On the Dedekind-MacNeille completion and formal concept analysis based on multilattices

J. Medina[a], M. Ojeda-Aciego[b], J. Pócs[c], E. Ramírez-Poussa[d]

[a]*Department of Mathematics, University of Cádiz. Spain*
Email: `jesus.medina@uca.es`
[b]*Universidad de Málaga. Dept Matemática Aplicada, Spain*
Email: `aciego@uma.es`
[c]*Dept of Algebra and Geometry, Palacký University Olomouc, Czech Republic, and Mathematical Institute, Slovak Academy of Sciences, Košice, Slovakia*
Email: `pocs@saske.sk`
[d]*Department of Statistic and O.R., University of Cádiz. Spain*
Email: `eloisa.ramirez@uca.es`

Abstract

The Dedekind-MacNeille completion of a poset P can be seen as the least complete lattice containing P. In this work, we analyze some results concerning the use of this completion within the framework of Formal Concept Analysis in terms of the poset of concepts associated with a Galois connection between posets. Specifically, we show an interesting property of the Dedekind-MacNeille completion, in that the completion of the concept poset of a Galois connection between posets coincides with the concept lattice of the Galois connection extended to the corresponding completions. Moreover, we study the specific case when P has multilattice structure and state and prove the corresponding representation theorem.

Key words: Multilattices, Dedekind-MacNeille completion, formal concept analysis

1. Introduction

The Dedekind-MacNeille completion of a partially ordered set P was introduced by H.M. MacNeille in [14] as a generalization of Dedekind's method for constructing the field of the real numbers from the rational numbers. In a few words, one can say that the Dedekind-MacNeille completion of a poset $(P, \leq)$ is the smallest complete lattice that contains P.

This construction has already played a role in the research topic of formal concept analysis in which, for instance, the concept lattice corresponding to the general ordinal scale associated to a poset is precisely the Dedekind-MacNeille completion of P, see [10]. The problem of actually constructing the completion of a finite poset is very interesting from a practical standpoint, and it is not surprising that several researchers have devised algorithms for constructing it.

On the other hand, multilattices are structures in which the restrictions imposed on a (complete) lattice, namely, the "existence of *least* upper (resp. *greatest* lower) bounds" is weakened to "existence of *minimal* upper (resp. *maximal* lower) bounds".

Multilattices are examples of hyperstructures which have proved to be useful in some areas of informatics [22]. Particularly, it is worth to note that the free monoid X^* over a set X has the structure of a multilattice when considering the *substring ordering* (see [12, 19]). Still in X^*, a multilattice can be obtained when considering the ordering between *subsequences* defined in terms that one sequence can be obtained from another by deleting some elements in the latter [8, 19].

Although introduced in a theoretical framework more than fifty years ago [3], they have been used as practical tools to handle uncertain information [15, 4]; specifically, they can be used as suitable structures capable of describing certain aspects of uncertainty, and reasoning with incomplete information. We will follow the algebraic formalization given in [5].

Precisely, it is in this respect where one finds the link between multilattices and Formal Concept Analysis (FCA); specifically, related to the many approaches that can be found in papers aimed at generalizing FCA in order to deal with uncertainty, imprecise data, or incomplete information, which have provided different abstract frameworks [20, 11, 2, 1, 17], ranging from residuated lattices, to non-commutative conjunctors, and to multi-adjoint lattices. Non-commutativity enables passing from adjoint pairs (generalization of conjunction and implication in a residuated lattice) to adjoint triples [6]. Adjoint triples on lattices have proven to be a useful tool when working in fuzzy formal concept analysis. Furthermore, in [18] it was shown that they can play an important role as well within the framework of multilattices, especially in order to form the Galois connections needed to build concepts in a multilattice-based framework.

This paper studies an extension of the usual theory of FCA, in that we seemingly assume the most general framework for the corresponding constructions. Firstly, we aim at showing that the Dedekind-MacNeille completion behaves adequately with respect to the FCA construction of the concepts, in that the completion of the concept poset coincides with the concept lattice of the corresponding completions of the initial posets. The second objective is to prove the representation theorem, so-called the basic theorem in [10], for the theory of multilattice-based FCA; in this respect, a number of intermediate technical results are stated and proved in order to serve as the theoretical tools for the proof of the basic theorem.

2. Preliminaries

In this section, in order to make this paper as self-contained as possible, we recall the preliminary definitions of multilattices, Formal Concept Analysis, and Dedekind-MacNeille completion.

2.1. Multilattices

To begin with, let us recall the definition of a complete lattice:

Definition 1. A *complete lattice* is a poset $(L, \leq)$ where every subset of L has supremum and infimum.

When the existence of supremum (infimum) element is replaced by the existence of minimal (maximal) elements of the upper (lower) bounds of a subset, the notion of multilattice arises. In order to formalize this definition, the following notions are needed.

Definition 2. Let $(P, \leq)$ be a poset and $K \subseteq P$, we say that:

- K is called a *chain* if for every two elements $x, y \in K$ we have that either $x \leq y$ or $y \leq x$.
- K is called an *antichain* if none of its elements are comparable, i.e., for every different $x, y \in K$ we have both $x \nleq y$ and $y \nleq x$.

Definition 3. A poset $(P, \leq)$ is called *chain-complete (also termed* coherent *in some references)* if every chain has supremum and infimum.

Once we have recalled these notions, the definition of a complete multilattice is given below.

Definition 4 ([5]). A chain-complete poset $(M, \leq)$ is said to be a *complete multilattice* if for each subset X the set of upper (resp. lower) bounds of X has minimal (resp. maximal) elements.

Each minimal (resp. maximal) element of the upper (resp. lower) bounds of a subset is called *multisupremum* (resp. *multinfimum*). The set of all multisuprema, resp. multinfima, of X will be denoted by $\mathrm{msup}(X)$, resp. $\mathrm{minf}(X)$.

Remark 5. Note that, by definition, the set $\mathrm{msup}(X)$, resp. $\mathrm{minf}(X)$, is never empty. Particularly, every complete multilattice has a bottom and a top element. Moreover, note that the two sets $\mathrm{msup}(X)$ and $\mathrm{minf}(X)$ are always antichains.

Some examples of multilattices (both with finitely and with infinitely many elements) are given below.

Example 6.

- The Hasse diagram of the smallest multilattice which is not a lattice can be seen in Figure 1; this multilattice is denoted as $M6$.

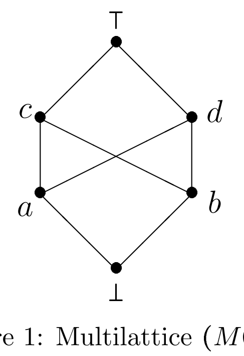

Figure 1: Multilattice $(M6, \leq)$

  If we consider the subset $\{a, b\}$, we have that $\mathrm{minf}\{a, b\}$ is a singleton $\{\bot\}$, however, the set $\mathrm{msup}\{a, b\}$ is formed by two incomparable elements, $\{c, d\}$. Analogously, $\mathrm{minf}\{c, d\} = \{a, b\}$ and $\mathrm{msup}\{c, d\} = \{\top\}$.

- Example of infinite multilattices are given in Figure 2.

- Figure 3 introduces a poset which is not a complete multilattice because it is not chain-complete. Notice that, in the diagram, the elements e and f are both upper bounds of the set $\{c_1, c_2, \ldots\}$.

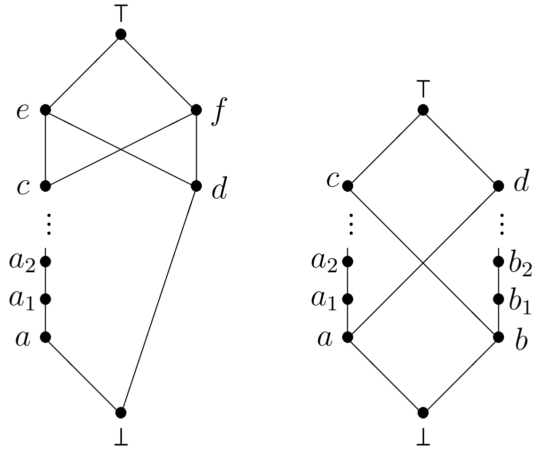

Figure 2: Infinite multilattices

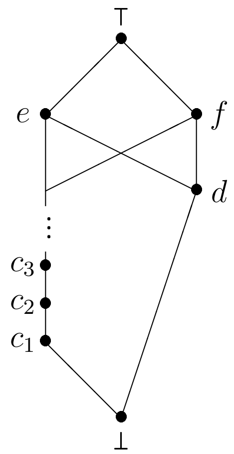

Figure 3: A poset which is not chain-complete

$\square$

The following result is obtained directly from the fact that each complete multilattice is a chain-complete poset.

Proposition 7 ([17]). *Given a complete multilattice* $(M, \leq)$*, every upper (resp. lower) bound of a subset* $X \subseteq M$ *is greater (resp. smaller) than at least one multisupremum (resp. multinfimum) of* X*.*

Although the following remark can be straightforwardly obtained, we prefer to formally state it since it will be used later.

Remark 8. Given $X \subseteq M$, if $\mathrm{minf}(X) \cap X \neq \varnothing$, then X has a minimum.

2.2. Closure operators and closure systems

As the concepts (that is, the basic constructions in FCA) are closed elements under certain constructions, we give here the preliminary notions needed in relation to closure operators and closure systems.

Definition 9. Given a poset $(P, \leq)$, a *closure operator* on P is a mapping $c\colon P \to P$ which is monotone, inflationary and idempotent. Specifically, this means the following conditions for all $x, y \in P$

1. $x \leq y$ implies $c(x) \leq c(y)$
2. $x \leq c(x)$
3. $c(x) = c(c(x))$

Let L be a complete lattice. A subset $S \subseteq L$ is a *closure system* if for all $X \subseteq S$ we have that $\inf(X) \in S$.

In this case, every closure operator gives rise to a closure system and vice versa, as the following proposition shows.

Proposition 10. *Let c be a closure operator on a complete lattice $(L, \sqcap, \sqcup)$. Then the family $S_c = \{x \in L \mid c(x) = x\}$ of closed elements of L is a closure system, and forms a complete lattice when ordered by inclusion, in which for any $X \subseteq S_c$ the supremum and infimum are defined by*

$$\bigwedge X = \bigsqcap X \qquad\qquad \bigvee X = c(\bigsqcup X).$$

Conversely, given a closure system S in L, then $E_S(x) = \bigsqcap\{y \in S \mid x \leq y\}$ defines a closure operator E_S on L.

2.3. Galois connections and Formal Concept Analysis

The notion of Galois connection, which we recall here, will play as well an important role hereafter.

Definition 11 ([7]). Let ${}^{\downarrow}\colon P \to Q$ and ${}^{\uparrow}\colon Q \to P$ be two mappings between the posets $(P, \leq)$ and $(Q, \leq)$. The pair $({}^{\uparrow}, {}^{\downarrow})$ is called a *Galois connection* if:

- $p_1 \leq p_2$ implies ${p_2}^{\downarrow} \leq {p_1}^{\downarrow}$, for every $p_1, p_2 \in P$;
- $q_1 \leq q_2$ implies ${q_2}^{\uparrow} \leq {q_1}^{\uparrow}$, for every $q_1, q_2 \in Q$;
- $p \leq p^{\downarrow\uparrow}$ and $q \leq q^{\uparrow\downarrow}$, for all $p \in P$ and $q \in Q$.

An interesting property of a Galois connection $({}^{\uparrow}, {}^{\downarrow})$ is that ${}^{\downarrow} = {}^{\downarrow\uparrow\downarrow}$ and ${}^{\uparrow} = {}^{\uparrow\downarrow\uparrow}$, where the chain of arrows means their composition.

Once we have a Galois connection, we can focus on the pairs of elements (p, q) which are the image of each other by the application of the corresponding arrow. These pairs can be seen as fixed points of the Galois connection, and are usually called *concepts*. We follow [10] for the main notions of FCA:

Definition 12. A pair (p, q) is said to be a *concept* of the Galois connection $({}^{\uparrow}, {}^{\downarrow})$ if $p^{\downarrow} = q$ and $q^{\uparrow} = p$.

The set of concepts can be ordered by defining $(p_1, q_1) \leq (p_2, q_2)$ if and only if $p_1 \leq p_2$ (or equivalently $q_2 \leq q_1$). The resulting poset will be denoted $\mathsf{CP}(P, Q, {}^{\uparrow}, {}^{\downarrow})$.

In the case that P and Q are lattices, the following result holds:

Theorem 13 ([10]). *Let $(L_1, \leq_1)$ and $(L_2, \leq_2)$ be two complete lattices and $({}^{\uparrow}, {}^{\downarrow})$ a Galois connection between them, then we have that $\mathsf{CP}(L_1, L_2, {}^{\uparrow}, {}^{\downarrow})$ is a complete lattice, and the constructions of infima and suprema are given below:*

$$\bigwedge_{i \in I}(x_i, y_i) = \left(\bigwedge_{i \in I} x_i, (\bigvee_{i \in I} y_i)^{\uparrow\downarrow}\right) \qquad \bigvee_{i \in I}(x_i, y_i) = \left((\bigvee_{i \in I} x_i)^{\downarrow\uparrow}, \bigwedge_{i \in I} y_i\right)$$

In this case, we will stress the fact that the set of concepts is a lattice by writing $\mathsf{CL}(L_1, L_2, {}^{\uparrow}, {}^{\downarrow})$.

The following definitions introduce the notion of supremum-dense (resp. infimum-dense) subset, and dual isomorphism, which will be useful later in relation to the basic theorem of FCA for multilattices.

Definition 14. Let $(L, \leq)$ be a lattice and let $Q \subseteq L$, we say that the subset Q is *supremum-dense* in L if for every element $a \in L$ there is a subset $A \subseteq Q$ such that a is the supremum of A. The dual of supremum-dense is *infimum-dense*.

Definition 15. Let $(P, \leq)$ and $(Q, \leq)$ be two posets and φ a mapping from P onto Q such that $x \leq y$ in P if and only if $\varphi(y) \leq \varphi(x)$ in Q. Then, the mapping φ is called *dual isomorphism*.

2.4. Adjoint triples and Formal Concept Analysis

Finally, we will recall some extensions of notions about formal concept analysis based on the so-called adjoint triples, which can be seen as operators that arise as a generalization of a triangular norm and its residuated implication. These operators will be considered later in Section 5, and are taken from [17].

Definition 16. Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets and consider mappings $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$, then $(\&, \swarrow, \nwarrow)$ is said to be an *adjoint triple* with respect to P_1, P_2, P_3, if $\&$, $\swarrow, \nwarrow$ satisfy the *adjoint property*: For all $x \in P_1$, $y \in P_2$, $z \in P_3$

$$x \leq_1 z \swarrow y \quad \text{iff} \quad x \,\&\, y \leq_3 z \quad \text{iff} \quad y \leq_2 z \nwarrow x$$

It is worth to recall that the conjunctor of an adjoint triple was called *biresiduated mapping* in [23].

A small example of a non-trivial adjoint triple defined on the multilattice $M6$ is given below.

Example 17. Considering $M6$, the triple $(\&, \swarrow, \nwarrow)$, defined in Table 1 forms an adjoint triple with respect to $M6$. Notice that we have considered $x \,\&\, b = x \,\&\, c$, $x \,\&\, d = x \,\&\, \top$ and $d \,\&\, x = \top \,\&\, x$, for all $x \in M6$.

Of course, the bigger the multilattice the more complex operators have to be defined. □

Table 1: Definition of $\&$, $\swarrow$ and $\nwarrow$

$\&$	$\bot$	a	b	c	d	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	a	a	a	a
b	$\bot$	$\bot$	c	c	$\top$	$\top$
c	$\bot$	a	c	c	$\top$	$\top$
d	$\bot$	a	$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	a	$\top$	$\top$	$\top$	$\top$

$\swarrow$	$\bot$	a	b	c	d	$\top$
$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\top$	$\top$	a	a	a	a
b	$\top$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
c	$\top$	$\top$	c	c	a	a
d	$\top$	$\top$	a	a	a	a
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\nwarrow$	$\bot$	a	b	c	d	$\top$
$\bot$	$\top$	$\bot$	a	$\bot$	$\bot$	$\bot$
a	$\top$	$\top$	a	a	a	a
b	$\top$	$\bot$	a	$\bot$	$\bot$	$\bot$
c	$\top$	$\top$	c	c	a	a
d	$\top$	$\top$	a	a	a	a
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

Based on the definition of adjoint triple we can define the notion of *frame*.[1]

Definition 18. A *frame* $\mathcal{L}$ is a tuple

$$(L_1, L_2, P, \preceq_1, \preceq_2, \leq, \&, \swarrow, \nwarrow)$$

where $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$ are complete lattices, $(P, \leq)$ is a poset and, $(\&, \swarrow, \nwarrow)$ is an adjoint triple with respect to L_1, L_2, P. These frames are denoted as $(L_1, L_2, P, \&)$.

Given a frame, a *context* is a tuple consisting of sets of objects, attributes and a fuzzy relation among them. Formally,

Definition 19. Let $(L_1, L_2, P, \&)$ be a frame, a *context* is a tuple (A, B, R) such that A and B are nonempty sets (interpreted as attributes and objects, respectively) and R is a P-fuzzy relation $R\colon A \times B \to P$.

L_1^A and L_2^B denote the set of fuzzy subsets $f\colon A \to L_1$, $g\colon B \to L_2$, respectively. From the partial orders in $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$, a pointwise partial order can be considered which provides L_1^A and L_2^B with the structure of complete lattice. Abusing notation, $(L_1^A, \preceq_1)$ and $(L_2^B, \preceq_2)$ are complete lattices where $\preceq_1$ and $\preceq_2$ are defined pointwise.

Given a fixed frame and a context for that frame, the concept-forming operators ${}^\uparrow\colon L_2^B \longrightarrow L_1^A$ and ${}^\downarrow\colon L_1^A \longrightarrow L_2^B$ are defined, for all $g \in L_2^B$, $f \in L_1^A$ and $a \in A$, $b \in B$, as

$$g^\uparrow(a) = \inf\{R(a, b) \swarrow g(b) \mid b \in B\} \tag{1}$$
$$f^\downarrow(b) = \inf\{R(a, b) \nwarrow f(a) \mid a \in A\} \tag{2}$$

These two arrows form a Galois connection [17]. Therefore, a *fuzzy concept* is a pair $\langle g, f\rangle$ satisfying that $g \in L_2^B$, $f \in L_1^A$ and that $g^\uparrow = f$ and $f^\downarrow = g$; with $({}^\uparrow, {}^\downarrow)$ being the Galois connection defined above.

[1] It is unfortunate that the term *frame* has become a consolidated keyword with different meanings within both Formal Concept Analysis and Lattice Theory. The only use of the term "*frame*" in this work is to represent the structure on which formal contexts will be interpreted.

Definition 20. The *fuzzy concept lattice* associated with a fuzzy frame $(L_1, L_2, P, \&)$ and a context (A, B, R) is the set

$$\mathfrak{B}(A,B,R) = \{\langle g, f\rangle \mid g \in L_2^B, f \in L_1^A \text{ and } g^{\uparrow} = f, f^{\downarrow} = g\}$$

in which the ordering is defined by $\langle g_1, f_1\rangle \le \langle g_2, f_2\rangle$ if and only if $g_1 \le_2 g_2$ (equivalently $f_2 \le_1 f_1$).

Since $\mathfrak{B}(A,B,R)$ coincides with $\mathsf{CL}(L_1^A, L_2^B, {}^{\uparrow}, {}^{\downarrow})$, by Theorem 13, we have that it is a complete lattice. Moreover, notice that the concept lattice $\mathfrak{B}(A,B,R)$ is a particular case of a multi-adjoint concept lattice [16] with only one adjoint triple, which turns out to be equivalent to the generalized concept lattice given by Krajči [13].

To finish this section, we introduce the important notion of left-continuity in the general framework of multilattices. This notion will be applied to the conjunctor in an adjoint triple.

Definition 21. Let $(M_1, \le_1)$, $(M_2, \le_2)$ be two multilattices and $(P, \le)$ a poset, and $\&\colon M_1 \times M_2 \to P$ a mapping among them. Given elements $m_2 \in M_2$ and $p \in P$, we consider the following set

$$X_{m_2}^p = \{m_1 \in M_1 \mid m_1 \,\&\, m_2 \le p\}$$

and define $\&$ to be *left-continuous* in M_1 if for all $m_2 \in M_2$ and $p \in P$, and for all nonempty subset $K_1 \subseteq X_{m_2}^p$, the inclusion $\operatorname{msup}(K_1) \subseteq X_{m_2}^p$ holds.

3. Closure systems in multilattices

This section introduces the definition of closure system in a multilattice, several properties are proved and, finally, the characterization in terms of a closure operator is given. A complete multilattice $(M, \le)$ will be fixed in all this section.

The first definition is clearly a natural generalization of a closure system in a lattice.

Definition 22. A set $S \subseteq M$ is a *closure system* in M, if for all[2] $X \subseteq S$ $\operatorname{minf}(X) \subseteq S$ holds.

[2]Note that the subset X can be empty.

The following results relate closure operators to closure systems on a multilattice. The first one shows that the set of fixed points of a closure operator gives rise to a closure system.

Lemma 23. *Let c be a closure operator on M, then the set of fixed points $S_c = \{x \in M \mid c(x) = x\}$ forms a closure system in M.*

Proof. We have to prove that $\operatorname{minf}(X) \subseteq S_c$, for all $X \subseteq S_c$. For that purpose, we have to distinguish two cases.

Firstly, if we consider $X = \varnothing \subseteq S_c$, then $\operatorname{minf}(X)$ consists of the top element of M, which is fixed point because c is inflationary (Definition 9(2)). Therefore, we obtain that $\operatorname{minf}(X) \subseteq S_c$.

Secondly, given a nonempty $X \subseteq S_c$, we can consider an element $m \in \operatorname{minf}(X)$, since $\operatorname{minf}(X)$ is always nonempty (see Remark 5). Then we have $c(m) \leq c(x) = x$, for all $x \in X$, i.e. $c(m)$ is a lower bound of X. Since $m \leq c(m)$ and m is a maximal lower bound of X, then $m = c(m)$. Therefore, $\operatorname{minf}(X) \subseteq S_c$ and S_c is a closure system. □

The following technical lemma will be fundamental in order to define a closure operator from a closure system.

Lemma 24. *Given a closure system $S \subseteq M$ and $y \in M$, then the set $\{x \in S \mid y \leq x\}$ has a minimum.*

Proof. From the definition of complete multilattice, since y is a lower bound, there exists at least one element $m \in \operatorname{minf}\{x \in S \mid y \leq x\}$ such that $y \leq m$; in addition, as S is a closure system, then $m \in S$. Thus, by Remark 8, the set $\{x \in S \mid y \leq x\}$ has a minimum, actually m is the minimum. □

Consequently, given a closure system $S \subseteq M$, the mapping $E_S\colon M \to M$, defined by $E_S(y) = \min\{x \in S \mid y \leq x\}$, is a closure operator on M.

The previous results provide the generalization of the well-known relationship between closure systems and closure operators.

Theorem 25. *Each closure operator on M induces a closure system in M. Conversely, any closure system determines a closure operator.*

Proof. Straightforwardly from Lemmas 23 and 24. □

Proposition 26. *The closure operator induced by the closure system S_c is c itself, and, similarly, the closure system induced by the closure operator E_S is S. That is,*

$$E_{S_c} = c \quad \textit{and} \quad S_{E_S} = S$$

Proof. The equality $E_{S_c}(y) = \min\{x \in S_c \mid y \leq x\} = c(y)$ holds, since the closure of y, $c(y)$, is the smallest closed element greater than y.

On the other hand, $S_{E_S} = S$ follows from the fact that $y \in S$ if and only if $E_S(y) = \min\{x \in S \mid y \leq x\} = y$. □

The next result recalls the relation between Galois connections and closure systems in multilattices.

Proposition 27. *Any Galois connection between complete multilattices induces dually isomorphic closure systems. Conversely, each pair of dually isomorphic closure systems S_1 and S_2 in complete multilattices M_1 and M_2 determines a Galois connection between S_1 and S_2.*

4. Dedekind-MacNeille completion on multilattices

After recalling the notion of Dedekind-MacNeille completion on posets, this section introduces two technical results which will be used later. The following definition presents the two operators used in the completion of an ordered set P.

Definition 28. Let $(P, \leq)$ be a poset and $A \subseteq P$, the *"upper"* set and the *"lower"* set of A are respectively defined by

$$A^u = \{x \in P \mid a \leq x, \text{for all } a \in A\} \quad \text{and} \quad A^l = \{x \in P \mid x \leq a, \text{for all } a \in A\}$$

The mappings ^{u} and ^{l} on the powerset of the poset P form a Galois connection. Hence, the following properties hold, for all $A, B \subseteq P$,

$$A \subseteq A^{ul} \text{ and } A \subseteq A^{lu} \tag{3}$$

$$\text{if } A \subseteq B \text{ then } B^u \subseteq A^u \text{ and } B^l \subseteq A^l \tag{4}$$

$$A^u = A^{ulu} \text{ and } A^l = A^{lul} \tag{5}$$

$$\bigcap_{i \in I}(A_i)^u = \Big(\bigcup_{i \in I} A_i\Big)^u, \text{ where } A_i \subseteq P, \text{ for all } i \in I \tag{6}$$

$$\bigcap_{i \in I}(A_i)^l = \Big(\bigcup_{i \in I} A_i\Big)^l, \text{ where } A_i \subseteq P, \text{ for all } i \in I \tag{7}$$

Considering the operators ^{u} and ^{l}, the Dedekind-MacNeille completion of a poset $(P, \leq)$ is defined as follows:

Definition 29 ([7]). Let $(P, \leq)$ be a poset. The *Dedekind-MacNeille completion* of P is the set $\mathsf{DM}(P) = \{A^{ul} \mid A \subseteq P\}$, which forms a complete lattice with respect to the inclusion ordering.

It is worth to note that $\mathsf{DM}(P)$ forms a closure system in the powerset of P; consequently, infimum coincides with the intersection and supremum is the closure of the union.

The following theorem characterizes the Dedekind-MacNeille completion of a poset $(P, \leq)$.

Theorem 30 ([7]). *Let $(P, \leq)$ be an ordered set and let $\iota\colon P \hookrightarrow \mathsf{DM}(P)$ be the order-embedding of P into its Dedekind-MacNeille completion given by $\iota(x) = x^l$.*

(i) $\iota(P)$ is both supremum-dense and infimum-dense in $\mathsf{DM}(P)$.

(ii) Let $(L, \leq)$ be a complete lattice and assume that P is a subset of L which is both supremum-dense and infimum-dense in L. Then $L \cong \mathsf{DM}(P)$ via an order-isomorphism which is an extension of ι.

As a result, given a poset $(P, \leq)$, the mapping $\iota\colon P \hookrightarrow \mathsf{DM}(P)$ above is an order-embedding of P into $\mathsf{DM}(P)$.

Another technical result, which will be useful later, is the following:

Proposition 31 ([21]). *For all $X \subseteq P$ the following[3] equalities hold in $\mathsf{DM}(P)$:*

$$\bigwedge_{x \in X} x^l = X^l \qquad\qquad \bigvee_{x \in X} x^l = X^{ul}$$

The following proposition introduces some useful equalities in the case that our underlying poset is indeed a multilattice.

Proposition 32. *For every $X \subseteq M$, the following equalities are satisfied:*

$$X^l = \bigcup_{y \in \operatorname{minf}(X)} y^l \quad \text{and} \quad X^u = \bigcup_{y \in \operatorname{msup}(X)} y^u$$

[3] In order to simplify the notation we will write x instead of $\{x\}$.

Proof. We will prove just the first equality, the second one is similar.

First of all, we will prove that $X^l \subseteq \bigcup_{y\in \mathrm{minf}(X)} y^l$. By definition of multilattice we have that M is a chain-complete poset, then for all $x \in X^l$ there exists $y \in \mathrm{minf}(X)$, such that $x \leq y$. From the last inequality we obtain that $x \in y^l$ and, as a consequence, $x \in \bigcup_{y\in \mathrm{minf}(X)} y^l$. Therefore, we can conclude that $X^l \subseteq \bigcup_{y\in \mathrm{minf}(X)} y^l$.

It remains to prove that $\bigcup_{y\in \mathrm{minf}(X)} y^l \subseteq X^l$. For that purpose, we will consider $z \in \bigcup_{y\in \mathrm{minf}(X)} y^l$, then $z \in y^l$ for some $y \in \mathrm{minf}(X)$, from which the following inequalities hold $z \leq y \leq x$ for all $x \in X$. Finally, we can state that $z \in X^l$ and, therefore $\bigcup_{y\in \mathrm{minf}(X)} y^l \subseteq X^l$. $\square$

As we know that all the elements in the Dedekind-MacNeille completion of P can be expressed as infima or suprema of elements of P, the following lemma describes how the elements in the completion of a multilattice M can be expressed in terms of elements in M.

Lemma 33. *Let $(M, \leq)$ be a complete multilattice, then for all $X \subseteq M$ the following equalities in $\mathsf{DM}(M)$ hold:*

$$\bigwedge_{x\in X} x^l = \bigvee_{y\in \mathrm{minf}(X)} \iota(y) \qquad\qquad \bigvee_{x\in X} x^l = \bigwedge_{y\in \mathrm{msup}(X)} \iota(y)$$

Proof. Given $X \subseteq M$, by Proposition 31 we have that $\bigwedge\limits_{x\in X} x^l = X^l$. Then, the following chain of equalities holds

$$\bigvee_{y\in \mathrm{minf}(X)} \iota(y) \overset{(1)}{=} \left(\bigcup_{y\in \mathrm{minf}(X)} y^l\right)^{ul} \overset{(2)}{=} (X^l)^{ul} = X^l = \bigwedge_{x\in X} x^l$$

where (1) is given by Proposition 10 and (2) by Proposition 32.

On the other hand, by Proposition 31 the equality $\bigvee_{x\in X} x^l = X^{ul}$ holds. Then, we have that

$$X^{ul} \overset{(1)}{=} \Big(\bigcup_{y\in \mathrm{msup}\, X} y^u\Big)^l \overset{(2)}{=} \bigcap_{y\in \mathrm{msup}\, X} y^{ul} \overset{(3)}{=} \bigcap_{y\in \mathrm{msup}\, X} y^l \overset{(4)}{=} \bigwedge_{y\in \mathrm{msup}\, X} \iota(y)$$

where (1) is given by Proposition 32, the equality (2) holds since $({}^u, {}^l)$ is a Galois connection, (3) because $y^{ul} = y^l$, for all $y \in M$, and (4) is due to Proposition 10. $\square$

5. Dedekind-MacNeille completion and FCA

As stated in the introduction, the Dedekind-MacNeille construction has already played an important role in FCA. As an example, it can be seen as the concept lattice associated to the general ordinal scale associated to a poset, see [10]. Several algorithms for constructing the Dedekind-MacNeille completion of a finite poset have been proposed, for instance, Ganter and Kuznetsov [9] introduced a stepwise method, with cubic complexity, which constructs one new element at a time.

Proposition 34. *Let $(L,\leq)$ be a complete lattice, $(P,\leq)$ be a poset and $\varphi\colon P\to L$ be an order-embedding such that $\varphi(P)$ is both supremum and infimum dense in L. Then $L\cong\mathfrak{B}(P,P,\leq)\cong\mathsf{DM}(P)$.*

Proof. Since $({}^u,{}^l)$ is the Galois connection given by the concept-forming operators associated with the context $(P,P,\leq)$, one easily deduces that $\mathfrak{B}(P,P,\leq)\cong\mathsf{DM}(P)$, since $\mathsf{DM}(P)$ is the set of extensions of the concept lattice $\mathfrak{B}(P,P,\leq)$, see [10, page 48].

On the other hand, due to the fact that $\varphi\colon P\to L$ is an order-embedding, we have that P and $\varphi(P)$ are isomorphic. Moreover, from Theorem 30 we have that $L\cong\mathsf{DM}(\varphi(P))$. As a result, we obtain the following chain of isomorphisms:

$$L\cong\mathsf{DM}(\varphi(P))\cong\mathsf{DM}(P)\cong\mathfrak{B}(P,P,\leq)$$

□

Our next goal is to prove that the Dedekind-MacNeille completion "distributes" with respect to the construction of the concept lattice associated to a Galois connection.

Let the pair of mappings $\varphi\colon P\to Q$ and $\psi\colon Q\to P$ be a Galois connection between posets. The following result states that it can be extended to the corresponding completions.

Proposition 35 ([23]). *Any Galois connection $\varphi\colon P\to Q$ and $\psi\colon Q\to P$ between posets, can be uniquely extended to a Galois connection between $\mathsf{DM}(P)$ and $\mathsf{DM}(Q)$.*

This extension $(\bar{\varphi},\bar{\psi})$ is given by

$$\bar{\varphi}(A^{ul})=\bigwedge_{x\in A}\iota_Q(\varphi(x)),\ \text{for all } A\subseteq P \text{ and } \bar{\psi}(B^{ul})=\bigwedge_{y\in B}\iota_P(\psi(y)),\ \text{for all } B\subseteq Q.$$

From now on, in order to simplify the notation, we will erase the subscripts from the mappings ι_P and ι_Q, that is, we will write ι instead of ι_P or ι_Q.

We can now state and prove the main result in this section:

Theorem 36. *Let $(P,\leq)$, $(Q,\leq)$ be posets and (φ,ψ) be a Galois connection between P and Q, the Dedekind-MacNeille completion of concept poset $\mathsf{CP}(P,Q,\varphi,\psi)$ is isomorphic to the concept lattice $\mathsf{CL}(\mathsf{DM}(P),\mathsf{DM}(Q),\bar{\varphi},\bar{\psi})$, that is*

$$\mathsf{DM}(\mathsf{CP}(P,Q,\varphi,\psi)) \cong \mathsf{CL}(\mathsf{DM}(P),\mathsf{DM}(Q),\bar{\varphi},\bar{\psi})$$

Proof. From Theorem 30, it is sufficient to show that $\mathsf{CP}(P,Q,\varphi,\psi)$ can be order-embedded as a supremum and infimum dense subset of the lattice $\mathsf{CL}(\mathsf{DM}(P),\mathsf{DM}(Q),\bar{\varphi},\bar{\psi})$.

Let $(X,Y)\in\mathsf{CL}(\mathsf{DM}(P),\mathsf{DM}(Q),\bar{\varphi},\bar{\psi})$ be an arbitrary element. By $X\in\mathsf{DM}(P)$ and Proposition 31, we have that $X = X^{ul} = \bigvee_{x\in X} x^l$. Moreover, since the Galois connection $(\bar{\varphi},\bar{\psi})$ is the extension of (φ,ψ), we obtain

$$\begin{aligned}(X,Y) &= \big(\bar{\psi}(\bar{\varphi}(X)),\bar{\varphi}(X)\big) = \big(\bar{\psi}(\bar{\varphi}(X)),\bigwedge_{x\in X}\iota(\varphi(x))\big)\\ \bigvee_{x\in X}\big(\iota(\psi(\varphi(x))),\iota(\varphi(x))\big) &= \big(\bar{\psi}(\bar{\varphi}(\bigvee_{x\in X}\iota(\psi(\varphi(x))))),\bigwedge_{x\in X}\iota(\varphi(x))\big)\end{aligned}$$

Since both are concepts of $\mathsf{CL}(\mathsf{DM}(P),\mathsf{DM}(Q),\bar{\varphi},\bar{\psi})$ and they have the same intension, they are the same concept and so $\mathsf{CP}(P,Q,\varphi,\psi)$ is supremum dense in $\mathsf{CL}(\mathsf{DM}(P),\mathsf{DM}(Q),\bar{\varphi},\bar{\psi})$.

The proof of infimum dense is similarly obtained.

□

Another extension result, similar to Proposition 35, can be stated in terms of adjoint triples:

Proposition 37 ([23]). *Given $(P_1,\leq_1)$, $(P_2,\leq_2)$ and $(P_3,\leq_3)$ three posets. Each conjunctor $T\colon P_1\times P_2\to P_3$ of an adjoint triple can be uniquely extended to the operator $\widehat{T}\colon\mathsf{DM}(P_1)\times\mathsf{DM}(P_2)\to\mathsf{DM}(P_3)$, which also is the conjunctor of an adjoint triple.*

This extension is given by

$$\widehat{T}(X,Y) = \bigvee_{\substack{x\in X\\ y\in Y}}\iota(T(x,y)),\ \text{for all } X\in\mathsf{DM}(P_1), Y\in\mathsf{DM}(P_2).$$

Lemma 38. *Let* $(P_1, \leq_1)$*,* $(P_2, \leq_2)$ *and* $(P_3, \leq_3)$ *be posets and* $(\&, \swarrow, \nwarrow)$ *be an adjoint triple and* $\&\colon \mathsf{DM}(P_1) \times \mathsf{DM}(P_2) \to \mathsf{DM}(P_3)$ *be its extension with corresponding residuations* $\Swarrow$ *and* $\Nwarrow$*. Then, for all* $p \in P_3$*,* $x \in P_1$*,* $y \in P_2$

$$\iota(p \swarrow y) = \iota(p) \Swarrow \iota(y) \qquad \text{and} \qquad \iota(p \nwarrow x) = \iota(p) \Nwarrow \iota(x)$$

i.e., the adjoint triple $(\&, \Swarrow, \Nwarrow)$ *is an extension of* $(\&, \swarrow, \nwarrow)$*.*

Proof. We will prove the first statement, the second one is analogously proved.

In order to prove the equality, we consider $x = p \swarrow y = \max\{x' \in P_1 \mid x' \,\&\, y \leq_3 p\}$ and we will prove that $x^l = \max\{X' \in \mathsf{DM}(P_1) \mid X' \& y^l \leq_3 p^l\} = p^l \Swarrow y^l$.

$$x^l \& y^l = \bigvee_{\substack{x' \in x^l \\ y' \in y^l}} \iota(x' \,\&\, y') = \iota(x \,\&\, y) \overset{(\star)}{\leq}_3 \iota(p) = p^l$$

Where $(\star)$ holds since the mapping ι is order-embedding (in particular, order-preserving) and $x \,\&\, y \leq_3 p$.

On the other hand, we consider $X \in \{X' \in \mathsf{DM}(P_1) \mid X' \& y^l \leq_3 p^l\}$ and we want to prove that $X \subseteq x^l$. From the following chain of inequalities, which holds for each $x_i \in X$,

$$\iota(x_i \,\&\, y) \leq_3 \bigvee_{\substack{x_k \in X \\ y_j \in y^l}} \iota(x_k \,\&\, y_j) = X \& y^l \leq_3 p^l$$

we have that $\iota(x_i \,\&\, y) \leq_3 p^l$, for all $x_i \in X$. Moreover, since the operator ^{l} is order-embedding we obtain that $x_i \,\&\, y \leq_3 p$, for all $x_i \in X$ and, therefore, $x_i \in \{x' \in P_1 \mid x' \,\&\, y \leq_3 p\}$ from which the inequality $x_i \leq_1 x$ holds, for all $x_i \in X$ (since $x = \max\{x' \in P_1 \mid x' \,\&\, y \leq_3 p\}$). Finally, since $x^l = \{x' \in P_1 \mid x' \leq_1 x\}$, we can conclude that $X \subseteq x^l$. □

The technical lemma below will be helpful in the proofs of Lemma 40 and the representation theorem.

Lemma 39. *Given* $E_1 \in \mathsf{DM}(M_1)$ *and* $E_2 \in \mathsf{DM}(M_2)$*,* $p \in P$*, where* $(M_1, \leq_1$ *),* $(M_2, \leq_2)$ *are complete multilattices and* $(P \leq)$ *is a poset. Then*

$$E_1 \& E_2 \leq p^l \qquad \text{if and only if} \qquad x \& y \leq p, \text{ for all } x \in E_1, y \in E_2$$

Moreover, if we write $E_1 = \bigvee_{x \in E_1} x^l$ *and* $E_2 = \bigvee_{y \in E_2} y^l$*, then*

$$E_1 \& E_2 \leq p^l \qquad \text{if and only if} \qquad E_1 \& y^l \leq p^l, \text{ for all } y \in E_2$$
$$E_1 \& E_2 \leq p^l \qquad \text{if and only if} \qquad x^l \& E_2 \leq p^l, \text{ for all } x \in E_1$$

PROOF. By Proposition 31, we have that $E_1 = \bigvee_{x \in E_1} x^l$ and $E_2 = \bigvee_{y \in E_2} y^l$. Then

$$E_1 \& E_2 = \bigvee_{\substack{x \in E_1 \\ y \in E_2}} \iota(x \& y) = \bigvee_{y \in E_2} \Big(\bigvee_{x \in E_1} \iota(x \& y) \Big) \leq p^l$$

Hence, for all $y \in E_2$, $x \in E_1$, the inequality $\iota(x \& y) \leq p^l$ holds, which is equivalent to $x \& y \leq p$ because ι is order-embeding.

The other implication is analogously obtained.

In order to prove the rest of equivalences, we just introduce the first one, since the other follows similarly. Indeed, this equivalence arises considering that both equalities are equivalent to $x \& y \leq p$, for all $x \in E_1, y \in E_2$, applying the previous proved equivalence. □

The following lemma, which requires left-continuity and, hence, at least the multilattice structure on the arguments of the conjunctor, shows that the values of the implications are obtained from one element in the multilattice, by the order-embeding mapping ι.

Lemma 40. *Let $(M_1, \leq_1)$ and $(M_2, \leq_2)$ be complete multilattices, $(P, \leq)$ be a poset, $\& \colon M_1 \times M_2 \to P$ be left-continuous and $\& \colon \mathsf{DM}(M_1) \times \mathsf{DM}(M_2) \to \mathsf{DM}(P)$ be its extension. Then for any $p \in P$, $E_1 \in \mathsf{DM}(M_1)$ and $E_2 \in \mathsf{DM}(M_2)$*

$$p^l \swarrow E_2 \in \iota(M_1) \quad \textit{and} \quad p^l \nwarrow E_1 \in \iota(M_2)$$

PROOF. We prove that if $X = \max\{X' \in \mathsf{DM}(M_1) \mid X' \& E_2 \leq p^l\}$, then there exists $x \in M_1$ such that $\iota(x) = X$, that is $X = x^l$.

Since $X \& E_2 \leq p^l$, by Lemma 39, for all $y_j \in E_2$, $x_i \in X$, the inequality $x_i \& y_j \leq p$ holds. This provides, applying that $\&$ is left continuous in M_1, that for all $y_j \in E_2$ and for all $x \in \operatorname{msup}\{x_i \mid x_i \in X\}$ we have that $x \& y_j \leq p$ and, therefore,

$$x^l \& y_j^l = \bigvee_{\substack{x' \in x^l \\ y' \in y_j^l}} \iota(x' \,\&\, y') = \iota(x \,\&\, y_j) \leq \iota(p) = p^l$$

Moreover, any $x \in \operatorname{msup}\{x_i \mid x_i \in X\}$ satisfies than $X = \bigvee_{x_i \in X} x_i^l \leq_1 x^l$ and so,

$$X \& y_j^l \leq x^l \& y_j^l \leq p^l$$

for all $y_j \in E_2$. Now, fixing $x_s \in \operatorname{msup}\{x_i \mid x_i \in X\}$, applying that & is left continuous in the right argument and a similar procedure as previously, we obtain that $x_s^l \& E_2 \leq p^l$. Therefore,

$$X \& E_2 \leq x_s^l \& E_2 \leq p^l$$

and, by the maximality of X, we obtain that $X = x_s^l$. □

Moreover, the following corollary is obtained from the proof.

Corollary 41. *Considering the same hypotheses as in Lemma 40, if $X = \max\{X' \in \mathsf{DM}(M_1) \mid X' \& E_2 \leq p^l\}$, then there exists $x \in M_1$ such that $X = x^l$ and, in particular, $\max(X) = x$.*

According to this lemma, for all $p \in P$ the mappings $p \swarrow$, $p^l \Swarrow$ and $p \nwarrow$, $p^l \Nwarrow$ respectively, have isomorphic range (image).

Proposition 42. *Given two coomplete multilattices $(M_1, \leq_1)$ and $(M_2, \leq_2)$, a poset $(P, \leq)$, a left-continuous conjunctor $\&: M_1 \times M_2 \to P$, its extension $\&: \mathsf{DM}(M_1) \times \mathsf{DM}(M_2) \to \mathsf{DM}(P)$ and the corresponding residuated implications $\swarrow$, $\nwarrow$, $\Swarrow$ and $\Nwarrow$. The sets*

- *$p \swarrow M_2 = \{p \swarrow y \mid y \in M_2\}$ and*
- *$p^l \Swarrow \mathsf{DM}(M_2) = \{p^l \Swarrow Y \mid Y \in \mathsf{DM}(M_2)\}$*

with the restrictive ordering in M_1 and $\mathsf{DM}(M_1)$, respectively, are isomorphic. Analogously, the sets $p \nwarrow M_1$ and $p^l \Nwarrow \mathsf{DM}(M_1)$ are similarly defined and they are isomorphic as well.

Proof. Firstly, we will define a mapping

$$\Phi: p \swarrow M_2 \to p^l \Swarrow \mathsf{DM}(M_2)$$

as $\Phi(p \swarrow y) = (p \swarrow y)^l = p^l \Swarrow y^l$, for all $y \in M_2$.

Clearly, this mapping is well defined. We will prove that Φ is an isomorphism, that is, it is mapping from $p \swarrow M_2$ onto $p^l \Swarrow \mathsf{DM}(M_2)$ and is an order-embedding.

Let us prove that Φ is onto. For that, we consider $E_2 \in \mathsf{DM}(M_2)$ and we will obtain an element $y \in M_2$ such that the equality $(p \swarrow y)^l = p^l \Swarrow E_2$ holds. By Lemma 40, we have that, given $E_2 \in \mathsf{DM}(M_2)$, there exists $x \in M_1$ such that $p^l \Swarrow E_2 = x^l$. Now, let us prove that the element $y = p \nwarrow x$

satisfies the required property. Since $(\&, \swarrow\!\!\!\swarrow, \nwarrow\!\!\!\nwarrow)$ forms an adjoint triple, the following chain holds:

$$\begin{aligned}(p^l \swarrow\!\!\!\swarrow E_2) \overset{(1)}{=} x^l &\overset{(2)}{\leq_1} \big(p^l \swarrow\!\!\!\swarrow (p^l \nwarrow\!\!\!\nwarrow x^l)\big) \\ &\overset{(3)}{=} \big(p^l \swarrow\!\!\!\swarrow (p \nwarrow x)^l\big) \overset{(4)}{=} (p^l \swarrow\!\!\!\swarrow y^l) \overset{(5)}{=} (p \swarrow y)^l\end{aligned}$$

where (1) is given from Lemma 40, as stated previously, (2) arises from the adjoint property [6], (3) and (5) hold by Lemma 38 and (4) from the considered definition of y.

Therefore, we have that $p^l \swarrow\!\!\!\swarrow E_2 \leq_1 (p \swarrow y)^l$.

It remains to prove the other inequality. Since $x^l = (p^l \swarrow\!\!\!\swarrow E_2)$, clearly we have that $x^l \leq_1 p^l \swarrow\!\!\!\swarrow E_2$, which is equivalent to $E_2 \leq_2 (p^l \nwarrow\!\!\!\nwarrow x^l)$, applying the adjoint property. In addition, by Lemma 38, we have that $(p^l \nwarrow\!\!\!\nwarrow x^l) = y^l$ and, by the properties of adjoint triples, we obtain the following chain of inequalities

$$E_2 \leq_2 y^l \leq_2 \big(p^l \nwarrow\!\!\!\nwarrow (p^l \swarrow\!\!\!\swarrow y^l)\big) = \big(p^l \nwarrow\!\!\!\nwarrow (p \swarrow y)^l\big)$$

By the adjoint property, once again, the inequality $(p \swarrow y)^l \& E_2 \leq p^l$ holds. Therefore, since x^l is the maximum element satisfying $x^l \& E_2 \leq p^l$, we have that $(p \swarrow y)^l \leq_1 x^l$. Hence, the inequality $(p \swarrow y)^l \leq_1 p^l \swarrow\!\!\!\swarrow E_2$ holds as well. As a consequence, we obtain $(p \swarrow y)^l = p^l \swarrow\!\!\!\swarrow E_2$, which proves that Φ is onto.

Now, we will show that Φ is an order-embedding, that is, $p \swarrow y_1 \leq_1 p \swarrow y_2$ if and only if $(p \swarrow y_1)^l \leq_1 (p \swarrow y_2)^l$:

By the adjoint property, $p \swarrow y_1 \leq_1 p \swarrow y_2$ is equivalent to $(p \swarrow y_1) \& y_2 \leq_1 p$ which, by the monotonicity of $\&$, is also equivalent to

$$x \& y \leq_1 p, \qquad \text{for all } x \leq_1 p \swarrow y_1,\ y \leq_2 y_2$$

Now, by Lemma 39, the inequality $(p \swarrow y_1)^l \& y_2^l \leq_1 p^l$ holds, which is equivalent to $(p \swarrow y_1)^l \leq_1 (p \swarrow y_2)^l$. □

Once again, we introduce below a technical result which will be used later when analyzing the concept multilattice.

Proposition 43. *Let $(P_i)_{i \in I}$ be a family of bounded posets. Then*

$$\mathsf{DM}\Big(\prod_{i \in I} P_i\Big) \cong \prod_{i \in I} \mathsf{DM}(P_i).$$

PROOF. If P_i is bounded for all $i \in I$, then $\prod_{i \in I} P_i$ can be densely embedded into $\mathsf{DM}(\prod_{i \in I} P_i)$. According to Proposition 34 we obtain that $\mathsf{DM}(\prod_{i \in I} P_i) \cong \prod_{i \in I} \mathsf{DM}(P_i)$. □

Note that for posets which are not bounded this is not valid in general. An example being the product of two open unit intervals of reals $(0,1) \times (0,1)$. Since $\mathsf{DM}((0,1)) \cong \langle 0,1 \rangle$ ($\langle 0,1 \rangle$ represents the closed unit interval), we obtain $\mathsf{DM}((0,1)) \times \mathsf{DM}((0,1)) \cong \langle 0,1 \rangle \times \langle 0,1 \rangle$, while $\mathsf{DM}((0,1) \times (0,1))$ is isomorphic to $(0,1) \times (0,1) \cup \{O, I\}$, O, I representing added universal bounds.

Theorem 44 ([18]). *Given two multilattices* $(M_1, \leq_1)$, $(M_2, \leq_2)$, *a poset* $(P, \leq)$, *an adjoint triple among them* $(\&, \swarrow, \nwarrow)$ *where* $\&$ *is left-continuous, a formal context* (A, B, R), *and* $g \in M_2^B$ *and* $f \in M_1^A$, *then we have that the following infima exist*

$$\inf\{R(a,b) \swarrow g(b) \mid b \in B\} \quad \textit{and} \quad \inf\{R(a,b) \nwarrow f(a) \mid a \in A\}.$$

Moreover, in [18] it was also proved that the mappings ${}^{\uparrow}\colon M_2^B \to M_1^A$ and ${}^{\downarrow}\colon M_1^A \to M_2^B$, defined as:

$$g^{\uparrow}(a) = \inf\{R(a,b) \swarrow g(b) \mid b \in B\} \tag{8}$$

$$f^{\downarrow}(b) = \inf\{R(a,b) \nwarrow f(a) \mid a \in A\} \tag{9}$$

for all $f \in M_1^A$, $g \in M_2^B$, $a \in A$ and $b \in B$, form a Galois connection between M_1^A and M_2^B.

Hence, from this particular case of Galois connection, the poset of concepts $\mathsf{CP}(M_1^A, M_2^B, {}^{\uparrow}, {}^{\downarrow})$ can be considered.

Definition 45. The poset $\mathsf{CP}(M_1^A, M_2^B, {}^{\uparrow}, {}^{\downarrow})$, which in principle need not be a lattice, will be denoted as $\mathfrak{C}(M_1, M_2)$ and called *concept multilattice.*

Theorem 46. *Given a formal context* (A, B, R), *any frame* $(M_1, M_2, P, \&)$ *admitting to define the concept forming operators by Equations* (8) *and* (9), *for all* $f \in M_1^A$, $g \in M_2^B$, $a \in A$ *and* $b \in B$, *induces a Galois connection between the complete lattices* $\mathsf{DM}(M_1)^A$ *and* $\mathsf{DM}(M_2)^B$.

PROOF. By Proposition 37, the adjoint triple $(\&, \swarrow, \nwarrow)$ can be extended to an adjoint triple $(\&, \Swarrow, \Nwarrow)$ in the corresponding Dedekind-MacNeille completions.

Furthermore, Proposition 43 guarantees that there are two isomorphisms $G\colon \mathsf{DM}(M_2)^B \to \mathsf{DM}(M_2^B)$ and $F\colon \mathsf{DM}(M_1)^A \to \mathsf{DM}(M_1^A)$. About the definitions of these isomophisms we simply recall that, given $g \in M_2^B$, we have that $G^{-1}(g^l) \in \mathsf{DM}(M_2)^B$ and is defined as $G^{-1}(g^l)(b) = (g(b))^l = \iota(g(b))$, for all $b \in B$. This fact also happens with respect to each $f \in M_1^A$ and F^{-1}.

Now, we will prove that the mappings $\Downarrow\colon \mathsf{DM}(M_1)^A \to \mathsf{DM}(M_2)^B$ and $\Uparrow\colon \mathsf{DM}(M_2)^B \to \mathsf{DM}(M_1)^A$ defined by

$$\begin{array}{rcl} \overline{g}^{\Uparrow}(a) & = & \inf\{R(a,b)^l \swarrow \overline{g}(b) \mid b \in B\} \\ \overline{f}^{\Downarrow}(b) & = & \inf\{R(a,b)^l \nwarrow \overline{f}(a) \mid a \in A\} \end{array}$$

for all $\overline{f} \in \mathsf{DM}(M_1)^A$ and $\overline{g} \in \mathsf{DM}(M_2)^B$, satisfy that $\overline{g}^{\Uparrow} = F^{-1}(G(\overline{g})^{\bar{\uparrow}})$ and $\overline{f}^{\Downarrow} = F^{-1}(G(\overline{f})^{\bar{\downarrow}})$, where $(^{\bar{\uparrow}}, ^{\bar{\downarrow}})$ are the extensions, given by Proposition 35, of the Galois connection $(^{\uparrow}, ^{\downarrow})$, defined in Eq. (8) and (9). As a consequence, since $(^{\bar{\uparrow}}, ^{\bar{\downarrow}})$ is a Galois connection, we obtain that $(^{\Uparrow}, ^{\Downarrow})$ is also a Galois connection.

Now, we will show the proof for the equality $\overline{g}^{\Uparrow} = F^{-1}(G(\overline{g})^{\bar{\uparrow}})$, the other equality follows similarly.

Since $G(\overline{g}) \in \mathsf{DM}(M_2^B)$, there exists a subset $Y \subseteq M_2^B$, such that $G(\overline{g}) = Y^{ul}$. Therefore, by the definition of $^{\bar{\uparrow}}$ (Proposition 35), we obtain that

$$G(\overline{g})^{\bar{\uparrow}} = (Y^{ul})^{\bar{\uparrow}} = \bigwedge_{g_i \in Y} \iota(g_i^{\uparrow})$$

Applying the isomorphism F^{-1} we have

$$F^{-1}(G(\overline{g})^{\bar{\uparrow}})(a) = F^{-1}\Big(\bigwedge_{g_i \in Y} \iota(g_i^{\uparrow})\Big)(a) \overset{(1)}{=} \bigwedge_{g_i \in Y} F^{-1}(\iota(g_i^{\uparrow}))(a) \overset{(2)}{=} \bigwedge_{g_i \in Y} \iota(g_i^{\uparrow}(a))$$

where (1) holds because, in particular, F^{-1} preserves infima and (2) holds by the definition of F^{-1} discussed above. Below we consider the definition of $^{\uparrow}$ and G^{-1}, Lemma 38 and Proposition 31.

$$\begin{array}{rcl}
\bigwedge_{g_i \in Y} (\iota(g_i^{\uparrow}(a)) & = & \bigwedge_{g_i \in Y} \iota(\inf\{R(a,b) \swarrow g_i(b) \mid b \in B\}) \\
& = & \bigwedge_{g_i \in Y} \inf\{\iota(R(a,b)) \Swarrow \iota(g_i(b)) \mid b \in B\} \\
& = & \bigwedge_{g_i \in Y} \inf\{R(a,b)^l \Swarrow G^{-1}(g_i^l)(b) \mid b \in B\} \\
& = & \inf\{R(a,b)^l \Swarrow \bigvee_{g_i \in Y} G^{-1}(g_i^l)(b) \mid b \in B\} \\
& = & \inf\{R(a,b)^l \Swarrow G^{-1}(\bigvee_{g_i \in Y} (g_i^l))(b) \mid b \in B\} \\
& = & \inf\{R(a,b)^l \Swarrow G^{-1}(Y^{ul})(b) \mid b \in B\} \\
& = & \inf\{R(a,b)^l \Swarrow G^{-1}(G(\overline{g}))(b) \mid b \in B\} \\
& = & \inf\{R(a,b)^l \Swarrow \overline{g}(b) \mid b \in B\} \\
& = & \overline{g}^{\Uparrow}(a)
\end{array}$$

Therefore, $F^{-1}(G(\overline{g})^{\bar{\uparrow}})(a) = \overline{g}^{\Uparrow}(a)$, for all $a \in A$.

□

From the previous development, we can state the following result:

Corollary 47. *The Galois connection* $(^{\uparrow}, ^{\downarrow})$ *is extended by the Galois connection formed by* $^{\Downarrow}\colon \mathsf{DM}(M_1)^A \to \mathsf{DM}(M_2)^B$ *and* $^{\Uparrow}\colon \mathsf{DM}(M_2)^B \to \mathsf{DM}(M_1)^A$.

Moreover, the following isomorphism holds.

Theorem 48. *In the setting of Theorem 46, we have that*

$$\mathfrak{C}(M_1, M_2) \cong \mathsf{CL}(\mathsf{DM}(M_1)^A, \mathsf{DM}(M_2)^B, {}^{\Uparrow}, {}^{\Downarrow})$$

Proof. First of all, we define a mapping

$$\Phi\colon \mathfrak{C}(M_1, M_2) \to \mathsf{CL}(\mathsf{DM}(M_1)^A, \mathsf{DM}(M_2)^B, {}^{\Uparrow}, {}^{\Downarrow})$$

as $\Phi(g, f) = (\overline{g}, \overline{f})$, where $\overline{g}(b) = g(b)^l$, $\overline{f}(a) = f(a)^l$, for all $a \in A$, $b \in B$ and $(g, f) \in \mathfrak{C}(M_1, M_2)$. Let us prove that Φ is well defined, that is, $(\overline{g}, \overline{f})$

is an element in $\mathsf{CL}(\mathsf{DM}(M_1)^A, \mathsf{DM}(M_2)^B, {\Uparrow}, {\Downarrow})$. Given $a \in A$, we obtain the following chain of equalities from definitions and Lemma 38.

$$\begin{aligned}
\overline{g}^{\Uparrow}(a) &= \inf\{R(a,b)^l \swarrow\!\!\!\swarrow \overline{g}(b) \mid b \in B\} \\
&= \inf\{R(a,b)^l \swarrow\!\!\!\swarrow g(b)^l \mid b \in B\} \\
&= \inf\{R(a,b) \swarrow g(b) \mid b \in B\}^l \\
&= (g^{\uparrow}(a))^l \\
&= f(a)^l \\
&= \overline{f}(a)
\end{aligned}$$

Analogously, $\overline{f}^{\Downarrow} = \overline{g}$. Hence, Φ is well defined.

Now, we will prove that Φ is an isomorphism. Firstly, we will see that it is onto, that is, any concept in $\mathsf{CL}(\mathsf{DM}(M_1)^A, \mathsf{DM}(M_2)^B, {\Uparrow}, {\Downarrow})$ has the form $(\overline{g}, \overline{f})$ for some $f \in {M_1}^A$ and $g \in {M_2}^B$. Formally, assume an element, without loss of generality we can write $(\overline{g}, \overline{f}) \in \mathsf{CL}(\mathsf{DM}(M_1)^A, \mathsf{DM}(M_2)^B, {\Uparrow}, {\Downarrow})$; by Proposition 42, we have that for each $a \in A$ there exists $x_a \in M_1$ satisfying that $(R(a,b) \nwarrow x_a)^l = R(a,b)^l \nwarrow\!\!\!\nwarrow \overline{f}(a)$. Then, we can define $f\colon A \to M_1$ by $f(a) = x_a$, which satisfies the following chain of equalities:

$$\begin{aligned}
\overline{f}^{\Downarrow}(b) &= \inf\{R(a,b)^l \nwarrow\!\!\!\nwarrow \overline{f}(a) \mid a \in A\} \\
&= \inf\{(R(a,b) \nwarrow x_a)^l \mid a \in A\} \\
&= \inf\{(R(a,b) \nwarrow f(a))^l \mid a \in A\} \\
&= \inf\{R(a,b)^l \nwarrow\!\!\!\nwarrow f(a)^l \mid a \in A\} \\
&= (\inf\{R(a,b) \nwarrow f(a) \mid a \in A\})^l \\
&= (f^{\downarrow}(b))^l
\end{aligned}$$

Therefore, given $(\overline{g}, \overline{f}) \in \mathsf{CL}(\mathsf{DM}(M_1)^A, \mathsf{DM}(M_2)^B, {\Uparrow}, {\Downarrow})$ there exists $(f^{\downarrow}, f) \in \mathfrak{C}(M_1, M_2)$ satisfying that $\Phi(f^{\downarrow}, f) = (\overline{g}, \overline{f})$.

Finally, the fact that Φ is order-embedding is a consequence of ι being order-embedding. □

Consequently, the concept multilattice $\mathfrak{C}(M_1, M_2)$ turns out to be a lattice when & is left-continuous. Moreover, $\mathfrak{C}(M_1, M_2) \cong \mathsf{DM}(\mathfrak{C}(M_1, M_2))$.

The following example illustrates the previous results in order to show that the completion of the concept poset of a Galois connection between multilattices coincides with the concept lattices of the Galois connection extended to the corresponding completions.

Example 49. The considered frame is $(M6, M6, M6, \&)$, where $\&$ is the conjunctor of the adjoint triple defined in Example 17, and the fixed context is (A, B, R), with $A = \{a_1, a_2\}$, $B = \{b_1, b_2, b_3, b_4, b_5\}$, and the relation R given by Table 2.

Table 2: Relation R of Example 49.

R	a_1	a_2
b_1	d	$\bot$
b_2	c	a
b_3	$\top$	b
b_4	a	d
b_5	b	$\top$

It is clear that the considered adjoint triple satisfies Equations (8) and (9), for all $f \in M_1^A$, $g \in M_2^B$, $a \in A$ and $b \in B$, therefore we can use Theorem 44 in order to obtain the poset of concepts $\mathsf{CP}(M_1^A, M_2^B, {}^\uparrow, {}^\downarrow)$. Moreover, by Theorem 48, it is already a complete lattice.

The poset of concepts $\mathsf{CP}(M_1^A, M_2^B, {}^\uparrow, {}^\downarrow)$, whose Hasse diagram is presented in Figure 4, has 14 concepts whose extensions are the following:

$$
\begin{aligned}
E_0 &= \{\bot/b_1, a/b_2, \bot/b_3, a/b_4, \bot/b_5\}\\
E_1 &= \{\bot/b_1, c/b_2, \bot/b_3, a/b_4, \bot/b_5\}\\
E_2 &= \{\bot/b_1, a/b_2, \bot/b_3, a/b_4, a/b_5\}\\
E_3 &= \{a/b_1, a/b_2, a/b_3, a/b_4, \bot/b_5\}\\
E_4 &= \{a/b_1, a/b_2, \top/b_3, a/b_4, \bot/b_5\}\\
E_5 &= \{\bot/b_1, c/b_2, \bot/b_3, a/b_4, a/b_5\}\\
E_6 &= \{a/b_1, c/b_2, \top/b_3, a/b_4, \bot/b_5\}\\
E_7 &= \{\bot/b_1, \top/b_2, \bot/b_3, \top/b_4, \bot/b_5\}\\
E_8 &= \{a/b_1, a/b_2, a/b_3, a/b_4, a/b_5\}\\
E_9 &= \{\bot/b_1, a/b_2, \bot/b_3, a/b_4, \top/b_5\}\\
E_{10} &= \{\top/b_1, \top/b_2, \top/b_3, \top/b_4, \bot/b_5\}\\
E_{11} &= \{a/b_1, c/b_2, \top/b_3, a/b_4, a/b_5\}\\
E_{12} &= \{\bot/b_1, \top/b_2, \bot/b_3, \top/b_4, \top/b_5\}\\
E_{13} &= \{a/b_1, a/b_2, a/b_3, a/b_4, \top/b_5\}\\
E_{14} &= \{\top/b_1, \top/b_2, \top/b_3, \top/b_4, \top/b_5\}
\end{aligned}
$$

It is easy to see that the poset of concepts has the structure of a

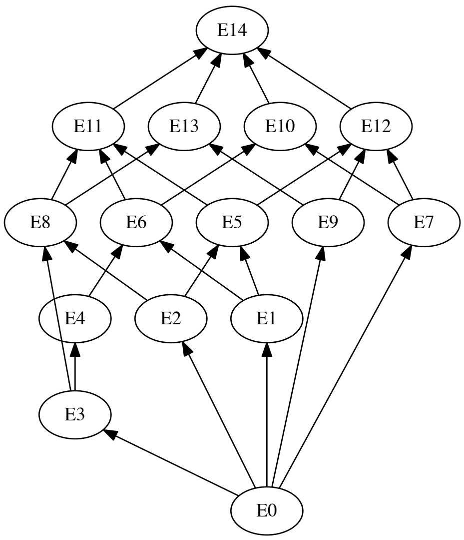

Figure 4: Poset of concepts $\mathsf{CP}(M_1^A, M_2^B, \uparrow, \downarrow)$

concept lattice, consequently, in this particular case, $\mathsf{CP}(M_1^A, M_2^B, \uparrow, \downarrow) \cong \mathsf{DM}(\mathsf{CP}(M_1^A, M_2^B, \uparrow, \downarrow))$.

Now, we will obtain the concept lattice using the Galois connection extended to the corresponding completions of $M6$, that is, we will obtain $\mathsf{CL}(\mathsf{DM}(M6), \mathsf{DM}(M6), \Downarrow, \Uparrow)$, where

$$\begin{array}{rcl} \overline{g}^{\Uparrow}(a) & = & \inf\{R(a,b)^l \swarrow \overline{g}(b) \mid b \in B\} \\ \overline{f}^{\Downarrow}(b) & = & \inf\{R(a,b)^l \nwarrow \overline{f}(a) \mid a \in A\} \end{array}$$

are the mappings defined in the proof of Theorem 46 and $\swarrow$, $\nwarrow$ are the extensions on the completion of $M6$ of the residuated implications of the original adjoint triple. Hence, based on Proposition 37 and Lemma 38, they are defined in Table 3. The corresponding concept lattice $\mathsf{CL}(\mathsf{DM}(M6), \mathsf{DM}(M6), \Downarrow, \Uparrow)$ appears in Figure 5 and it can be computed by using any mechanism to obtain the whole set of concepts (in this case, we have computed the concept lattice from the irreducible elements). It is worth to remark that, contrariwise, in the previous case we could not use any existing algorithm. Finally, the following concepts are obtained:

$$
\begin{aligned}
E'_0 &= \{\bot^l/b_1, a^l/b_2, \bot^l/b_3, a^l/b_4, \bot^l/b_5\}\\
E'_1 &= \{\bot^l/b_1, c^l/b_2, \bot^l/b_3, a^l/b_4, \bot^l/b_5\}\\
E'_2 &= \{\bot^l/b_1, a^l/b_2, \bot^l/b_3, a^l/b_4, a^l/b_5\}\\
E'_3 &= \{a^l/b_1, a^l/b_2, a^l/b_3, a^l/b_4, \bot^l/b_5\}\\
E'_4 &= \{a^l/b_1, a^l/b_2, \top^l/b_3, a^l/b_4, \bot^l/b_5\}\\
E'_5 &= \{\bot^l/b_1, c^l/b_2, \bot^l/b_3, a^l/b_4, a^l/b_5\}\\
E'_6 &= \{a^l/b_1, c^l/b_2, \top^l/b_3, a^l/b_4, \bot^l/b_5\}\\
E'_7 &= \{\bot^l/b_1, \top^l/b_2, \bot^l/b_3, \top^l/b_4, \bot^l/b_5\}\\
E'_8 &= \{a^l/b_1, a^l/b_2, a^l/b_3, a^l/b_4, a^l/b_5\}\\
E'_9 &= \{\bot^l/b_1, a^l/b_2, \bot^l/b_3, a^l/b_4, \top^l/b_5\}\\
E'_{10} &= \{\top^l/b_1, \top^l/b_2, \top^l/b_3, \top^l/b_4, \bot^l/b_5\}\\
E'_{11} &= \{a^l/b_1, c^l/b_2, \top^l/b_3, a^l/b_4, a^l/b_5\}\\
E'_{12} &= \{\bot^l/b_1, \top^l/b_2, \bot^l/b_3, \top^l/b_4, \top^l/b_5\}\\
E'_{13} &= \langle\{a^l/b_1, a^l/b_2, a^l/b_3, a^l/b_4, \top^l/b_5\}\\
E'_{14} &= \langle\{\top^l/b_1, \top^l/b_2, \top^l/b_3, \top^l/b_4, \top^l/b_5\}
\end{aligned}
$$

We can easily notice that both concept lattices are isomorphic, see Figures 4 and 5 .

Table 3: Definition of $\swarrow$ and $\nwarrow$

$\nwarrow$	$\bot^l$	a^l	b^l	$\{\bot, a, b\}$	c^l	d^l	$\top^l$
$\bot^l$	$\top^l$	$\bot^l$	a^l	$\bot^l$	$\bot^l$	$\bot^l$	$\bot^l$
a^l	$\top^l$	$\top^l$	a^l	a^l	a^l	a^l	a^l
b^l	$\top^l$	$\bot^l$	a^l	$\bot^l$	$\bot^l$	$\bot^l$	$\bot^l$
$\{\bot, a, b\}$	$\top^l$	$\top^l$	a^l	a^l	a^l	a^l	a^l
c^l	$\top^l$	$\top^l$	c^l	c^l	c^l	a^l	a^l
d^l	$\top^l$	$\top^l$	a^l	a^l	a^l	a^l	a^l
$\top^l$	$\top^l$	$\top^l$	$\top^l$	$\top^l$	$\top^l$	$\top^l$	$\top^l$

$\swarrow$	$\bot^l$	a^l	b^l	$\{\bot, a, b\}$	c^l	d^l	$\top^l$
$\bot^l$	$\top^l$	$\bot^l$	$\bot^l$	$\bot^l$	$\bot^l$	$\bot^l$	$\bot^l$
a^l	$\top^l$	$\top^l$	a^l	a^l	a^l	a^l	a^l
b^l	$\top^l$	$\bot^l$	$\bot^l$	$\bot^l$	$\bot^l$	$\bot^l$	$\bot^l$
$\{\bot, a, b\}$	$\top^l$	$\top^l$	a^l	a^l	a^l	a^l	a^l
c^l	$\top^l$	$\top^l$	c^l	c^l	c^l	a^l	a^l
d^l	$\top^l$	$\top^l$	a^l	a^l	a^l	a^l	a^l
$\top^l$	$\top^l$	$\top^l$	$\top^l$	$\top^l$	$\top^l$	$\top^l$	$\top^l$

6. Representation theorem for the multilattice-based case

In this section, we give the corresponding adaptation of the basic theorem of Formal Concept Analysis to the case of multilattices as underlying truth-values structures and a left-continuous adjoint triple.

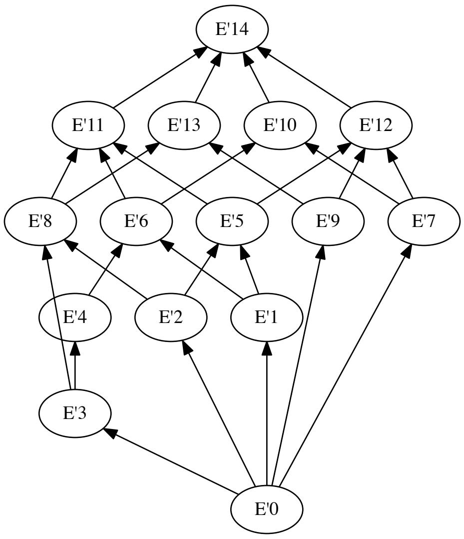

Figure 5: Concept lattice $\mathsf{CL}(\mathsf{DM}(M6), \mathsf{DM}(M6), ^{\Downarrow}, ^{\Uparrow})$

Theorem 50. *Let $(M_1, \leq_1)$, $(M_2, \leq_2)$ be complete multilattices, $(P, \leq)$ be a poset, $\&: M_1 \times M_2 \to P$ be left continuous and (A, B, R) be a formal context. A complete lattice $(L, \sqsubseteq)$ is isomorphic to concept multilattice (it is always lattice) $\mathfrak{C}(M_1, M_2)$ if and only if there are mappings $\beta: B \times M_2 \to L$ and $\alpha: A \times M_1 \to L$ such that:*

(i) *$\beta[B \times M_2]$ is supremum-dense in L.*
(ii) *$\alpha[A \times M_1]$ is infimum-dense in L.*
(iii) *For every $b \in B$, $a \in A$ and $x \in M_1$, $y \in M_2$*

$$\beta(b, x) \sqsubseteq \alpha(a, y) \quad \textit{if and only if} \quad x \& y \leq R(b, a).$$

Proof. Since $\mathsf{DM}(\mathfrak{C}(M_1, M_2)) \cong \mathfrak{C}(M_1, M_2)$ we can use the basic theorem for "complete lattice". There are mappings $\bar{\beta}: B \times \mathsf{DM}(M_2) \to \mathfrak{C}(M_1, M_2)$ and $\bar{\alpha}: A \times \mathsf{DM}(M_1) \to \mathfrak{C}(M_1, M_2)$ satisfying conditions (i)-(iii) of theorem. Since M_2 is supremum dense in $\mathsf{DM}(M_2)$ and M_1 is supremum dense in $\mathsf{DM}(M_1)$ the restriction of $\bar{\beta}$ to the set $B \times M_2$ and the restriction of $\bar{\alpha}$ to the set $A \times M_1$ have also properties (i)-(iii) of the theorem.

Conversely, suppose that there is a complete lattice L and there is a pair of mappings β, α satisfying conditions of the theorem. We show that $\mathsf{DM}(\mathfrak{C}(M_1, M_2)) \cong L$. Define classical (binary) formal context (G, M, I) in the following way: we put $G = B \times \mathsf{DM}(M_2)$, $M = A \times \mathsf{DM}(M_1)$ and we define

$$\big((b,E_2),(a,E_1)\big) \in I \quad \text{iff} \quad E_1 \& E_2 \leq R(a,b)^l.$$

According to the Basic Theorem for classical Concept Lattices, applied to $\mathfrak{B}(G,M,I)$ and itself, there is a pair of mappings $\gamma: G = B \times \mathsf{DM}(M_2) \to \mathfrak{B}(G,M,I)$ given by $\gamma(b,E_2) = ((b,E_2)'', (b,E_2)')$ and $\mu: M = A \times \mathsf{DM}(M_1) \to \mathfrak{B}(G,M,I)$ given by $\mu(a,E_1) = ((a,E_1)', (a,E_1)'')$ such that $\gamma(G)$ is supremum dense and $\mu(M)$ is infimum dense and

$$E_1 \& E_2 \leq R(a,b)^l \text{ iff } \big((b,E_2),(a,E_1)\big) \in I \text{ iff } \gamma(b,E_2) \leq \mu(a,E_1)$$

Now, using the Basic Theorem for fuzzy Concept Lattices with respect to $\mathsf{CL}(\mathsf{DM}(M_1)^A, \mathsf{DM}(M_2)^B, \Uparrow, \Downarrow)$ and $\mathfrak{B}(G,M,I)$, and Theorem 36, we obtain that

$$\mathfrak{C}(M_1,M_2) \cong \mathsf{CL}(\mathsf{DM}(M_1)^A, \mathsf{DM}(M_2)^B, \Uparrow, \Downarrow) \cong \mathfrak{B}(G,M,I)$$

Now, as a next step, we describe a reduction of the formal context (G,M,I).

Considering $E_2 \in \mathsf{DM}(M_2) \smallsetminus \iota(M_2)$ and an arbitrary object $b \in B$, by Lemma 39,

$$E_1 \& E_2 \leq p^l \qquad \text{if and only if} \qquad E_1 \& y^l \leq p^l, \text{ for all } y \in E_2$$

for all $E_1 \in \mathsf{DM}(M_1)$ and $p \in P$. Therefore, we obtain $(a,E_1) \in (b,E_2)'$ if and only if $(a,E_1) \in (b,y^l)'$, for all $y \in E_2$.

Consequently, $((b,E_2)'', (b,E_2)')$ is not $\bigvee$-irreducible and row determined by object (b,E_2) can be omitted from the context.

In similar way, one can show that columns determined by attribute (a,E_1) where $E_1 \in \mathsf{DM}(M_1) \smallsetminus \iota(M_1)$ can be omitted too.

Therefore, we obtain the reduction (G_r, M_r, I_r) of formal context (G,M,I), where $G_r = B \times \iota(M_2)$, $M_r = A \times \iota(M_1)$ and so, $\mathfrak{B}(G,M,I) \cong \mathfrak{B}(G_r,M_r,I_r)$.

Moreover, defining $\beta^*: B \times \iota(M_2) \to L$, $\alpha^*: A \times \iota(M_1) \to L$, as $\beta^*(b,\iota(y)) = \beta(b,y)$, $\alpha^*(a,\iota(x)) = \alpha(a,x)$, respectively. The ranges of mappings β^* and α^* are clearly supremum-dense and infimum-dense in L. Furthermore, the following chain of equivalences are obtained

$$\begin{aligned}\big((b,\iota(x)),(a,\iota(y)\big) \in I_r \quad & \text{iff} \quad x^l \& y^l \leq R(b,a)^l \text{ iff } x \& y \leq p \\ & \text{iff} \quad \beta(b,x) \sqsubseteq \alpha(a,y) \text{ iff } \beta^*(b,\iota(x)) \sqsubseteq \alpha^*(a,\iota(y))\end{aligned}$$

Thus, due to the Basic Theorem on classical Concept Lattices with respect to $\mathfrak{B}(G_r, M_r, I_r)$ and L and using the properties of the mappings β^* and α^* we obtain that $\mathfrak{B}(G_r, M_r, I_r)$ is isomorphic to L. This last isomorphism, together with the others proven throughout the proof, provides the final isomorphism:

$$\mathfrak{C}(M_1, M_2) \cong \mathsf{DM}(\mathfrak{C}(M_1, M_2)) \cong \mathfrak{B}(G, M, I) \cong \mathfrak{B}(G_r, M_r, I_r) \cong L$$

□

7. Conclusions

After recalling the basic notions about FCA, multilattices, and the Dedekind-MacNeille completion, we have studied the properties of the Dedekind-MacNeille completion of a multilattice in terms of the elements of the multilattice. Moreover, we have proved that the effect of interspersing the Dedekind-MacNeille completion with respect to the construction of the concepts is, somehow, distributive. Finally, the representation theorem of multilattice-based FCA is stated and proved.

The different intermediate results have been stated at the maximum level of generality; this way, the reader can find lemmas about posets or about multilattices, depending on the complexity of the required properties. Note, however, that the important notion of left-continuity needed in the statement of the basic theorem makes that we have to consider, at least, a multilattice.

As future work, we will keep studying the algebraic properties of multilattices in relation to the theory of Formal Concept Analysis; in this respect, it might be interesting considering the potential implications of the *soft left-continuity* introduced in [18].

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