Multi-lattices as a Basis for Generalized Fuzzy Logic Programming

Jesús Medina, Manuel Ojeda-Aciego, and Jorge Ruiz-Calviño

Dept. Matemática Aplicada. Universidad de Málaga*
{jmedina,aciego,jorgerucal}@ctima.uma.es

Abstract. A prospective study of the use of ordered multi-lattices as underlying sets of truth-values for a generalised framework of logic programming is presented. Specifically, we investigate the possibility of using multi-lattice-valued interpretations of logic programs and the theoretical problems that this generates with regard to its fixed point semantics.

1 Introduction

Weakening the structure of the underlying set of truth-values for logic programming has been studied extensively in the recent years. There are approaches which are based either on the structure of lattice (residuated lattice [4, 13] or multi-adjoint lattice [9]), or more restrictive structures, such as bilattices or trilattices [7], or even more general structures such as algebraic domains [11]. One can also find some attempts aiming at weakening the restrictions imposed on a (complete) lattice, namely, the "existence of least upper bounds and greatest lower bounds" is relaxed to the "existence of *minimal* upper bounds and *maximal* lower bounds". In this direction, Benado [1] and Hansen [5] proposed definitions of a structure so-called multi-lattice.

Recently an alternative notion of multi-lattice was introduced [2, 8] as a theoretical tool to deal with some problems in the theory of mechanised deduction in temporal logics. This kind of structure also arises in the research area concerning fuzzy extensions of logic programming: for instance, one of the hypotheses of the main termination result for sorted multi-adjoint logic programs [3] can be weakened only when the underlying set of truth-values is a multi-lattice (the question of providing a counter-example on a lattice remains open).

As far as we know, there have been no attempts to use multi-lattices in the context of extended fuzzy logic programming; our aim in this work is precisely to study the computational capabilities of this new structure in that framework and, specifically, in relation to its fixed point semantics.

The structure of the paper is as follows: In Section 2 the definition and preliminary theoretical results about multi-lattices are introduced; later, the syntax and semantics of our extended logic programs are presented in Section 3; then, an initial proposal for fixed point semantics for these extended logic programs is given in Section 4. Finally, in the last section we present some conclusions and prospects for future work.

* Partially supported by Spanish DGI project TIC2003-09001-C02-01.

2 Preliminary Results

Recall that a lattice is a poset such that the set of upper (lower) bounds has a unique minimal (maximal) element, that is, a *minimum* (*maximum*). In a multi-lattice, this property is relaxed in the sense that minimal elements for the set of upper bounds should exist, but the uniqueness condition is dropped.

Definition 1. *A* complete multi-lattice *is a partially ordered set, $\langle M, \leq \rangle$, such that for every subset $X \subseteq M$, the set of upper (lower) bounds of X has minimal (maximal) elements, which are called* multi-suprema *(*multi-infima*).*

Note that, by definition, it follows that the sets $\mathrm{multinf}(X)$ and $\mathrm{multisup}(X)$ are *antichains* (non-empty sets consisting of pair-wise incomparable elements).

It is remarkable that, under suitable conditions, the set of fixed points of a mapping from M to M does have a minimum and a maximum.

Definition 2. *A mapping $f : P \longrightarrow Q$ between two posets is said to be* isotone *if $x \leq y$ implies $f(x) \leq f(y)$; a mapping $g : P \longrightarrow P$ is* inflationary *if $x \leq g(x)$ for all $x \in P$.*

Theorem 1. *Let $f : M \longrightarrow M$ be an isotone and inflationary mapping on a multi-lattice, then its set of fixed points is non-empty and has a minimum element.*

Proof. Let us write $X = \{x \mid f(x) = x\}$, this set is nonempty since inflation forces $\top$ to be a fixed point; now, consider $a \in \mathrm{multinf}(X)$ a maximal lower bound of X, and let us prove that a is a fixed point of f.

As a is a lower bound for all $x \in X$, we have $a \leq x$ and, by isotonicity, $f(a) \leq f(x) = x$ for all $x \in X$ (the equality follows by definition of X); thus, $f(a)$ is also a lower bound of X. Moreover, a is maximal and, by inflation, we have $a \leq f(a)$; thus, we also have $f(a) \leq a$ and a should be a fixed point, that is $a \in X$.

Consider $a, b \in \mathrm{multinf}(X)$, and recall that we have just proved that $a, b \in X$. As both are lower bounds of X, then $a \leq b$ and $b \leq a$. Thus, $\mathrm{multinf}(X)$ is a singleton consisting of the minimum element of X, that is, the minimum fixed point. □

As by assumption, our sets will not necessarily have a supremum but a *set* of multi-suprema, we will need to work with some ordering between subsets of posets. Three different (pre-)orderings are usually considered in the literature, the Hoare ordering, the Smyth ordering and the Egli-Milner ordering:

Definition 3. *Consider $X, Y \subseteq 2^M$:*

- *$X \sqsubseteq_H Y$ iff for all $x \in X$ exists $y \in Y$ such that $x \leq y$.*
- *$X \sqsubseteq_S Y$ iff for all $y \in Y$ exists $x \in X$ such that $x \leq y$.*
- *$X \sqsubseteq_{EM} Y$ iff $X \sqsubseteq_H Y$ and $X \sqsubseteq_S Y$*

Regarding computational properties of multi-lattices, it is interesting to impose certain conditions on the sets of upper (lower) bounds of a given set X. Specifically, we would like to ensure that any upper (lower) bound is greater (less) than a minimal (maximal); this condition enables to work on the set of multi-suprema (multi-infima) as a set of "generators" of the bounds of X. The formalisation of these concepts is given as follows, where $UB(X)$ (resp. $LB(X)$) denotes the set of upper (lower) bounds of X:

Definition 4. *A multi-lattice is said to be* consistent *if the following set of inequalities hold for all* $X \subseteq M$*:*

$$LB(X) \sqsubseteq_{EM} \text{multinf}(X) \qquad\qquad \text{multisup}(X) \sqsubseteq_{EM} UB(X)$$

Note that in the two items above, one part of the Egli-Milner ordering is trivial, since any multi-infimum is a lower bound and any multi-supremum is an upper bound. It is not difficult to provide examples of non-consistent multi-lattices:

Example 1. A non-consistent multi-lattice is showed on the left of Fig. 1, where

$$UB(\{a,b\}) = \{\top, d\} \cup \{c_n \mid n \in \mathbb{N}\}$$

in which element d is minimal in $UB(\{a,b\})$; however, the elements c_n fail to be greater than one minimal upper bound.

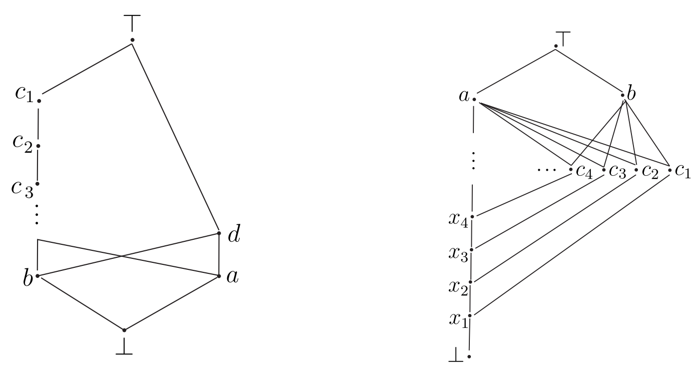

Fig. 1.

Another reasonable condition to require on a multi-lattice is that it should not contain infinite sets of mutually incomparable elements (antichains) since, semantically, it makes little sense to consider infinitely many incomparable truth-values. Consistent multi-lattices without infinite antichains have interesting computational properties: to begin with, recall that the sets of multi-suprema or multi-infima for totally ordered subsets (also called *chains*) always have a supremum and an infimum.

Lemma 1. *Let M be a consistent multi-lattice without infinite antichains, then any chain in M has a supremum and an infimum.*

Proof. Let $\{x_i\}_{i\in I} \subset M$ be a chain and, assume that $a, b \in \mathrm{multisup}(\{x_i\})$. We will show that there is an element $c \in \mathrm{multinf}(\{a,b\})$ which is an upper bound of the chain.

As there are no infinite antichains in M, the set $\mathrm{multinf}(\{a,b\})$ is finite, and we can write

$$\mathrm{multinf}(\{a,b\}) = \{c_1,\ldots,c_n\}$$

If $n = 1$, as any x_i is a lower bound of $\{a,b\}$, by the hypothesis of consistency we would have $x_i \leq c_1$ for all $i \in I$.

If $n > 1$, by contradiction, assume that no c_j, with $j = 1,\ldots,n$, is an upper bound of the chain; then, for all j we choose an element x_j which is not upper bounded by c_j. Now, as $\{x_i\}$ is a chain, let us consider the greatest of $x_1,\ldots,x_n$, say x_{j_0}. By consistency, there is c_k which is greater than x_{j_0}, but then

$$x_k \leq x_{j_0} \leq c_k$$

which would contradict the choice of x_k.

Summarising, we have proved the existence of $c \in \mathrm{multinf}(\{a,b\})$ which, moreover, is an upper bound of the chain. Now, $c \in \mathrm{multinf}(\{a,b\})$ implies the inequalities $c \leq a$ and $c \leq b$; on the other hand, as c is also an upper bound of $\{x_i\}$ and a and b are multi-suprema of $\{x_i\}$, then $a \leq c$ and $b \leq c$, resulting that $a = b = c$, which proves that $\mathrm{multisup}(\{x_i\})$ is a singleton, hence the supremum of the chain.

The proof for the infimum is similar. □

All the hypotheses are necessary for the existence of supremum and infimum of chains; in particular, the condition on infinite antichains cannot be dropped.

Example 2. The poset on the right of Fig. 1 is a consistent multi-lattice; however, the set of upper bounds of the increasing sequence $\{x_n\}$ does not have a minimum, but two minimals, namely, a and b.

We will assume in the rest of the paper that our underlying multi-lattices are complete, consistent and without infinite antichains.

3 Extended logic programs

In this section we provide a first approximation of the definition of an extended logic programming paradigm in which the underlying set of truth-values is assumed to have structure of multi-lattice. The proposed schema is an extension of the monotonic logic programs of [4]. The definition of logic program is given, as usual, as a set of rules and facts.

Definition 5. *An* extended logic program *is a set $\mathbb{P}$ of rules of the form $A \leftarrow \mathcal{B}$ such that:*

1. *A is a propositional symbol of Π, and*
2. *$\mathcal{B}$ is a formula of $\mathfrak{F}$ built from propositional symbols and elements of M by using isotone operators.*

An interpretation is an assignment of truth-values to every propositional symbol in the language.

Definition 6. *An* interpretation *is a mapping $I\colon \Pi \to M$. The set of all interpretations is denoted $\mathcal{I}$.*

Note that by the unique homomorphic extension theorem, any interpretation I can be uniquely extended to the whole set of formulas (the extension will be denoted as $\hat{I}$). The ordering $\leq$ of the truth-values M can be extended point-wise to the set of interpretations as usual.

A rule of an extended logic program is satisfied whenever the truth-value of the head of the rule is greater or equal than the truth-value of its body. Formally:

Definition 7. *Given an interpretation I, a rule $A \leftarrow \mathcal{B}$ is* satisfied *by I iff $\hat{I}(\mathcal{B}) \leq I(A)$. An interpretation I is said to be a* model *of an extended logic program $\mathbb{P}$ iff all rules in $\mathbb{P}$ are satisfied by I, then we write $I \models \mathbb{P}$.*

Example 3. Let us consider the following program on the multi-lattice in Fig. 2:

$$
\begin{aligned}
E &\leftarrow A \\
E &\leftarrow B \\
A &\leftarrow a \\
B &\leftarrow b
\end{aligned}
$$

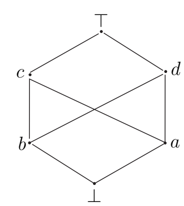

Fig. 2.

It is easy to check that the interpretation defined as $I(E) = \top$, $I(A) = a$, $I(B) = b$ is a model of the program.

Every extended program $\mathbb{P}$ has the top interpretation $\triangledown$ as a model; regarding minimal models, it is possible to prove the following technical lemma.

Lemma 2. *A chain of models $\{I_k\}_{k\in K}$ of $\mathbb{P}$ has an infimum in $\mathcal{I}$ which is a model of $\mathbb{P}$.*

Proof. Given a propositional symbol A, the existence of $\inf_k\{I_k(A)\}$ is guaranteed by Lemma 1, thus we can safely define an interpretation I_ω as follows:

$$I_\omega(A) = \inf_{k\in K}\{I_k(A)\}$$

Now, let us show that I_ω is a model of $\mathbb{P}$:

Given a rule $A \leftarrow @[B_1, \ldots, B_n]$ in $\mathbb{P}$, where @ denotes the composition of the operators occurring in the body of the rule, and the B_i's are the variables occurring in it; by isotonicity of @ we obtain the following chain of inequalities for all $i \in K$:

$$\hat{I}_i(\mathcal{B}) = @[I_i(B_1), ..., I_i(B_n)] \geq @\left[\inf_{k\in K}\{I_k(B_1)\}, ..., \inf_{k\in K}\{I_k(B_n)\}\right] = \hat{I}_\omega(\mathcal{B})$$

As I_i is a model for all i we obtain:

$$I_i(A) \geq \hat{I}_i(\mathcal{B}) \geq \hat{I}_\omega(\mathcal{B})$$

thus, by definition of infimum, we have

$$I_\omega(A) = \inf_{k\in K}\{I_k(A)\} \geq \hat{I}_\omega(\mathcal{B})$$

so I_ω is a model of $\mathbb{P}$. □

Theorem 2. *There exist minimal models for any extended logic program* $\mathbb{P}$.

Proof. Let $\mathcal{M}$ be the set of models of $\mathbb{P}$. By Zorn's lemma, we only have to prove that any chain in $\mathcal{M}$ is lower bounded, but this follows from the previous lemma since the infimum of a chain of models is also a model. □

Example 4. Continuing with the program in the previous example, it is easy to check that the program does not have a minimum model but two minimal ones:

$I_1(E) = c$	$I_2(E) = d$
$I_1(A) = a$	$I_2(A) = a$
$I_1(B) = b$	$I_2(B) = b$

4 Fix-point semantics

An interesting technical problem arises when trying to extend the definition of the immediate consequences operators to the framework of multi-lattice-based logic programs. One of the several possible approaches to provide a fixed point semantics for the extended logic programs is presented and analysed.

The main theoretical tool for the study of the fixed point semantics of programming languages is Knaster-Tarski theorem in some of its constructive versions, although some other fixed point theorems are also of use, see [6].

Given a logic program $\mathbb{P}$ valued on a *lattice*, the operator $T_{\mathbb{P}}: \mathcal{I} \to \mathcal{I}$, maps interpretations to interpretations, and can be defined by considering

$$T_{\mathbb{P}}(I)(A) = \sup\{\hat{I}(\mathcal{B}) \mid A \leftarrow \mathcal{B} \in \mathbb{P}\}$$

Note that all the suprema involved in the definition do exist provided that we are assuming a complete lattice structure on the underlying set of truth-values; however, this needs not hold for a multi-lattice.

In order to work this problem out, we consider the following definition

Definition 8. *Given an extended logic program $\mathbb{P}$, an interpretation I and a propositional symbol A; we can define $T_{\mathbb{P}}(I)(A)$ as*

$$\text{multisup}\Big(\{I(A)\} \cup \{\hat{I}(\mathcal{B}) \mid A \leftarrow \mathcal{B} \in \mathbb{P}\}\Big)$$

Some properties of this definition of the $T_{\mathbb{P}}$ operator are stated below, where $\sqsubseteq_S$ denotes the Smyth-ordering between subsets of a poset:

Lemma 3. *If $I \sqsubseteq J$, then $T_{\mathbb{P}}(I)(A) \sqsubseteq_S T_{\mathbb{P}}(J)(A)$ for all propositional symbol A.*

Proof. Let us write X_I to denote the set $\{I(A)\} \cup \{\hat{I}(\mathcal{B}) \mid A \leftarrow \mathcal{B} \in \mathbb{P}\}$, then the hypothesis states that $X_J^{\uparrow} \subseteq X_I^{\uparrow}$, where the $\uparrow$ denotes the upwards-closure of a set. Now, consider $b \in T_{\mathbb{P}}(J)(A)$, then b is an element of $X_J^{\uparrow} \subseteq X_I^{\uparrow}$; thus, by consistency, considering any minimal a of $X_I^{\uparrow}$ below b leads to the existence of an element $a \in T_{\mathbb{P}}(I)(A)$. □

The definition of $T_{\mathbb{P}}$ proposed above generates some coherence problems, in that the resulting 'value' is not an element, but a subset of the multi-lattice. A possible solution to this problem would be to consider a *choice function* $()^*$ which, given an interpretation, for any propositional symbol A selects an element in $T_{\mathbb{P}}(I)(A)$; this way, $T_{\mathbb{P}}(I)^*$ represents actually an interpretation.

Regarding particular properties of the composition of the $T_{\mathbb{P}}$ operator with suitable choice functions, the first property one can obtain, directly from the definition, is that the composition leads to an inflationary operator.

Lemma 4. *Given an interpretation I and a choice function $()^*$, then $I(A) \leq T_{\mathbb{P}}(I)^*(A)$ for all propositional symbol A.*

Note that, for some choice functions, the resulting operator ${T_{\mathbb{P}}}^*$ might not be monotone in the set of interpretations, since it can lead to incomparable interpretations; the multi-lattice of Fig. 2 can be used to construct a counter-example.

Example 5. Consider the following program with just two facts $\{A \leftarrow a, A \leftarrow b\}$ and interpretations $I(A) = \bot$ and $J(A) = c$; obviously $I \sqsubseteq J$. Now, we have that $T_{\mathbb{P}}(I)(A) = \{c, d\}$ and $T_{\mathbb{P}}(J)(A) = \{c\}$. Thus, the choice function $()^*$ which selects d in $T_{\mathbb{P}}(I)(A)$ generates incomparable interpretations $T_{\mathbb{P}}(I)^*$ and $T_{\mathbb{P}}(J)^*$.

We are interested in computing models of our extended programs by successive iteration of ${T_{\mathbb{P}}}^*$. Therefore, we should characterise the models of $\mathbb{P}$ in terms $T_{\mathbb{P}}$. The following result, which characterises the models of our extended programs in terms of properties of $T_{\mathbb{P}}$, can be proved:

Proposition 1. *The four statements below are equivalent:*

1. *I is a model of $\mathbb{P}$.*
2. *$T_{\mathbb{P}}(I)(A) = \{I(A)\}$ for all $A \in \Pi$.*
3. *$T_{\mathbb{P}}(I)^* = I$ for all choice function.*
4. *$I \in T_{\mathbb{P}}(I)$,[1] (i.e. I is a fixed point of $T_{\mathbb{P}}$ as a non-deterministic operator).*

[1] Abusing notation this means that $I(A) \in T_{\mathbb{P}}(I)(A)$ for all $A \in \Pi$.

Proof.
(1 ⇒ 2). Let us assume that I is a model of $\mathbb{P}$; then, we have that $I(A) \geq \hat{I}(\mathcal{B})$ for all rule $A \leftarrow \mathcal{B} \in \mathbb{P}$. This implies that $I(A)$ is the maximum of the set

$$\{I(A)\} \cup \{\hat{I}(\mathcal{B}) \mid A \leftarrow \mathcal{B} \in \mathbb{P}\}$$

hence, the only multi-supremum.
(2 ⇒ 1). The hypothesis implies that $I(A)$ is an upper bound of

$$\{\hat{I}(\mathcal{B}) \mid A \leftarrow \mathcal{B} \in \mathbb{P}\}$$

as a result, $I(A) \geq \hat{I}(\mathcal{B})$ for all rule $A \leftarrow \mathcal{B} \in \mathbb{P}$ and $I \models \mathbb{P}$.
(2 ⇔ 3 ⇔ 4). Trivial. □

Regarding the iterated application of the $T_{\mathbb{P}}$ operator, the use of choice functions is essential. Let us consider a model I, that is, a fixed point of $T_{\mathbb{P}}$, then for all propositional variable A, we have that $T_{\mathbb{P}}(I)(A) = \{I(A)\}$. Lemma 3 guides us in the choice after each application of $T_{\mathbb{P}}$ as follows:

- For the base case, we have[2] $\triangle \sqsubseteq I$, then $T_{\mathbb{P}}(\triangle)(A) \sqsubseteq_S T_{\mathbb{P}}(I)(A) = \{I(A)\}$. This means that there exists an element $m_1(A) \in T_{\mathbb{P}}(\triangle)(A)$ such that

$$m_1(A) \leq I(A)$$

  This way we obtain an interpretation m_1 satisfying $m_1 \sqsubseteq I$ such that for any propositional variable A, $m_1(A)$ is an element of $T_{\mathbb{P}}(\triangle)(A)$.
- This argument applies also to any successor ordinal: given $m_k \sqsubseteq I$, there exists an element $m_{k+1}(A) \in T_{\mathbb{P}}(m_k)(A)$ such that

$$m_k(A) \leq m_{k+1}(A) \leq I(A)$$

  where the first inequality holds by the definition of $T_{\mathbb{P}}$ and the second inequality follows from Lemma 3.
- For a limit ordinal α, Lemma 1 states that for all A the increasing sequence $\{m_n(A)\}$ has a supremum, which is considered, by definition, to be $m_\alpha(A)$.

As a result of the discussion above we obtain that we can choose suitable elements in the sets generated by the application of $T_{\mathbb{P}}$ in such a way that we can construct a transfinite sequence of interpretations m_k satisfying

$$m_1 \sqsubseteq m_2 \sqsubseteq \cdots \sqsubseteq m_k \sqsubseteq \cdots \sqsubseteq I$$

Note that the sequence of interpretations above, can be interpreted as the Kleene sequence which allows to reach the least fixed point of $T_{\mathbb{P}}$ in the classical case.

Interestingly enough, if I is a minimal model of $\mathbb{P}$, the previous sequence of interpretations can be proved to converge to I.

[2] Here, as usual, $\triangle$ denotes the minimum interpretation.

Theorem 3. *Let I be a minimal model of $\mathbb{P}$, then the previous construction leads to a Kleene sequence $\{m_\lambda\}$ which converges to I.*

Proof. A cardinality-based argument suffices to show that $\{m_\lambda\}$ is eventually constant and equal to I:

Let β be the least ordinal greater than the cardinal of the set of interpretations, for all $\lambda < \beta$ we can consider the interpretation m_λ and, thus, define the following map

$$\begin{aligned} h\colon \beta &\longrightarrow \mathcal{I} \\ \lambda &\mapsto m_\lambda \end{aligned}$$

If the transfinite sequence were strictly increasing, then h would be injective, obtaining a contradiction with the choice of β. As a result, we have proved the existence of an ordinal α such that $m_\alpha = m_{\alpha+1}$.

Recall that, by definition, we have $m_\alpha \sqsubseteq I$ and $m_{\alpha+1} \in T_{\mathbb{P}}(m_\alpha)$, therefore $m_\alpha \in T_{\mathbb{P}}(m_\alpha)$ and, by Proposition 1, m_α is a model of $\mathbb{P}$. By minimality of I we have that $m_\alpha = I$. □

Example 6. Continuing with the previous example, let us consider the minimal model I_1, and let us construct a sequence of approximating interpretations as stated in the theorem above.

	$\triangle$	$T_{\mathbb{P}}(\triangle)$	m_1	$T_{\mathbb{P}}(m_1)$	m_2
A	$\bot$	$\{a\}$	a	$\{a\}$	a
B	$\bot$	$\{b\}$	b	$\{b\}$	b
E	$\bot$	$\{\bot\}$	$\bot$	$\{c, d\}$	c

5 Conclusions and future work

A fixed point semantics has been presented for multi-lattice-based logic programming, together with some initial and encouraging results: in particular, we have proved the existence of minimal models for any extended program and that any minimal model can be attained by some Kleene-like sequence.

However, a number of theoretical problems have to be investigated in the future: such as the constructive nature of minimal models (is it possible to construct suitable choice functions which generate convergent sequence of interpretations with limit a minimal model?). Possible answers should on a general theory of fixed points, relying on some of the ideas related to fixed points in partially ordered sets [10] or, perhaps, in fuzzy extensions of Tarski's theorem [12].

References

1. M. Benado. Les ensembles partiellement ordonnés et le théorème de raffinement de Schreier, II. Théorie des multistructures. *Czech. Math. J.*, 5(80):308–344, 1955.

2. P. Cordero, G. Gutiérrez, J. Martínez, and I. P. de Guzmán. A new algebraic tool for automatic theorem provers. *Ann. Math. and Artif. Intelligence*, 42(4):369–398, 2004.
3. C. Damásio, J. Medina, and M. Ojeda-Aciego. Sorted multi-adjoint logic programs: termination results and applications. *Lect. Notes in Artificial Intelligence* 3229:260–273, 2004.
4. C. Damásio and L. Pereira. Monotonic and residuated logic programs. *Lecture Notes in Computer Science*, 2143:748–759, 2001.
5. D. Hansen. An axiomatic characterization of multi-lattices. *Discrete Mathematics*, 1:99–101, 1981.
6. M.A. Khamsi and D. Misane. Fixed point theorems in logic programming. *Ann. Math. and Artif. Intelligence*, 21:231–243, 1997.
7. L. Lakhsmanan and F. Sadri. On a theory of probabilistic deductive databases. *Theory and Practice of Logic Programming*, 1(1):5–42, 2001.
8. J. Martínez, G. Gutiérrez, I.P. de Guzmán and P. Cordero. Generalizations of lattices looking at computation. *Discrete Mathematics*, 295:107–141, 2005.
9. J. Medina, M. Ojeda-Aciego, and P. Vojtáš. Multi-adjoint logic programming with continuous semantics. *Lect. Notes in Artificial Intelligence*, 2173:351–364, 2001.
10. A. Ran and M. Reurings. A fixed point theorem in partially ordered sets and some applications to matrix equations. *Proc. of the AMS*, 132(5):1435–1443, 2003.
11. W. Rounds and G.-Q. Zhang. Clausal logic and logic programming in algebraic domains. *Inform. and Computation*, 171:183–200, 2001.
12. A. Stouti. A fuzzy version of Tarski's fixpoint theorem. *Archivum mathematicum*, 40:273–279, 2004.
13. P. Vojtáš. Fuzzy logic programming. *Fuzzy sets and systems*, 124(3):361–370, 2001.