\beta(b, \sup Y) = \sup\{\beta(b, y) \mid y \in Y\}$, for all $Y \subseteq L_2$ and $b \in B$.

(2) $\alpha(a, \sup X) = \inf\{\alpha(a, x) \mid x \in X\}$, for all $X \subseteq L_1$ and $a \in A$.

Now, the following property shows that any subset of $A \times L_1$ is related to a concept. This result will be used in the next section in order to generalize the framework introduced in [15].

Proposition 20 *Given a multi-adjoint concept lattice $(\mathcal{M}, \preceq)$ represented by a complete lattice $(V, \sqsubseteq)$ and the mappings $\alpha\colon A \times L_1 \to V$, $\beta\colon B \times L_2 \to V$, we have that for each $K \subseteq A \times L_1$, there exists a unique concept $\langle g, f \rangle \in \mathcal{M}$ such that*

$$\inf\{\alpha(a, x) \mid (a, x) \in K\} = \sup\{\beta(b', g(b')) \mid b' \in B\}$$
$$= \inf\{\alpha(a', f(a')) \mid a' \in A\}$$

Analogously, for each $N \subseteq B \times L_2$, there exists a unique concept $\langle g, f \rangle \in \mathcal{M}$ such that

$$\sup\{\beta(b, y) \mid (b, y) \in N\} = \sup\{\beta(b', g(b')) \mid b' \in B\}$$
$$= \inf\{\alpha(a', f(a')) \mid a' \in A\}$$

PROOF. Given $K \subseteq A \times L_1$, let us consider the sets $K_a = \{x \mid (a, x) \in K\}$, and the function $h\colon A \to L_1$ defined as $h(a) = \sup K_a$.

By Corollary 19, we have that, for all $a' \in A$, the following equality holds

$$\alpha(a', h(a')) = \inf\{\alpha(a', x) \mid x \in K_{a'}\}$$

Therefore:

$$\inf\{\alpha(a', h(a')) \mid a' \in A\} = \inf\{\inf\{\alpha(a', x) \mid x \in K_{a'}\} \mid a' \in A\}$$
$$= \inf\{\alpha(a', x) \mid (a', x) \in K\}$$

Finally, we obtain the following chain of equalities:

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$$\inf\{\alpha(a,x) \mid (a,x) \in K\} = \inf\{\alpha(a', h(a')) \mid a' \in A\}$$
$$\overset{(1)}{=} \sup\{\beta(b', h^{\downarrow}(b')) \mid b' \in B\}$$
$$\overset{(2)}{=} \varphi(\langle h^{\downarrow}, h^{\downarrow\uparrow}\rangle)$$
$$\overset{(3)}{=} \inf\{\alpha(a', h^{\downarrow\uparrow}(a')) \mid a' \in A\}$$

where equality (1) follows by Proposition 14 (items 4 and 5), and equalities (2), (3) by Proposition 16. This means that the concept whose existence is postulated in the statement is $\langle h^{\downarrow}, h^{\downarrow\uparrow}\rangle$.

Now, the uniqueness follows from the isomorphism φ:

As $\varphi(\langle h^{\downarrow}, h^{\downarrow\uparrow}\rangle) = \sup\{\beta(b, h^{\downarrow}(b)) \mid b \in B\}$, if there would exist another concept $\langle g, f\rangle$ such that $\sup\{\beta(b, g(b)) \mid b \in B\} = \inf\{\alpha(a,x) \mid (a,x) \in K\}$, we would have:

$$\varphi(\langle h^{\downarrow}, h^{\downarrow\uparrow}\rangle) = \sup\{\beta(b, h^{\downarrow}(b)) \mid b \in B\}$$
$$= \inf\{\alpha(a,x) \mid (a,x) \in K\}$$
$$= \sup\{\beta(b, g(b)) \mid b \in B\}$$
$$= \varphi(\langle g, f\rangle)$$

Thus, $\langle h^{\downarrow}, h^{\downarrow\uparrow}\rangle = \langle g, f\rangle$.

The second statement follows similarly. □

4   Multi-adjoint t-concept lattice

In this section a new construction based on the previous notion of multi-adjoint concept lattice is presented. The t-concepts are introduced as a generalisation of the approach given in [15] for non-commutative conjunctors, which provides greater flexibility and, hence, allows for specifying and solving a greater number of problems in more complex knowledge-based systems.

The basic structure we will work with is that of multi-adjoint frame, where the complete lattices $\langle L_1, \preceq_1\rangle, \langle L_2, \preceq_2\rangle$ coincide, and we will denote $\langle L, \preceq\rangle$. This way, given a context (A, B, R, σ), besides the Galois connection $(\uparrow, \downarrow)$ defined for the multi-adjoint concept lattice, it is possible to define an alternative version as follows:

$$g^{\uparrow_{op}}(a) = \inf\{R(a,b) \searrow_b g(b) \mid b \in B\}$$
$$f^{\downarrow_{op}}(b) = \inf\{R(a,b) \nearrow^b f(a) \mid a \in A\}$$

The definition above is indeed a Galois connection because of Proposition 8, since it coincides with the Galois connection defined by equations (1), (2), on the multi-adjoint frame $(L, L, P, \&_1^{op}, \ldots, \&_n^{op})$ and context (A, B, R, σ), being $\&_i^{op} \colon L \times L \to P$ and $x \&_i^{op} y = y \&_i x$ for all $i \in \{1, \ldots, n\}$. Since the implications are permuted, if the initial adjoint triples are $(\&_i, \searcdot_i, \nearrow^i)$, then the adjoint triples considered are $(\&_i^{op}, \nearrow^i, \searrow_i)$. Now, we have two Galois connections $(\uparrow, \downarrow)$, $(\uparrow^{op}, \downarrow^{op})$, on which two different multi-adjoint concept lattices $(\mathcal{M}, \preceq)$, $(\mathcal{M}^{op}, \preceq)$ can be defined. [4]

Both lattices are different if at least one conjunctor $\&_i$ is non-commutative, but are certainly related. This suggests to consider the following subsets of $\mathcal{M} \times \mathcal{M}^{op}$:

$$\mathcal{N}_1 = \{(\langle g, f_1 \rangle, \langle g, f_2 \rangle) \mid \langle g, f_1 \rangle \in \mathcal{M}, \langle g, f_2 \rangle \in \mathcal{M}^{op}\}$$
$$\mathcal{N}_2 = \{(\langle g_1, f \rangle, \langle g_2, f \rangle) \mid \langle g_1, f \rangle \in \mathcal{M}, \langle g_2, f \rangle \in \mathcal{M}^{op}\}$$

which, together with the orderings

$$(\langle g, f_1 \rangle, \langle g, f_2 \rangle) \preceq (\langle g', f_1' \rangle, \langle g', f_2' \rangle) \quad \text{if and only if} \quad g \preceq g'$$
$$(\langle g_1, f \rangle, \langle g_2, f \rangle) \preceq (\langle g_1', f' \rangle, \langle g_2', f' \rangle) \quad \text{if and only if} \quad f' \preceq f$$

are sublattices of $\mathcal{M} \times \mathcal{M}^{op}$ and, thus, are complete lattices.

Now we will show that Georgescu and Popescu's framework can be reproduced by means of the multi-adjoint framework. In [26] it has been proved that the concept lattices defined in [15] can be constructed in terms of generalized concept lattices [19]. On the other hand, in [27] it was proved that generalized concept lattices can be embedded into the multi-adjoint framework. As a result, we can obtain that every non-commutative fuzzy concept lattice $\mathcal{L}$, in the sense of Georgescu and Popescu, can be embedded into a specific product, $\mathcal{M} \times \mathcal{M}^{op}$, of multi-adjoint concept lattices. Specifically, the theorem below shows that, there exists a particular choice of multi-adjoint frame and context such that $\mathcal{L}$ is isomorphic to the sublattice $\mathcal{N}_1$ of $\mathcal{M} \times \mathcal{M}^{op}$.

Theorem 21 *Given a complete generalized residuated lattice $(L, \preceq, \&, \nearrow, \searrow)$ and a residuated context (A, B, R), then there exists a multi-adjoint concept lattice, $\mathcal{M}$, such that the sublattice $\mathcal{N}_1$ is isomorphic to the non-commutative fuzzy concept lattice $\mathcal{L}$.*

PROOF. Follows by from [26, Thm 6] and [27, Thm 14]. □

---

[4] Note that the ordering relation is the same for both lattices, although its domain might differ from one to another.

The theorem above justifies considering the general construction of $\mathcal{N}_1$ as a generalized concept lattice. However, it is important to note that the theorem above shows a strict embedding of the framework by Georgescu-Popescu into our framework. Their construction explicitly assumes that $(L, \&, \top)$ should be a commutative monoid, but $\mathcal{N}_1$ can be defined as well by directly considering an adjoint triple which, obviously needs not be either commutative nor associative.

The particular form of the elements of $\mathcal{N}_1$ suggests to abuse a little bit the notation, denote them as $\langle g, f_1, f_2 \rangle$, and use the term *t-concept* to refer to them (t- for triple).

Note that we will concentrate hereafter on the lattice $\mathcal{N}_1$, but similar results can be obtained for $\mathcal{N}_2$.

Example 22 *In Example 11 the attributes have been evaluated in the left argument $\&$, and the objects in the right argument. Due to the non-commutativity of $\&$, we obtain greater values for the attributes than for the objects (although, in this trivial example, it only appears in one case) and it seems that we'd rather consider the other possibility, the attributes in the right of $\&$ and the objects in the left. Therefore, starting again from the mapping g_0, the concept in Figure 6 is obtained.*

	g_0		$g_0^{\uparrow^{op}\downarrow^{op}}$		$g_0^{\uparrow^{op}}$
Mercury	0		$1/2$	size	1
Earth	0		$1/2$	dist	1
Jupiter	0		$3/4$	temp	1
Neptune	0		$3/4$		

Fig. 6. Concept obtained from g_0 and the opposite conjunction.

We can see that the concepts $\langle g_0^{\uparrow\downarrow}, g_0^{\uparrow} \rangle$ and $\langle g_0^{\uparrow^{op}\downarrow^{op}}, g_0^{\uparrow^{op}} \rangle$ are different. Now, if we would like to maintain both results until we have enough information on how to decide what result better fits our needs, it is reasonable to consider the corresponding t-concept, which considers the two possible results arising from the non-commutative character of conjunction. The t-concept $\langle g, g^{\uparrow}, g^{\uparrow^{op}} \rangle$ generated from g_0 is shown in Figure 7. Note that, in this particular case, $g^{\uparrow} = g^{\uparrow^{op}}$, but this is not the rule, as we will see in the more complex example in Section 6.

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	g_0		g		$g^\uparrow$	$g^{\uparrow op}$
Mercury	0		$1/2$	size	$3/4$	$3/4$
Earth	0		$3/4$	dist	$3/4$	$3/4$
Jupiter	0		1	temp	$3/4$	$3/4$
Neptune	0		1			

Fig. 7. t-concept obtained from g_0

5   The representation theorem of the lattice of t-concepts $\mathcal{N}_1$

We start this section by introducing some preliminary definitions and results needed for the statement and proof of the representation theorem.

The technical notion of characteristic mapping, introduced below, is not related to the statement of the representation theorem, but to its proof.

Definition 23 *Given a set A, a poset P with bottom element $\bot$, and elements $a \in A$, $x \in P$, the characteristic mapping $@_a^x \colon A \to P$, read "at point a the value is x", is defined as:*

$$@_a^x(a') = \begin{cases} x, & \text{if } a' = a \\ \bot, & \text{otherwise} \end{cases}$$

The following lemma will be used in the proof of the representation theorem.

Lemma 24 (see [27]) *Given the multi-adjoint concept lattice $(\mathcal{M}, \preceq)$, and given $a \in A$, $b \in B$, $x \in L_1$ and $y \in L_2$, the following equalities hold:*

$$@_a^{x\downarrow}(b') = R(a,b') \searrow_{b'} x \qquad \text{for all} \quad b' \in B$$
$$@_b^{y\uparrow}(a') = R(a',b) \nearrow^b y \qquad \text{for all} \quad a' \in A$$

Definition 25 *Given two complete lattices $(V_1, \sqsubseteq_1)$, $(V_2, \sqsubseteq_2)$ which represent the multi-adjoint concept lattices $(\mathcal{M}, \preceq)$, $(\mathcal{M}^{op}, \preceq)$, respectively, the sublattice $\mathcal{V}$ of $V_1 \times V_2$ is defined as:*

$$\mathcal{V} = \left\{ \left(\inf_{(a,x) \in K_1} \alpha_1(a,x), \inf_{(a,x) \in K_2} \alpha_2(a,x) \right) \mid (K_1, K_2) \in \mathcal{K} \right\}$$

where $\mathcal{K} = \{(K_1, K_2) \mid K_1, K_2 \subseteq A \times L$ and $\inf_{(a,x) \in K_1} @_a^{x\downarrow} = \inf_{(a,x) \in K_2} @_a^{x\downarrow op}\}$ and α_1, α_2 are the maps given in Definition 13 associated to $(\mathcal{M}, \preceq)$, $(\mathcal{M}^{op}, \preceq)$ respectively.

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The following result proves that the sublattice above is isomorphic to the complete sublattice of t-concepts $\mathcal{N}_1$ of $\mathcal{M} \times \mathcal{M}^{op}$:

Proposition 26 *Let* $(V_1, \sqsubseteq_1)$, $(V_2, \sqsubseteq_2)$ *be the complete lattices which represent two multi-adjoint concept lattices* $(\mathcal{M}, \preceq_1)$, $(\mathcal{M}^{op}, \preceq_2)$ *respectively, then* $\mathcal{N}_1$ *is isomorphic to the sublattice* $\mathcal{V}$ *of* $V_1 \times V_2$.

PROOF. From Proposition 16 we have that there exist two isomorphisms $\varphi_1 \colon \mathcal{M} \to V_1$, $\varphi_2 \colon \mathcal{M}^{op} \to V_2$ defined as:

$$\varphi_1(\langle g, f \rangle) = \inf\{\alpha_1(a, f(a)) \mid a \in A\};$$
$$\varphi_2(\langle g, f \rangle) = \inf\{\alpha_2(a, f(a)) \mid a \in A\}$$

Hence, we have only to show that the image of the restriction to $\mathcal{N}_1$ of the isomorphism $\varphi_1 \times \varphi_2 \colon \mathcal{M} \times \mathcal{M}^{op} \to V_1 \times V_2$ is $\mathcal{V}$, that is, $\varphi_1 \times \varphi_2(\mathcal{N}_1) = \mathcal{V}$.

Firstly, we will check that if $\langle g, f_1, f_2 \rangle \in \mathcal{N}_1$ then $\varphi_1 \times \varphi_2(\langle g, f_1, f_2 \rangle) \in \mathcal{V}$. Let us consider the subsets $K_1 = \{(a, f_1(a)) \mid a \in A\}$ and $K_2 = \{(a, f_2(a)) \mid a \in A\}$ of $A \times L$, we have that:

$$\inf_{(a,x) \in K_1} @_a^{x\downarrow} = \inf_{a \in A}(@_a^{f_1(a)})^\downarrow \overset{(\triangle)}{=} f_1^\downarrow \overset{(*)}{=} f_2^{\downarrow op} \overset{(\triangle)}{=} \inf_{a \in A}(@_a^{f_2(a)})^{\downarrow op} = \inf_{(a,x) \in K_2} @_a^{x \downarrow op}$$

where $(\triangle)$ follows by Equation (2) and Lemma 24, and $(*)$ is given because $\langle g, f_1, f_2 \rangle$ is a t-concept, thus $(K_1, K_2) \in \mathcal{K}$. Moreover,

$$\inf_{(a,x) \in K_1} \alpha_1(a, x) = \inf_{a \in A} \alpha_1(a, f_1(a)) = \varphi_1(\langle g, f_1 \rangle)$$
$$\inf_{(a,x) \in K_2} \alpha_2(a, x) = \inf_{a \in A} \alpha_2(a, f_2(a)) = \varphi_2(\langle g, f_2 \rangle)$$

hence the pair $(\varphi_1(\langle g, f_1 \rangle), \varphi_2(\langle g, f_2 \rangle))$ is in the required subset $\mathcal{V}$ of $V_1 \times V_2$.

Let us now consider an arbitrary element of $\mathcal{V}$, that is, a pair of the form $(\inf_{(a,x) \in K_1} \alpha_1(a, x), \inf_{(a,x) \in K_2} \alpha_2(a, x))$, with $(K_1, K_2) \in \mathcal{K}$. By Proposition 20 we have that there are (unique) concepts $\langle g_1, f_1 \rangle \in \mathcal{M}$, $\langle g_2, f_2 \rangle \in \mathcal{M}^{op}$ satisfying that

$$(\inf_{(a,x) \in K_1} \alpha_1(a, x), \inf_{(a,x) \in K_2} \alpha_2(a, x)) = (\inf_{a \in A} \alpha_1(a, f_1(a)), \inf_{a \in A} \alpha_2(a, f_2(a)))$$
$$= (\varphi_1(\langle g_1, f_1 \rangle), \varphi_2(\langle g_2, f_2 \rangle))$$

where the last equality is given by definition of φ_1 and φ_2, and g_1 satisfies that

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$$g_1(a) \overset{(1)}{=} \inf\{R(a,b) \nwarrow \sup_{(a,x_i)\in K_1}\{x_i\} \mid a \in A\}$$

$$\overset{(2)}{=} \inf\{\inf_{(a,x_i)\in K_1}\{R(a,b)\nwarrow x_i\} \mid a \in A\}$$

$$= \inf_{(a,x)\in K_1}\{R(a,b)\nwarrow x\}$$

$$\overset{(3)}{=} \inf_{(a,x)\in K_1}\{@_a^{x\downarrow}(b)\}$$

where (1) follows by the construction of g_1 (given in the proof of Proposition 20), (2) follows as a consequence of the adjoint property, and (3) follows because of Lemma 24. In a similar way, we can prove that $g_2(b) = \inf_{(a,x)\in K_2}\{@_a^{x\downarrow^{op}}(b)\}$.

Recalling that $(K_1, K_2) \in \mathcal{K}$, we obtain that $g_1 = g_2$ and, as a consequence

$$(\inf_{(a,x)\in K_1}\alpha_1(a,x), \inf_{(a,x)\in K_2}\alpha_2(a,x)) = \varphi_1 \times \varphi_2(\langle g_1, f_1, f_2\rangle) \qquad \square$$

We can now introduce the representation theorem to the multi-adjoint t-concept lattice $\mathcal{N}_1$ as a generalization of that by Georgescu and Popescu.

Theorem 27 *A lattice $(V, \sqsubseteq)$ is isomorphic to a complete lattice of t-concepts $(\mathcal{N}_1, \preceq)$ if and only if there exist two complete lattices $(V_1, \sqsubseteq_1)$ and $(V_2, \sqsubseteq_2)$ such that they represent the multi-adjoint concept lattices $(\mathcal{M}, \preceq)$, $(\mathcal{M}^{op}, \preceq)$, and there exists an isomorphism ν from V to the sublattice $\mathcal{V}$ of $V_1 \times V_2$.*

PROOF. Firstly, let $\psi\colon V \to \mathcal{N}_1$ be an isomorphism and $(\mathcal{M}, \preceq)$, $(\mathcal{M}^{op}, \preceq)$ the multi-adjoint concept lattices associated to the Galois connections $(^\uparrow, ^\downarrow)$ and $(^{\uparrow_{op}}, ^{\downarrow^{op}})$ respectively. Now, considering $V_1 = \mathcal{M}$, $V_2 = \mathcal{M}^{op}$ and, by the representation theorem on multi-adjoint concept lattices, $(V_1, \sqsubseteq_1)$ and $(V_2, \sqsubseteq_2)$ represent the multi-adjoint concept lattices $(\mathcal{M}, \preceq_1)$ and $(\mathcal{M}^{op}, \preceq_2)$; and, by Proposition 26, there exists an isomorphism φ from $\mathcal{N}_1$ to $\mathcal{V}$, thus $\nu = \varphi \circ \psi$ is an isomorphism from V to $\mathcal{V}$.

Conversely, we have two complete lattices $(V_1, \sqsubseteq_1)$ and $(V_2, \sqsubseteq_2)$ which represent the multi-adjoint concept lattices $(\mathcal{M}, \preceq)$, $(\mathcal{M}^{op}, \preceq)$, and an isomorphism ν from V to the sublattice $\mathcal{V}$ of $V_1 \times V_2$. Then, from Proposition 26, there exists an isomorphism $\varphi\colon \mathcal{V} \to \mathcal{N}_1$ and therefore $\varphi \circ \nu\colon V \to \mathcal{N}_1$ is an isomorphism. $\square$

As a consequence of the previous results, we can obtain the representation theorem of the framework presented in [15].

Corollary 28 *Let $I\colon B \times A \to P$ be a relation. A lattice $(V, \sqsubseteq)$ is isomorphic to $\mathcal{L}$ if and only if there exist two complete lattices $(V_1, \sqsubseteq_1)$ and $(V_2, \sqsubseteq_2)$ and five applications:*

$$\alpha_1\colon A \times L \to V_1; \quad \beta_1\colon B \times L \to V_1; \quad \nu\colon V \to V_1 \times V_2$$
$$\alpha_2\colon A \times L \to V_2; \quad \beta_2\colon B \times L \to V_2;$$

such that:

(1) $\alpha_1[A \times L]$ is infimum dense in V_1 and $\alpha_2[A \times L]$ is infimum dense in V_2.

(2) $\beta_1[B \times L]$ is supremum-dense in V_1 and $\beta_2[B \times L]$ is supremum-dense in V_2.

(3) For each $a \in A$, $b \in B$, $x, y \in L$:

$$\beta_1(b, y) \sqsubseteq_1 \alpha_1(a, x) \quad \text{iff} \quad x \,\&\, y \preceq I(b, a)$$
$$\beta_2(b, y) \sqsubseteq_2 \alpha_2(a, x) \quad \text{iff} \quad x \,\&^{op} y \preceq I(b, a)$$

(4) ν is a join-preserving monomorphism from V onto $V_1 \times V_2$ such that, for any $v \in V$, there exist $K_1, K_2 \subseteq A \times L$ satisfying that $\inf_{(a,x)\in K_1} @_a^{x\downarrow} = \inf_{(a,x)\in K_2} @_a^{x\downarrow^{op}}$ and such that $\nu(v)$ is equal to the pair:

$$\left(\inf_{(a,x)\in K_1} (\sup_{b\in B} \beta_1(b, I(b,a) \searrow x)), \inf_{(a,x)\in K_2} (\sup_{b\in B} \beta_2(b, I(b,a) \searrow x)) \right)$$

PROOF. Items (1), (2) and (3) are equivalent to the fact that V_1 and V_2 represent the concept lattices $\mathcal{L}$, $\mathcal{L}^{op}$, respectively, considering the relation $R\colon A \times B \to P$ defined as $R(a, b) = I(b, a)$ and Theorem 21.

For item (4) we will prove that the image of V by ν is $\mathcal{V}$. For this, we have only to show the following equalities:

$$\sup\{\beta_1(b, R(a, b) \searrow x) \mid b \in B\} = \alpha_1(a, x)$$
$$\sup\{\beta_2(b, R(a, b) \nearrow x) \mid b \in B\} = \alpha_2(a, x)$$

For the first one (the proof for the second is analogous) we have, by Lemma 24, that $R(a, b) \searrow x = @_a^{x\downarrow}(b)$; now, using Proposition 14, we obtain that

$$\sup\{\beta_1(b, @_a^{x\downarrow}(b)) \mid b \in B\} = \inf\{\alpha_1(a', @_a^x(a')) \mid a' \in A\} \overset{(*)}{=} \alpha_1(a, x)$$

where the equality $(*)$ holds by Proposition 18. $\square$

Finally, we present some preliminary facts about the computation of the t-concepts.

It is well known that, given a Galois connection $(^{\uparrow}, ^{\downarrow})$ the elements of the corresponding concept lattice can be generated from the fixpoints of $^{\uparrow\downarrow}$ (which are also the fixpoints of $^{\downarrow\uparrow}$). Now, we will show that the t-concepts can be obtained either from the fixpoints of the mapping $^{\uparrow\downarrow\uparrow_{op}\downarrow^{op}}$ or $^{\uparrow_{op}\downarrow^{op}\uparrow\downarrow}$. Similarly, t-concepts of $\mathcal{N}_2$ can be obtained from the fixpoints of either $^{\downarrow\uparrow\downarrow_{op}\uparrow^{op}}$ or $^{\downarrow_{op}\uparrow^{op}\downarrow\uparrow}$. As a consequence, we have a tool to obtain the minimum t-concept containing a given subset g or f and, in particular, a method to obtain all t-concepts in either $\mathcal{N}_1$ or $\mathcal{N}_2$.

From now on, we will write $^{\Uparrow}$, $^{\Downarrow}$ instead of $^{\uparrow_{op}}$, $^{\downarrow^{op}}$, respectively. Firstly, let us start with the following result, which is applicable to any pair of Galois connections.

Proposition 29 *Consider two lattices $\langle L_1, \preceq_1 \rangle$, $\langle L_2, \preceq_2 \rangle$, two Galois connections $(^{\uparrow}, ^{\downarrow})$, $(^{\Uparrow}, ^{\Downarrow})$ between them, and one element $g \in L_2$, then the following statements are equivalent:*

(1) g is a fixpoint of $^{\uparrow\downarrow\Uparrow\Downarrow} \colon L_2 \to L_2$.
(2) g is a fixpoint of $^{\uparrow\downarrow} \colon L_2 \to L_2$, and of $^{\Uparrow\Downarrow} \colon L_2 \to L_2$.
(3) g is a fixpoint of $^{\Uparrow\Downarrow\uparrow\downarrow} \colon L_2 \to L_2$.

PROOF. $(1 \implies 2)$. We have to show that $g = g^{\uparrow\downarrow}$. This results as a consequence of the following chain of inequalities

$$g \le g^{\uparrow\downarrow} \le g^{\uparrow\downarrow\Uparrow\Downarrow} = g$$

which hold because $(^{\uparrow}, ^{\downarrow})$ and $(^{\Uparrow}, ^{\Downarrow})$ are Galois connections.

The result for the other Galois connection, as g is a fixpoint of $^{\uparrow\downarrow\Uparrow\Downarrow}$, one can write $g^{\Uparrow\Downarrow} = g^{\uparrow\downarrow\Uparrow\Downarrow\Uparrow\Downarrow} \overset{(*)}{=} g^{\uparrow\downarrow\Uparrow\Downarrow} = g$, where $(*)$ follows from the properties of the Galois connection $(^{\Uparrow}, ^{\Downarrow})$.

$(2 \implies 1)$. As $g = g^{\uparrow\downarrow} = g^{\Uparrow\Downarrow}$, we have $g^{\uparrow\downarrow\Uparrow\Downarrow} = g^{\Uparrow\Downarrow} = g$.

The equivalence between (2) and (3) follows similarly. $\square$

The following corollary states that $\mathcal{N}_1$ coincides with the set of fixpoints of $^{\uparrow\downarrow\Uparrow\Downarrow} \colon L^B \to L^B$.

Corollary 30 *Given the lattice of t-concepts $\mathcal{N}_1$, we have $\langle g, g^{\uparrow}, g^{\Uparrow} \rangle \in \mathcal{N}_1$ if and only if $g \in L^B$ is a fixpoint of $^{\uparrow\downarrow\Uparrow\Downarrow} \colon L^B \to L^B$.*

6 A working example

A store has three different departments (decor, wear, and a travel agency) and four candidates from the staff which will be promoted. The owners of the store want to designate one general manager out of these four, whereas the other three will be promoted to head of each department.

Assume that there exists a fuzzy relation between each worker and the different departments (which could have been made by using, for instance, the time that each one has worked for each department, the sales obtained in the last evaluation period on each department, etc). According to the discussion in the board of directors, the choice can be made on the basis of giving more importance to values associated to each worker, or to those associated to each department. Although it was clear that neither condition was determinant, finally, there was no agreement on which of the two conditions is more important.

A non-commutative operator such as $\& \colon [0,1] \times [0,1] \to [0,1]$ defined by $x \,\&\, y = x^2 \cdot y$, might be useful for the purposes of this example.

It is not difficult to check that the operators $\nearrow \colon [0,1] \times [0,1] \to [0,1]$ and $\nwarrow \colon [0,1] \times [0,1] \to [0,1]$ defined by

$$
z \nearrow y = \begin{cases} 1 & \text{if } y = 0; \\ \min\left\{\sqrt{\dfrac{z}{y}}, 1\right\} & \text{otherwise.} \end{cases}
\qquad
z \nwarrow x = \begin{cases} 1 & \text{if } x = 0; \\ \min\left\{\dfrac{z}{x^2}, 1\right\} & \text{otherwise.} \end{cases}
$$

for all $x, y, z \in [0,1]$, allow to build an adjoint triple $(\&, \nearrow, \nwarrow)$. Note that the two different implications are suitable representations of the two possible biases between worker-based or department-based reasoning. This way, if we consider the corresponding t-concept, we would obtain a unique evaluation of the candidates integrating both criteria.

R	Anne	Beth	Chris	Daphne
Decor	0.2	0.5	0.8	0.6
Wear	0.6	0.65	0.7	0.4
Travel	0.4	0.6	0.7	0.2

Fig. 8. Fuzzy relation between the objects and the attributes.

As a result, we can interpret the initial problem as that of finding a t-concept by starting from a mapping representing the initial state and considering the set $A = \{\text{Decor}, \text{Wear}, \text{Travel}\}$ as the set of attributes, and the set $B = \{\text{Anne}, \text{Beth}, \text{Chris}, \text{Daphne}\}$ as the set of objects, which are related by the fuzzy relation R in Figure 8.

	g_0		g	$g^{\uparrow}$	$g^{\Uparrow}$	
Anne	0		0.49	0.64	0.83	Decor
Beth	0		0.78	0.91	1	Wear
Chris	0		0.84	0.67	0.99	Travel
Daphne	0		0.45			

Fig. 9. t-concept to obtain the manager.

Obtaining the best candidate for Decor

	f_0^D		f		$f^{\downarrow}$	$f^{\Downarrow}$
Decor	1		1	Anne	0.2	0.45
Wear	0		0.87	Beth	0.5	0.71
Travel	0		0.62	Chris	0.8	0.89
				Daphne	0.52	0.57

Obtaining the best candidate for Wear

	f_0^W		f		$f^{\downarrow}$	$f^{\Downarrow}$
Decor	0		0.58	Anne	0.6	0.59
Wear	1		1	Beth	0.65	0.81
Travel	0		0.71	Chris	0.7	0.84
				Daphne	0.4	0.53

Obtaining the best candidate for Travel

	f_0^T		f		$f^{\downarrow}$	$f^{\Downarrow}$
Decor	0		0.71	Anne	0.4	0.53
Wear	0		1	Beth	0.6	0.77
Travel	1		1	Chris	0.7	0.84
				Daphne	0.2	0.45

Fig. 10. t-concepts for obtaining the chief of each department.

In order to detect the best candidate, we start from the constantly zero mapping g_0 in order to obtain, by iteration, a least fixpoint and obtain the extension of the corresponding t-concept, as stated in the previous section. The obtained values are presented in Figure 9. As a result, we notice that Chris is globally the best candidate, and should be promoted to general manager.

Now, in order to perform the choice of the head of each department, we start

from an evaluation which focuses on each department separately. In this case, we calculate the t-concepts starting from the evaluations f_0^D, f_0^W, f_0^T. Attending to the obtained results, see Figure 10, the final decision can be made.

One notices that Chris gets the best scores, if we discard this result, since she will be promoted to General Manager, we obtain that Decor will be headed by Daphne, Wear by Anne, and Travel by Beth.

t-concept related to Anne

	g_0		g		$g^{\uparrow}$	$g^{\Uparrow}$
Anne	1		1	Decor	0.45	0.2
Beth	0		1	Wear	3/4	0.6
Chris	0		1	Travel	0.53	0.4
Daphne	0		0.71			

t-concept related to Beth

	g_0		g		$g^{\uparrow}$	$g^{\Uparrow}$
Anne	0		0.63	Decor	0.56	0.5
Beth	1		1	Wear	0.81	0.65
Chris	0		1	Travel	0.59	0.6
Daphne	0		0.58			

t-concept related to Chris

	g_0		g		$g^{\uparrow}$	$g^{\Uparrow}$
Anne	0		0.59	Decor	0.58	0.58
Beth	0		0.93	Wear	0.84	0.7
Chris	1		1	Travel	0.61	0.7
Daphne	0		0.54			

t-concept related to Daphne

	g_0		g		$g^{\uparrow}$	$g^{\Uparrow}$
Anne	0		0.63	Decor	0.56	0.5
Beth	0		1	Wear	0.63	0.4
Chris	0		1	Travel	0.45	0.2
Daphne	1		1			

Fig. 11. t-concepts for each candidate

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Assigning Anne the alternative adjoint triple

	g_0		g		$g^{\uparrow}$	$g^{\Uparrow}$
Anne	0		0.58	Decor	0.84	0.8
Beth	0		0.71	Wear	0.84	0.7
Chris	0		0.84	Travel	0.67	0.7
Daphne	0		0.45			

Assigning Beth the alternative adjoint triple

	g_0		g		$g^{\uparrow}$	$g^{\Uparrow}$
Anne	0		0.53	Decor	0.61	0.71
Beth	0		1	Wear	0.87	0.81
Chris	0		0.93	Travel	0.63	0.78
Daphne	0		0.51			

Fig. 12. Alternative t-concepts for obtaining the manager.

Another approach might have been performed in order to confirm the decisions before the announcement. For instance, consider a different starting point in which each candidate is evaluated separately against each department; this way, one starts with evaluations g_0^A, g_0^B, g_0^C, g_0^D being the characteristic functions for each of the candidates. The results are presented in Figure 11.

After analyzing the results, one obtains again the same results, in that Chris should be the general manager, Decor will be headed by Daphne, Wear by Anne, and Travel by Beth.

Now, consider that a particular worker receives some added value due, for instance, to a policy of equality in order to balance the staff. We can implement a uniform extra score to this particular worker by assigning her a different adjoint triple. It is worth to remark that the assignment of a different adjoint triple to a worker needs not modify the final results. In Figure 12, alternative results concerning the choice of manager are presented; in each case, a worker has been assigned the adjoint triple associated to the conjunctor x^3y^2. We see that, although Anne's data are treated with the alternative conjunctor, the final choice for the manager does not get modified; however, Beth becomes the best candidate when she is assigned the alternative conjunctor.

7 Conclusions

The concept lattice of t-concepts has been introduced as a generalization of [15]. Continuing the study of the multi-adjoint concept lattices, we showed that the common information to the two sided concept lattices generated from the two possible residual implications associated to a non-commutative conjunctor, can be seen as a sublattice of the Cartesian product of both concept lattices. Such common information can be thought of as "neutral" information with regard to the non-commutativity of the conjunctor. The resulting theory allows for obtaining a simpler proof of a general representation theorem for t-concepts, which can be easily instantiated to obtain the representation theorem in [15]. A working example has been presented which shows the flexibility and expressive power of the use of t-concepts.

References

[1] A. Abdel-Hamid and N. Morsi. Associatively tied implicacions. *Fuzzy Sets and Systems*, 136(3):291–311, 2003.

[2] C. Alcalde, A. Burusco, R. Fuentes-González, and I. Zubia. Treatment of L-fuzzy contexts with absent values. *Information Sciences*, 179:1–15, 2009.

[3] R. Bělohlávek. Fuzzy concepts and conceptual structures: induced similarities. In *Joint Conference on Information Sciences*, pages 179–182, 1998.

[4] R. Bělohlávek. Lattices of fixed points of fuzzy Galois connections. *Mathematical Logic Quartely*, 47(1):111–116, 2001.

[5] R. Bělohlávek. Concept lattices and order in fuzzy logic. *Annals of Pure and Applied Logic*, 128:277–298, 2004.

[6] R. Bělohlávek. A note on variable threshold concept lattices: Threshold-based operators are reducible to classical concept-forming operators. *Information Sciences*, 177(15):3186–3191, 2007.

[7] R. Bělohlávek, V. Vychodil. What is a fuzzy concept lattice? In *Proc of 3rd Intl Conf on Concept Lattices and Their Applications*, CLA 2005, Olomouc, Czech Republic, pp. 34-45, 2005.

[8] A. Burusco and R. Fuentes-González. The study of *L*-fuzzy concept lattice. *Mathware & Soft Computing*, 3:209–218, 1994.

[9] B. Davey and H. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, second edition, 2002.

[10] P. du Boucher-Ryana and D. Bridge. Collaborative recommending using formal concept analysis. *Knowledge-Based Systems*, 19(5):309–315, 2006.

[11] S.-Q. Fan, W.-X. Zhang, and W. Xu. Fuzzy inference based on fuzzy concept lattice. *Fuzzy Sets and Systems*, 157(24):3177–3187, 2006.

[12] A. Formica. Ontology-based concept similarity in formal concept analysis. *Information Sciences*, 176(18):2624–2641, 2006.

[13] A. Formica. Concept similarity in formal concept analysis: An information content approach. *Knowledge-Based Systems*, 21(1):80–87, 2008.

[14] B. Ganter and R. Wille. *Formal Concept Analysis: Mathematical Foundation.* Springer Verlag, 1999.

[15] G. Georgescu and A. Popescu. Concept lattices and similarity in non-commutative fuzzy logic. *Fundamenta Informaticae*, 55(1):23–54, 2002.

[16] G. Jiang, K. Ogasawara, A. Endoh, and T. Sakurai. Context-based ontology building support in clinical domains using formal concept analysis. *International Journal of Medical Informatics*, 71(1):71–81, 2003.

[17] P. Julián, G. Moreno, and J. Penabad. On fuzzy unfolding: A multi-adjoint approach. *Fuzzy Sets and Systems*, 154:16–33, 2005.

[18] S. Krajči. Every concept lattice with hedges is isomorphic to some generalized concept lattice. In *Intl Workshop on Concept Lattices and their Applications*, pages 1–9, 2005.

[19] S. Krajči. A generalized concept lattice. *Logic Journal of IGPL*, 13(5):543–550, 2005.

[20] S. O. Kuznetsov. Complexity of learning in concept lattices from positive and negative examples. *Discrete Applied Mathematics*, 142:111–125, 2004.

[21] Y. Lei and M. Luo. Rough concept lattices and domains. *Annals of Pure and Applied Logic* 159(3):333–340, 2009.

[22] M. Liu, M. Shao, W. Zhang, and C. Wu. Reduction method for concept lattices based on rough set theory and its application. *Computers & Mathematics with Applications*, 53(9):1390–1410, 2007.

[23] X. Liu, W. Wang, T. Chai, and W. Liu. Approaches to the representations and logic operations of fuzzy concepts in the framework of axiomatic fuzzy set theory I. *Information Sciences*, 177(4):1007–1026, 2007.

[24] X. Liu, W. Wang, T. Chai, and W. Liu. Approaches to the representations and logic operations of fuzzy concepts in the framework of axiomatic fuzzy set theory II. *Information Sciences*, 177(4):1027–1045, 2007.

[25] J. Medina, M. Ojeda-Aciego, and J. Ruiz-Calviño. On multi-adjoint concept lattices: definition and representation theorem. *Lect. Notes in Computer Science*, 4390:197–209, 2007.

[26] J. Medina, M. Ojeda-Aciego, and J. Ruiz-Calviño. Relating generalized concept lattices with concept lattices for non-commutative conjunctors. *Applied Mathematics Letters*, 21(12):1296–1300, 2008.

[27] J. Medina, M. Ojeda-Aciego, and J. Ruiz-Calviño. Formal concept analysis via multi-adjoint concept lattices. *Fuzzy Sets and Systems*, 160(2):130–144, 2009.

[28] J. Medina, M. Ojeda-Aciego, and P. Vojtáš. Towards biresiduated multi-adjoint logic programming. *Lect. Notes in Artificial Intelligence*, 3040:608–617, 2003.

[29] J. Medina, M. Ojeda-Aciego, and P. Vojtáš. Similarity-based unification: a multi-adjoint approach. *Fuzzy Sets and Systems*, 146(1):43–62, 2004.

[30] V. Phan-Luong. A framework for integrating information sources under lattice structure. *Information Fusion*, 9:278–292, 2008.

[31] S. Pollandt. *Fuzzy Begriffe*. Springer, Berlin, 1997.

[32] K.-S. Qu and Y.-H. Zhai. Generating complete set of implications for formal contexts. *Knowledge-Based Systems*, 21:429–433, 2008.

[33] M.-W. Shao, M. Liu, and W.-X. Zhang. Set approximations in fuzzy formal concept analysis. *Fuzzy Sets and Systems*, 158(23):2627–2640, 2007.

[34] L. Wang and X. Liu. Concept analysis via rough set and afs algebra. *Information Sciences*, 178(21):4125–4137, 2008.

[35] X. Wang and W. Zhang. Relations of attribute reduction between object and property oriented concept lattices. *Knowledge-Based Systems*, 21(5):398–403, 2008.

[36] Q. Wu and Z. Liu. Real formal concept analysis based on grey-rough set theory. *Knowledge-Based Systems*, 22(1):38–45, 2009.

[37] W.-X. Zhang, J.-M. Ma, and S.-Q. Fan. Variable threshold concept lattices. *Information Sciences*, 177(22):4883–4892, 2007.