Received: Added at production | Revised: Added at production | Accepted: Added at production

DOI: xxx/xxxx

ARTICLE TYPE

Relational Galois connections between transitive fuzzy digraphs

I.P. Cabrera*[1] | P. Cordero[1] | E. Muñoz-Velasco[1] | M. Ojeda-Aciego[1] | B. De Baets[2]

[1]Dept. Matemática Aplicada, Universidad de Málaga, Spain

[2]KERMIT, Dept. of Data Analysis and Mathematical Modelling, Ghent University, Belgium

Correspondence
*Inma P. Cabrera,
Dept. Matemática Aplicada.
ETSI Informática. Universidad de Málaga
Blv. Louis Pasteur 35. 29071-Málaga, Spain
Email: ipcabrera@uma.es

Abstract

Fuzzy directed graphs are often chosen as the data structure to model and implement solutions to several problems in the applied sciences. Galois connections have also shown to be useful both in theoretical and in practical problems.

In this paper, the notion of relational Galois connection is extended to be applied between transitive fuzzy directed graphs. In this framework, the components of the connection are crisp relations satisfying certain reasonable properties given in terms of the so-called full powering.

KEYWORDS:
Galois connection; fuzzy digraph

1 | INTRODUCTION

The notion of Galois connection has shown its applicability since its introduction seventy-five years ago[20]. Recent papers can still be found showing the use of Galois connections (together with their siblings, the *adjunctions*), both in theoretical and in practical problems. For instance, Kycia[17] demonstrates how to construct a Galois connection between two systems with entropy, where the connection transfers changes in one system to the other one, preserving the ordering structure induced by entropy, thus opening a new area of abstract modeling of systems in presence of entropy; Brattka[4] considers the characterization of limit computable functions either by Turing jumps on the input side or by limits on the output side; this pair of adjoint operations leads to a formal Galois connection in a certain lattice of representation spaces. Faul[11] uses adjunctions to provide a balanced study of two apparently different approaches to broadcast domination of product graphs. Moraschini[18] introduces a logical and algebraic description of right adjoint functors between generalized quasi-varieties, developing a correspondence between the concept of adjunction and a new notion of translation between relative equational consequences. Gibbons et al.[15] use adjunctions to elegantly explain the relational algebra constructs (selections, projections, join, grouping and joins) on bulk types such as sets, bags, and lists.

In this paper, we continue our research line on the construction of Galois connections between sets with unbalanced structures initiated in[13], where we attempted to characterize the existence of the right part of a Galois connection of a given mapping between sets with a different structure (it is precisely this condition of different structure that makes this problem to be outside the scope of Freyd's adjoint functor theorem). Since then, we have obtained results in several frameworks: for instance, in[14], given a mapping from a (pre-)ordered set $(A, \leq_A)$ into an unstructured set B, we solved the problem of completing the structure of B, namely, defining a suitable (pre-)ordering relation $\leq_B$ on B, such that there exists a mapping such that the pair of mappings forms an isotone Galois connection (or adjunction) between the (pre-)ordered sets $(A, \leq_A)$ and $(B, \leq_B)$. Later, in[6] we moved to the fuzzy framework by considering the corresponding problem between a fuzzy preposet (A, ρ_A) and an unstructured B; this work was recently extended in[7], by considering that equality is expressed by a fuzzy equivalence relation, so that the problem considers a mapping between a fuzzy preordered structure $(A, \approx_A, \rho_A)$ and a fuzzy structure $(B, \approx_B)$. These two papers satisfactorily extend the problem to the fuzzy case in both the domain and range of the Galois connection but, in both cases, the components of the Galois connection are (crisp) functions. A first attempt aiming at obtaining a notion of Galois connection whose components are, in fact, fuzzy functions was given in[8]; there, we shifted our attention by considering that the domain and range are just sets endowed with arbitrary relations and considering connections whose components are (proper) relations, obtaining what we called *relational Galois connections*.

We attempt here a first generalization of the notion of relational Galois connection to the fuzzy case. The focus is put on transitive fuzzy directed graphs, fuzzy T-digraphs for short, because of their interest for applications. One can find interesting theoretical applications of digraphs, for instance, Ceballos et al. [10] use the link between digraphs and Leibniz algebras in order to obtain the complete classification of 2- and 3-dimensional Leibniz non-Lie algebras. Moreover, Akram et al. [1] use bipolar fuzzy digraphs for designing and implementing algorithms of decision support systems for practical problems such as vulnerability assessment of gas pipeline networks. Akram et al[2] introduce the notion of fuzzy rough digraph and consider its application in decision making. In[3], Baykasoglu applies a fuzzy digraph model to quantify manufacturing flexibility. In[16], Koulouriotis and Ketipi develop a fuzzy digraph method for robot evaluation and selection, according to a given industrial application. The preceding examples just show the interest for the structure of fuzzy digraphs, but the interest of including, at least, transitivity is undeniable. There are studies about the theoretical analysis of the transitivity editing problem on digraphs[21], and also about finding a sensible and useful definition of transitive closure of a bipolar weighted digraph[19], since it has been noted in the literature that a transitive closure of a bipolar weighted digraph contains useful new information for the fuzzy cognitive map it models.

In this work, we focus specifically on providing an adequate notion of relational Galois connection between fuzzy T-digraphs which inherits most of the interesting equivalent characterizations of the notion of crisp Galois connection.

The structure of this paper is the following: in Section 2, the preliminary notions on relational Galois connections between T-digraphs are introduced; then, in Section 3, the required notions about fuzzy T-digraphs are given, together with the notion of relational Galois connection between fuzzy T-digraphs and its characterization in terms of a suitable Galois condition. Finally, in Section 4, some conclusions are drawn and prospects for future work are given.

2 | PRELIMINARY NOTIONS ON RELATIONAL GALOIS CONNECTIONS

We consider the usual framework of (crisp) relations. Namely, a binary relation $\mathcal{R}$ between two sets A and B is a subset of the Cartesian product $A \times B$ and it can also be seen as a function $\mathcal{R}$ from the set A to the powerset 2^B. For an element $(a, b) \in \mathcal{R}$, it is said that a is related to b and denoted as $a\mathcal{R}b$.

Given a binary relation $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{B}$, the *afterset* $a^{\mathcal{R}}$ of an element $a \in A$ is defined as $\{b \in B \mid a\mathcal{R}b\}$. The *domain* of $\mathcal{R}$ is the set $\mathrm{dom}(\mathcal{R}) = \{a \in A \mid a^{\mathcal{R}} \neq \varnothing\}$. A binary relation $\mathcal{R}$ is said to be *total* if $\mathrm{dom}(\mathcal{R}) = A$.

Given an arbitrary set A and a preorder relation $\leq$ defined on A, it is possible to lift $\leq$ to the powerset 2^A by defining

$$X \lll Y \iff \forall x \in X\ \exists y \in Y \text{ such that } x \leq y$$

$$X \Subset Y \iff \forall y \in Y\ \exists x \in X \text{ such that } x \leq y$$

$$X \propto Y \iff \forall x \in X\ \forall y \in Y\ \ x \leq y$$

We will use the term *powering* to refer to the lifting of a preorder to the powerset; thus, $\lll$, $\Subset$ and $\propto$ above are powerings of $\leq$. Note that the first two relations are actually preorder relations, specifically those used in the construction of the Hoare and Smyth powerdomains, respectively; the third one neither needs to be reflexive nor transitive, and was introduced in[8] as a convenient tool to develop relational Galois connections. Furthermore, it is worth noting that the powerings can be defined for any relation not necessarily being a preorder.

Naturally, each of the extensions above induces a particular notion of isotony, inflation, etc. For instance, given two preordered sets $(A, \leq)$ and $(B, \leq)$,[1] a binary relation $\mathcal{R} \subseteq A \times B$ is said to be:

- $\Subset$-isotone if $a_1 \leq a_2$ implies $a_1^{\mathcal{R}} \Subset a_2^{\mathcal{R}}$, for all $a_1, a_2 \in \mathrm{dom}(\mathcal{R})$;
- $\Subset$-antitone if $a_1 \leq a_2$ implies $a_2^{\mathcal{R}} \Subset a_1^{\mathcal{R}}$, for all $a_1, a_2 \in \mathrm{dom}(\mathcal{R})$.

A binary relation $\mathcal{R} \subseteq A \times A$ is said to be:

- $\Subset$-inflationary if $\{a\} \Subset a^{\mathcal{R}}$, for all $a \in \mathrm{dom}(\mathcal{R})$;

[1] Note that, as usual, we use the same symbol to denote both binary relations which need not be equal.

- $\Subset$-idempotent if $a^{\mathcal{R}\circ\mathcal{R}} \Subset a^{\mathcal{R}}$ and $a^{\mathcal{R}} \Subset a^{\mathcal{R}\circ\mathcal{R}}$, for all $a \in \mathrm{dom}(\mathcal{R})$.

We use the prefix to distinguish the powering used in the different definitions. Note that the previous notions are defined for elements in the domain of the relation; hence, for simplicity and without loss of generality, we will assume that all our relations are total.

Traditionally, a Galois connection is understood as a pair of antitone mappings whose compositions are both inflationary, and has a number of different alternative characterizations. In our generalized relational setting, we will work with the usual notion of relational composition. Let $\mathcal{R}$ be a binary relation between A and B and $\mathcal{S}$ be a binary relation between B and C. The composition of $\mathcal{R}$ and $\mathcal{S}$ is defined as follows

$$\mathcal{R}\circ\mathcal{S} = \{(x, z) \in A \times C \mid \exists b \in B \text{ with } x\mathcal{R}b \text{ and } b\mathcal{S}z\}$$

Observe that for an element $a \in A$, the afterset $a^{\mathcal{R}\circ\mathcal{S}}$ can be written as $\bigcup_{b\in a^{\mathcal{R}}} b^{\mathcal{S}}$.

A well-known characterization of a Galois connection (f, g) between two posets is the so-called *Galois condition*

$$a \leq g(b) \iff b \leq f(a).$$

As stated above, in our general framework there are several possible choices, which we will distinguish by using the corresponding prefix. For instance, given two relations $\mathcal{R}$ and $\mathcal{S}$, the $\ll$-Galois condition is given by

$$\{a\} \ll b^{\mathcal{S}} \iff \{b\} \ll a^{\mathcal{R}}.$$

In [5], we studied the properties of the different extensions obtained in terms of the powerings $\ll$ and $\Subset$ used in the corresponding Galois condition. The resulting notion was investigated within the framework of preordered structures in [9]; later, in [8], we focused our attention on another desirable characterization, the definition of a Galois connection in terms of closures.

Given a transitive directed graph $\mathbb{A} = (A, \tau)$, T-digraph for short, a powering $*$ of τ, and $C \subseteq A \times A$, we say that C is a $*$-*closure relation*, if C is $*$-isotone, $*$-inflationary and $*$-idempotent.

Given two T-digraphs $\mathbb{A}$ and $\mathbb{B}$, in [8] we defined the notion of *relational Galois connection* between $\mathbb{A}$ and $\mathbb{B}$ as a pair of relations $(\mathcal{R}, \mathcal{S})$ where $\mathcal{R} \subseteq A \times B$ and $\mathcal{S} \subseteq B \times A$ such that the following properties hold:

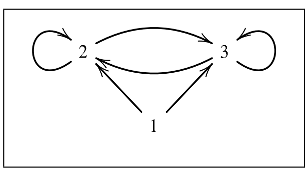
i. $\mathcal{R}$ and $\mathcal{S}$ are $\Subset$-antitone.

ii. $\mathcal{R}\circ\mathcal{S}$ and $\mathcal{S}\circ\mathcal{R}$ are $\Subset$-inflationary.

We can see below an example in which both $\mathcal{R}$ and $\mathcal{S}$ are proper (non-functional) relations.

Example 1. Consider $\mathbb{A} = (A, \tau)$ where $A = \{1, 2, 3\}$ and τ is the transitive relation $\{(1, 2), (1, 3), (2, 2), (2, 3), (3, 2), (3, 3)\}$. The pair of relations $(\mathcal{R}, \mathcal{S})$ given by the tables below constitutes a relational Galois connection between $\mathbb{A}$ and $\mathbb{A}$.

$\mathbb{A}$



x	$x^{\mathcal{R}}$
1	$\{2,3\}$
2	$\{2\}$
3	$\{3\}$

x	$x^{\mathcal{S}}$
1	$\{2,3\}$
2	$\{2\}$
3	$\{2,3\}$

□

The interesting point is that the $\Subset$-powering guarantees that both compositions in a relational Galois connection lead to a $\Subset$-closure relation. Formally, we have the following result.

Theorem 1 ([8]). Given a relational Galois connection $(\mathcal{R}, \mathcal{S})$ between two T-digraphs $\mathbb{A}$ and $\mathbb{B}$, we have that $\mathcal{R}\circ\mathcal{S}$ and $\mathcal{S}\circ\mathcal{R}$ are $\Subset$-closure relations.

Our characterization of relational Galois connections is based on the fact that the direct images of singletons should be *cliques* for both components of the relational Galois connection (given a T-digraph $\mathbb{A}$ and $X \subseteq A$ it is said that X is a clique if $X \propto X$).

The formal result is as follows.

Theorem 2 ([8]). Given two T-digraphs (A, τ) and (B, τ), the following statements are equivalent:

1. $(\mathcal{R}, \mathcal{S})$ is a relational Galois connection.

2. The two conditions below hold:

   (i) $\{a\} \Subset b^{\mathcal{S}}$ iff $\{b\} \Subset a^{\mathcal{R}}$ for all $a \in A$ and $b \in B$.

   (ii) $a^{\mathcal{R}}$ and $b^{\mathcal{S}}$ are cliques for all $a \in A$ and $b \in B$.

3. The two conditions below hold:

   (i) $\mathcal{R}$ and $\mathcal{S}$ are $\propto$-antitone.

   (ii) $\mathcal{R} \circ \mathcal{S}$ and $\mathcal{S} \circ \mathcal{R}$ are $\propto$-inflationary.

3 | RELATIONAL GALOIS CONNECTIONS BETWEEN FUZZY T-DIGRAPHS

Our framework in this work is relational at the level of Galois connections and fuzzy at the level of their domain and codomain; hence, the necessary definitions in the fuzzy case are introduced below.

Given a complete residuated lattice $\mathbb{L} = (L, \leq, \bot, \top, \otimes, \rightarrow)$, an $\mathbb{L}$-fuzzy set is a mapping from the universe to the membership values structure $X : U \rightarrow L$ where $X(u)$ expresses the degree to which u belongs to X.

An $\mathbb{L}$-fuzzy binary relation on U is an $\mathbb{L}$-fuzzy subset of $U \times U$, that is $\rho : U \times U \rightarrow L$, and it is said to be:

- *Reflexive* if $\rho(a, a) = \top$ for all $a \in U$.
- $\otimes$-*Transitive* if $\rho(a, b) \otimes \rho(b, c) \leq \rho(a, c)$ for all $a, b, c \in U$.

Definition 1. Given $\mathbb{A} = (A, \rho)$ we introduce the following notions:

- $\mathbb{A}$ is said to be a *fuzzy T-digraph* if ρ is a $\otimes$-transitive fuzzy relation on A.
- $\mathbb{A}$ is said to be a *fuzzy preposet* if ρ is a reflexive and $\otimes$-transitive fuzzy relation on A.

The adaptation of the different powerings to the fuzzy framework is the following.

Definition 2. Let (A, ρ) be a fuzzy T-digraph and $X, Y \subseteq A$, we define the Hoare, Smyth and full fuzzy powerings as follows:

i. $\rho_H(X, Y) = \bigwedge_{x \in X} \bigvee_{y \in Y} \rho(x, y)$

ii. $\rho_S(X, Y) = \bigwedge_{y \in Y} \bigvee_{x \in X} \rho(x, y)$

iii. $\rho_{\propto}(X, Y) = \bigwedge_{x \in X} \bigwedge_{y \in Y} \rho(x, y)$

Remark 1. We will often need to use the powerings above with one of the arguments being a singleton; we will omit the braces in the singleton to simplify the notation. Moreover, if the singleton appears on the left-hand side, it holds that $\rho_S(a, X) = \rho_{\propto}(a, X)$, whereas if the singleton appears on the right-hand side, it holds that $\rho_H(X, a) = \rho_{\propto}(X, a)$.

The notion of clique was crucial for the characterisation of relational Galois connection in the crisp case, the fuzzy adaptation is given below.

Definition 3. Let (A, ρ) be a fuzzy T-digraph and $X \subseteq A$. We say that a nonempty set X is a clique if for all $x, y \in X$ it holds $\rho(x, y) = \top$ or, equivalently, $\rho_{\propto}(X, X) = \top$.

The following technical result will be used later.

Lemma 1. Let (A, ρ) be a fuzzy T-digraph, $X \subseteq A$ and $a \in A$. If X is a clique, then $\rho_S(a, X) = \rho_{\propto}(a, X) = \rho_H(a, X)$.

Proof. As X is nonempty, it is straightforward that $\rho_S(a, X) \leq \rho_H(a, X)$.

By X being a clique, for all $x, y \in X$ we have $\rho(x, y) = \top$; moreover, by transitivity of ρ we have that

$$\rho(a, x) = \rho(a, x) \otimes \rho(x, y) \leq \rho(a, y).$$

As a result, we obtain

$$\rho_H(a, X) = \bigvee_{x \in A} \rho(a, x) \leq \bigwedge_{y \in A} \rho(a, y) = \rho_S(a, X).$$

□

In order to adapt the definition of relational Galois connection to our fuzzy framework, we need the notions of antitone relation and inflationary relation between fuzzy T-digraphs. The following definition states these notions in terms of the powering $\propto$.

Definition 4. Let (A, ρ) and (B, ρ) be fuzzy T-digraphs.

i. A relation $\mathcal{R} \subseteq A \times B$ is antitone if $\rho(a_1, a_2) \leq \rho(b_2, b_1)$ for all $b_1 \in a_1^{\mathcal{R}}$ and $b_2 \in a_2^{\mathcal{R}}$, or equivalently, $\rho(a_1, a_2) \leq \rho_\propto(a_2^{\mathcal{R}}, a_1^{\mathcal{R}})$.

ii. A relation $\mathcal{R} \subseteq A \times A$ is inflationary if $\rho(a_1, a_2) = \top$ for all $a_2 \in a_1^{\mathcal{R}}$ or, equivalently, $\rho_\propto(a, a^{\mathcal{R}}) = \top$.

The corresponding definition of relational Galois connection in our fuzzy framework is given below.

Definition 5. Let (A, ρ) and (B, ρ) be fuzzy T-digraphs. Given $\mathcal{R} \subseteq A \times B$ and $\mathcal{S} \subseteq B \times A$, we say that $(\mathcal{R}, \mathcal{S})$ is a *relational Galois connection* if $\mathcal{R}$ and $\mathcal{S}$ are antitone and $\mathcal{R} \circ \mathcal{S}$ and $\mathcal{S} \circ \mathcal{R}$ are inflationary.

Given the previous notion of relational Galois connection between fuzzy T-digraphs, it is worth considering the existence of a possible characterization in terms of a suitable generalization of the Galois condition.

A first result that can be obtained is the following.

Proposition 1. Let (A, ρ) and (B, ρ) be fuzzy T-digraphs and $\mathcal{R} \subseteq A \times B$ and $\mathcal{S} \subseteq B \times A$ be relations. If $(\mathcal{R}, \mathcal{S})$ is a relational Galois connection, then the following holds for all $a \in A$ and $b \in B$:

$$\rho_H(a, b^{\mathcal{S}}) \leq \rho_S(b, a^{\mathcal{R}}) \qquad \text{and} \qquad \rho_H(b, a^{\mathcal{R}}) \leq \rho_S(a, b^{\mathcal{S}})\,. \tag{1}$$

Proof. Assume that $(\mathcal{R}, \mathcal{S})$ is a relational Galois connection, let us prove the first inequality.

Given $a_1, a_2 \in A$, $b_1, b_2 \in B$ such that $b_1 \in a_1^{\mathcal{R}}$ and $a_2 \in b_2^{\mathcal{S}}$, as $\mathcal{S}$ is total, there exists $b_3 \in a_2^{\mathcal{R}}$. This implies, using the fact that $\mathcal{S} \circ \mathcal{R}$ is inflationary, that $\rho(b_2, b_3) = \top$. Consequently, using the fact that $\mathcal{R}$ is antitone and ρ is transitive, we get

$$\rho(a_1, a_2) = \rho(a_1, a_2) \otimes \rho(b_2, b_3) \leq \rho(b_2, b_3) \otimes \rho(b_3, b_1) \leq \rho(b_2, b_1)\,.$$

By taking supremum in the left-hand side and infimum in the right-hand side, we obtain that $\rho_H(a, b^{\mathcal{S}}) \leq \rho_S(b, a^{\mathcal{R}})$. The second inequality can be proved similarly.

□

Condition (1) turns out to be equivalent to $(\mathcal{R}, \mathcal{S})$ being a relational Galois connection between fuzzy preposets.

Proposition 2. Let (A, ρ) and (B, ρ) be fuzzy preposets and $\mathcal{R} \subseteq A \times B$ and $\mathcal{S} \subseteq B \times A$ be relations. If condition (1) holds, then $(\mathcal{R}, \mathcal{S})$ is a relational Galois connection.

Proof. For all $a \in A$ and $b \in a^{\mathcal{R}}$, as ρ is reflexive, one has $\rho(b, b) = \top$. Now, by using condition (1), we obtain

$$\top = \rho(b, b) \leq \rho_H(b, a^{\mathcal{R}}) \leq \rho_S(a, b^{\mathcal{S}}) = \rho_\propto(a, b^{\mathcal{S}})$$

and this proves that $\rho_\propto(a, a^{\mathcal{R} \circ \mathcal{S}}) = \top$, that is, $\mathcal{R} \circ \mathcal{S}$ is inflationary. The proof for $\mathcal{S} \circ \mathcal{R}$ is similar.

Let us prove now that $\mathcal{R}$ is antitone. Consider $a_1, a_2 \in A$, $b_1 \in a_1^{\mathcal{R}}$ and $b_2 \in a_2^{\mathcal{R}}$. For $x \in b_2^{\mathcal{S}}$, as $\mathcal{R} \circ \mathcal{S}$ is inflationary, we have that $\rho(a_2, x) = \top$. By transitivity of ρ, we have $\rho(a_1, a_2) \otimes \rho(a_2, x) \leq \rho(a_1, x)$, which implies $\rho(a_1, a_2) \leq \rho(a_1, x)$. Now, by (1), one has $\rho(a_1, a_2) \leq \rho(a_1, x) \leq \rho(b_2, b_1)$, proving that $\mathcal{R}$ is antitone. The proof that $\mathcal{S}$ is antitone is similar.

□

The following example shows that condition (1) is not enough to have a relational Galois connection for general fuzzy T-digraphs.

Example 2. Consider the following fuzzy T-digraphs $\mathbb{A} = (\{a_1, a_2, a_3\}, \rho_A)$ and $\mathbb{B} = (\{b_1, b_2, ab_3\}, \rho_B)$, and the relations $\mathcal{R} \subseteq A \times B$ and $\mathcal{S} \subseteq B \times A$ defined below:

ρ_A	a_1	a_2	a_3
a_1	1	1	1/2
a_2	0	0	0
a_3	0	1/2	1

ρ_B	b_1	b_2	b_3
b_1	1	0	0
b_2	1	0	0
b_3	1/2	0	1

x	$x^{\mathcal{R}}$
a_1	$\{b_1\}$
a_2	$\{b_2\}$
a_3	$\{b_3\}$

x	$x^{\mathcal{S}}$
b_1	$\{a_1\}$
b_2	$\{a_1, a_2\}$
b_3	$\{a_3\}$

It is routine to check that $(\mathcal{R}, \mathcal{S})$ verifies condition (1), but it is not a relational Galois connection, because $\{a_1\} \in a_2^{\mathcal{R} \circ \mathcal{S}}$, while $\rho(a_2, a_1) = 0$ and $\rho_S(a_2, a_2^{\mathcal{R} \circ \mathcal{S}}) = \rho_S(a_1, \{a_1, a_2\}) = 0$, which contradicts the fact that $\mathcal{R} \circ \mathcal{S}$ is inflationary.

□

Another important condition in the crisp case was the fact that the aftersets of the components of a relational Galois connection are cliques. In this framework, we obtain the following result.

Proposition 3. Let (A, ρ) and (B, ρ) be fuzzy T-digraphs. Given $\mathcal{R} \subseteq A \times B$ and $\mathcal{S} \subseteq B \times A$, if $(\mathcal{R}, \mathcal{S})$ is a relational Galois connection, then $a^{\mathcal{R}}$ and $b^{\mathcal{S}}$ are cliques for all $a \in A$, $b \in B$.

Proof. As $\mathcal{R} \circ \mathcal{S}$ is inflationary, it holds $\rho_A(a, x) = \top$ for all $x \in a^{\mathcal{R} \circ \mathcal{S}}$, that is, for all $b \in a^{\mathcal{R}}$ and for all $x \in b^{\mathcal{S}}$ one has $\rho_A(a, x) = \top$. This implies, by Proposition 1 already proved, that $\rho_B(b, y) = \top$, for all $y \in a^{\mathcal{R}}$, proving that $a^{\mathcal{R}}$ is a clique. The fact that $b^{\mathcal{S}}$ is a clique can be proved similarly.

□

As a consequence of this proposition, taking into account Lemma 1, we have that $\rho_S(a, X) = \rho_\propto(a, X) = \rho_H(a, X)$ if X is a clique. As a result, the inequalities in Proposition 1 collapse into equalities and, hence we are entitled to define the relational Galois condition (RG) as follows.

Definition 6. Let (A, ρ) and (B, ρ) be fuzzy T-digraphs and $\mathcal{R} \subseteq A \times B$ and $\mathcal{S} \subseteq B \times A$ be relations. We say that the pair $(\mathcal{R}, \mathcal{S})$ verifies the relational Galois condition (RG) if the following holds, for all $a \in A$ and $b \in B$:

$$\rho_\propto(a, b^{\mathcal{S}}) = \rho_\propto(b, a^{\mathcal{R}}). \tag{RG}$$

The theorem below shows that condition (RG) complemented with the fact that all the aftersets are cliques characterizes relational Galois connections between fuzzy T-digraphs.

Theorem 3. Let (A, ρ) and (B, ρ) be fuzzy T-digraphs. Given $\mathcal{R} \subseteq A \times B$ and $\mathcal{S} \subseteq B \times A$ then, $(\mathcal{R}, \mathcal{S})$ is a relational Galois connection between (A, ρ) and (B, ρ) if and only if the following conditions hold:

(i) $(\mathcal{R}, \mathcal{S})$ satisfies condition (RG).

(ii) $a^{\mathcal{R}}$ and $b^{\mathcal{S}}$ are cliques for all $a \in A$ and $b \in B$.

Proof. The direct implication is just a consequence of Propositions 1 and 3 and Lemma 1.

Conversely, if items (i) and (ii) hold, let us prove firstly that $\mathcal{R} \circ \mathcal{S}$ is inflationary. Given $x \in a^{\mathcal{R} \circ \mathcal{S}}$, there exists $b \in a^{\mathcal{R}}$ such that $x \in b^{\mathcal{S}}$. By (ii) it holds $\rho_B(b, y) = \top$, for all $y \in a^{\mathcal{R}}$, which implies, by (i), that $\rho_A(a, x) = \top$, proving that $\mathcal{R} \circ \mathcal{S}$ is inflationary. The proof of $\mathcal{S} \circ \mathcal{R}$ being inflationary is similar. Let us prove now that $\mathcal{R}$ is antitone. Consider $a_1, a_2 \in A$, $b_1 \in a_1^{\mathcal{R}}$ and $b_2 \in a_2^{\mathcal{R}}$. For $x \in b_2^{\mathcal{S}}$, as $\mathcal{R} \circ \mathcal{S}$ is inflationary, we have that $\rho_A(a_2, x) = \top$. By transitivity of ρ_A, we have that $\rho_A(a_1, a_2) \otimes \rho_A(a_2, x) \le \rho_A(a_1, x)$, which implies $\rho_A(a_1, a_2) \le \rho_A(a_1, x)$. Now, by (i), we have that $\rho_A(a_1, a_2) \le \rho_A(a_1, x) \le \rho_B(b_2, b_1)$, proving that $\mathcal{R}$ is antitone. The proof that $\mathcal{S}$ is antitone is similar.

□

It is worth noting that conditions (i) and (ii) in the theorem above are, indeed, independent. Example 2 can be used to show that condition (RG) holds but $b_2^{\mathcal{S}}$ is not a clique. Furthermore, the corresponding version of condition (RG) with the $\ll$-powering does not imply the clique condition either, since it holds as well in that example. Last but not least, there is no need to consider the $\Subset$-powering in this context since it coincides with the condition (RG) because the left argument is a singleton.

4 | CONCLUSIONS AND FUTURE WORK

A suitable notion of relational Galois connection between T-digraphs has been introduced. The definition follows the recently introduced one in[8], in that the connection is specified in terms of the antitonicity of its components and the inflationarity of both compositions. Once again, the clique condition turns out to be fundamental for the characterization as an indispensable complement to the fulfilment of the corresponding Galois condition (either in terms of the $\Subset$ powering or the most convenient, due to its better computational complexity, the $\propto$ powering).

Some applications of the notion of Galois connection have been published, such as the abstract modelling of systems in presence of entropy, the characterization of limit computable functions, and the study of broadcast domination of product graphs, among others. We believe that our generalization will allow to broaden the scope of these applications taking into account the suitability of fuzzy directed graphs to, inter alia, decision support systems for practical problems, quantification of manufacturing flexibility, and robot evaluation and selection.

This approach opens the way for a "completely fuzzy" notion of relational Galois connection in which the components of the connection are not just (crisp) relations but fuzzy relations. In that general context, it will be worth studying the minimal properties needed to define such a notion in its full generality, while maintaining (some of) the most useful equivalent characterisations of the notion of Galois connection[12].

5 | ACKNOWLEDGEMENTS

This work has been partially supported by the Spanish Ministry of Science, Innovation, and Universities (MCIU), the State Agency of Research (AEI) and the European Social Fund (FEDER) through the research projects PGC2018-095869-B-I00 and TIN2017-89023-P (MCIU/AEI/FEDER, UE).

References

1. Akram M, Alshehri Noura O., Davvaz B., Ashraf A. Bipolar Fuzzy Digraphs in Decision Support Systems. *Multiple-Valued Logic and Soft Computing.* 2016;27(5-6):531–551.

2. Akram M, Shumaiza, Arshad M. A new approach based on fuzzy rough digraphs for decision-making. *Journal of Intelligent and Fuzzy Systems.* 2018;35(2):2105–2121.

3. Baykasoglu A. A Practical Fuzzy Digraph Model for Modeling Manufacturing Flexibility. *Cybernetics and Systems.* 2009;40(6):475–489.

4. Brattka V. A Galois connection between Turing jumps and limits. *Logical Methods in Computer Science.* 2018;14(3:13).

5. Cabrera I.P., Cordero P., Ojeda-Aciego M. Relation-based Galois connections: towards the residual of a relation. In *Proc. of Computational and Mathematical Methods in Science and Engineering*, 2017.

6. Cabrera I.P., Cordero P., Garcia-Pardo F., Ojeda-Aciego M., De Baets B. On the construction of adjunctions between a fuzzy preposet and an unstructured set. *Fuzzy Sets and Systems.* 2017;320:81–92.

7. Cabrera I.P., Cordero P., Garcia-Pardo F., Ojeda-Aciego M., De Baets B. Galois connections between a fuzzy preordered structure and a general fuzzy structure. *IEEE Transactions on Fuzzy Systems.* 2018;26(3):1274–1287.

8. Cabrera I.P., Cordero P., Muñoz-Velasco E., Ojeda-Aciego M. A relational extension of Galois Connections. *Lecture Notes in Artificial Intelligence.* 2019;11511:290–303.

9. Cabrera I.P., Cordero P., Muñoz-Velasco E., Ojeda-Aciego M. Relational Connections between Preordered Sets. *Lecture Notes in Electrical Engineering.* 2019;574:163–169.

10. Ceballos M, Núñez J, Tenorio ÁF. (Pseudo)digraphs and Leibniz algebra isomorphisms. *Mathematical Methods in the Applied Sciences.* 2018;41:7481-7497.

11. Faul P. F. Adjunctions in the study of broadcast domination with a cost function. *The Australasian Journal on Combinatorics.* 2018;72:70–81.

12. García-Pardo F., Cabrera I.P., Cordero P., Ojeda-Aciego M. On Galois Connections and Soft Computing. *Lect. Notes in Computer Science.* 2013;7903:224–235.

13. García-Pardo F., Cabrera I.P., Cordero P., Ojeda-Aciego M., Rodríguez F.J. On the Existence of Isotone Galois Connections between Preorders. *Lect. Notes in Computer Science.* 2014;8478:67–79.

14. García-Pardo F., Cabrera I.P., Cordero P., Ojeda-Aciego M., Rodríguez F.J. On the definition of suitable orderings to generate adjunctions over an unstructured codomain. *Information Sciences.* 2014;286:173–187.

15. Gibbons J, Henglein F, Hinze R, Wu N. Relational algebra by way of adjunctions. *Proceedings of the ACM on Programming Languages.* 2018;2:86:1–86:28.

16. Koulouriotis D. E., Ketipi Maria K. A fuzzy digraph method for robot evaluation and selection. *Expert Systems and Applications.* 2011;38(9):11901–11910.

17. Kycia R. Landauer's principle as a special case of Galois connection. *Entropy.* 2018;20(12):971.

18. Moraschini T. A Logical and algebraic characterization of adjunctions between generalized quasi-Varieties. *J. Symbolic Logic.* 2018;83(3):899–919.

19. Niesink P, Poulin K, Šajna M. Computing transitive closure of bipolar weighted digraphs. *Discrete Applied Mathematics.* 2013;161:217-243.

20. Ore Ø. Galois Connections. *Transactions of the American Mathematical Society.* 1944;55:493–513.

21. Weller M, Komusiewicz C, Niedermeier R, Uhlmann J. On making directed graphs transitive. *J. Computer and System Sciences.* 2012;78(2):559–574.

□