On Reachability of Minimal Models of Multilattice-Based Logic Programs*

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Abstract. In this paper some results are obtained regarding the existence and reachability of minimal fixed points for multiple-valued functions on a multilattice. The concept of inf-preserving multi-valued function is introduced, and shown to be a sufficient condition for the existence of minimal fixed point; then, we identify a sufficient condition granting that the immediate consequence operator for multilattice-based fuzzy logic programs is sup-preserving and, hence, computes minimal models in at most ω iterations.

1 Introduction

Multilattice-based logic programs have been recently introduced as an extended paradigm for fuzzy logic programming in which the underlying set of truth-values for the propositional variable is considered to have a more relaxed structure than that of a complete lattice.

This line of research follows the trend of generalising the structure of the underlying set of truth-values for fuzzy logic programming, which has attracted the attention of a number of researchers in the recent years. For instance, there are approaches to fuzzy logic programming which are based either on the structure of lattice (residuated lattice [1, 2] or multi-adjoint lattice [3]), or on more restrictive structures, such as bilattices [4, 5], specially suited for the treatment of non-isotonicity, or even trilattices [6], in which points can be ordered according to truth, information, or precision. More general structures such as algebraic domains [7] have been used as well.

The first definition of multilattices seems to have been introduced in [8], although, much later, other authors proposed slightly different approaches [9, 10], the later being more appealing to computation.

The crucial point in which a complete multilattice differs from a complete lattice is that a given subset does not necessarily has a least upper bound (resp. greatest lower bound) but some minimal (resp. maximal) ones. As far as we know, the first paper which used multilattices in the context of extended fuzzy logic programming was [11], which was later generalized in [13]. In these

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papers, the meaning of programs was defined by means of a fixed point semantics. In particular, the non-existence of suprema in general, but a set of minimal upper bounds, suggested the possibility of developing a non-deterministic fixed point theory in the form of a multi-valued immediate consequences operator. Essentially, the results presented were the existence of minimal models below any model of a program, and that any minimal model can be attained by the iteration of a suitable version of the immediate consequence operator, existence of minimal models was proved independently of the fixed-point semantics used to reach them; but some other problems remained open, such as the constructive nature of minimal models or the reachability of minimal models after at most countably many iterations.

The first contribution of this paper is a theoretical one, related to the existence of minimal fixed points: obviously, the main theoretical problem can be stated simply in terms of a suitable version of fixed point theorem for multi-valued functions on a multilattice. Here, we provide an existence result for minimal fixed-points in such a general context.

The second contribution relates to the reachability of minimal models; specifically, we introduce conditions guaranteeing that minimal models can be reached by a suitable iteration of the immediate consequences operator, the underlying idea is to give a general version of a related result presented in [13] but for single-valued functions.

The structure of the paper is as follows: in Section 2, the definition and some preliminary results about multilattices are presented; later, in Section 3, we move to the context of multi-valued functions and orbits on a multilattice, in order to set the basic results about the existence of fixed-point for such functions; then, in Section 4, we concentrate on the case of fuzzy logic programs evaluated on a multilattice, the main result being the introduction of sufficient conditions granting computability for its fixed-point semantics; finally, Section 5 concludes.

2 Preliminaries

We provide in this section the basic notions of the theory of multilattices, together with some preliminary results which will be used later in this paper.

Definition 1. *A* complete multilattice *is a partially ordered set, $\langle M, \preceq \rangle$, such that for every subset $X \subseteq M$, the set of upper (resp. lower) bounds of X has minimal (resp. maximal) elements, which are called* multi-suprema *(resp.* multi-infima*).*

The sets of multi-suprema and multi-infima of a set X are denoted by $\text{multisup}(X)$ and $\text{multinf}(X)$. It is straightforward to note that these sets consist of pairwise incomparable elements (also called antichains).

Example 1. The simplest example of proper multilattice (i.e. one which is not a lattice) is called $M6$ and is shown in Fig. 1.

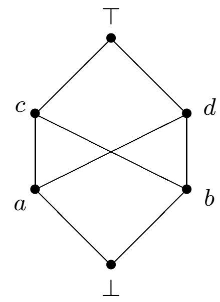

Fig. 1. The multilattice $M6$

An arbitrary complete multilatice needs not have nice computational properties. As an example of counter-intuitive behaviour, simply note that an upper bound of a set X needs not be greater than any minimal upper bound (multi-supremum); such a condition (and its dual, concerning lower bounds and multi-infima) has to be explicitly required.

The fulfilment of this condition is called *coherence*, and is formally introduced in the following definition, where we use the Egli-Milner pre-ordering relation, i.e., $X \sqsubseteq_{EM} Y$ if and only if for every $y \in Y$ there exists $x \in X$ such that $x \preceq y$ and for every $x \in X$ there exists $y \in Y$ such that $x \preceq y$.

Definition 2. *A complete multilattice M is said to be* coherent *if the following pair of inequations hold for all $X \subseteq M$:*

$$LB(X) \sqsubseteq_{EM} \text{multinf}(X); \qquad \text{multisup}(X) \sqsubseteq_{EM} UB(X)$$

Coherence together with the non-existence of infinite antichains (so that the sets multisup(X) and multinf(X) are always finite) have been shown to be useful conditions when working with multilattices. Under these hypotheses, the following important result was obtained in [11]:

Lemma 1. *Let M be a coherent complete multilattice without infinite antichains, then any chain in M has a supremum and an infimum.*

Now that we have given the basic results for a multilattice, we turn our attention to preliminary results for functions defined on a multilattice.

The definitions of isotone and inflationary function are the standard ones also in the framework of multilattices. We recall these definitions below:

Definition 3. *Let $f: M \longrightarrow M$ be a function on a multilattice, then:*

- *f is* isotone *if and only if for every $x, y \in M$ such that $x \preceq y$ we have that $f(x) \preceq f(y)$.*
- *f is* inflationary *if and only if $x \preceq f(x)$ for every $x \in M$*

For isotone and inflationary functions on a multilattice we have the following result concerning fixed points, introduced in [13]:

Theorem 1. *Let M be a coherent complete multilattice without antichains, let $f: M \longrightarrow M$ be an isotone and inflationary mapping on a multilattice, then its set of fixed points is non-empty and has a minimum element.*

As stated in the introduction, the main theoretical problem in this paper is to extend the previous theorem to the framework of multiple-valued functions.

3 Multi-valued Functions and Orbits on Multilattices

In this section we will recall some important results included in [12] about how to reach minimal fixed points of multi-valued functions. But another important point is the existence of this minimal fixed points. Therefore, we will search sufficient conditions to ensure the existence of these points.

Firstly we need recall some preliminary definitions.

Definition 4. *Given a multilattice $(M, \leq)$, by a* multi-valued function *we mean a function $f: M \longrightarrow 2^M$ (we do not require that $f(x) \neq \varnothing$ for every $x \in M$).*

We say that $x \in M$ is a fixed point *of f if and only if $x \in f(x)$.*

Although there exist different definitions of orders in 2^M, we will consider in this paper just the Smyth pre-ordering among sets, and we will write $X \sqsubseteq_S Y$ if and only if for every $y \in Y$ there exists $x \in X$ such that $x \leq y$. This pre-order is used to define the isotonicity and inflation for multi-valued functions.

Definition 5. *Given a multilattice $(M, \leq)$, a multi-valued function $f: M \to 2^M$ it is called:*

- Isotone *if and only if $x \leq y$ implies $f(x) \sqsubseteq_S f(y)$, for all $x, y \in M$.*
- Inflationary *if and only if $\{x\} \sqsubseteq_S f(x)$ for every $x \in M$.*

The concept of orbit has proven to be an important tool for studying reachability of minimal fixed points, see [18].

Definition 6. *Let $f: M \longrightarrow 2^M$ be a multi-valued function, an* orbit of f *is a transfinite sequence $(x_i)_{i \in I}$ of elements $x_i \in M$ where the cardinality of M is less than the cardinality of I ($|M| < |I|$) and:*

$$\begin{aligned} x_0 &= \bot \\ x_{i+1} &\in f(x_i) \\ x_\alpha &\in \operatorname{multisup}\{x_i \mid i < \alpha\}, \text{ for limit ordinals } \alpha \end{aligned}$$

As $f(x_i)$ is a set we have many possible choices for x_{i+1} so we have many possible orbits. Note the following straightforward consequences of the definition:

1. In an orbit, we have $f(x_i) \neq \varnothing$ for every $i \in I$.
2. If $(x_i)_{i \in I}$ is an orbit of f and there exists $k \in I$ such that $x_k = x_{k+1}$, then x_k is a fixed point of f.

3. Any increasing orbit eventually reaches a fixed point (this follows from the inequality $|M| < |I|$).

From the third point above, if we can show the existence of such orbits, then we ensure the existence of fixed points.

To begin with, if f is inflationary, any orbit $(x_i)_{i\in I}$ is increasing, for successor ordinals the inequality $\{x_i\} \sqsubseteq_S f(x_i)$ follows by inflation, hence $x_i \preceq x_{i+1}$. The definition for limit ordinals, directly implies that it is greater than any of its predecessors.

Furthermore, any orbit converges to a fixed point of f. This follows directly, since every transfinite increasing sequence is eventually stationary, and an ordinal α such that $x_\alpha = x_{\alpha+1} \in f(x_\alpha)$ is a fixed point.

Propositions 1 and 2 below were introduced in [12] and show conditions under which any minimal fixed point is attained by means of an orbit:

Proposition 1. *For an inflationary and isotone multi-valued function f we have that: for any minimal fixed point there is an orbit converging to it.*

Proposition 2. *If a multi-valued function f is inflationary, isotone and sup-preserving, then at most countably many steps are necessary to reach a minimal fixed point (provided that some exists).*

In order to find conditions to the existence of minimal fixed points of multi-valued functions we will follow the usual practice of considering the sets of pre-fixed points and post-fixed points:

$$\Phi(f) = \{x \in M \mid f(x) \sqsubseteq_S \{x\}\}$$
$$\Psi(f) = \{x \in M \mid \{x\} \sqsubseteq_S f(x)\}$$

Note that, $\Psi(f)$ is always nonempty, since $\bot \in \Psi(f)$. However, this does not hold for $\Phi(f)$, since it is not always the case that $\top \in \Phi(f)$. Actually, it is easy to see that $\top \in \Phi(f)$ if and only if $f(\top) \neq \varnothing$ (but recall that $f(\top)$ can be empty). For a general element x, the previous equivalence does not hold, but only one implication: if $f(x) = \varnothing$ then $x \notin \Phi(f)$ so if $x \in \Phi(f)$ then $f(x) \neq \varnothing$.

The following general result gives us a characterisation of the fixed point of a multi-valued function in terms of $\Phi(f)$:

Proposition 3. *Let M be a multilattice and $f\colon M \longrightarrow 2^M$ an inflationary multi-valued function. Then $x \in \Phi(f)$ if and only if x is a fixed point of f.*

Proof. Let $x \in \Phi(f)$ then, as f is inflationary, we have that $\{x\} \sqsubseteq_S f(x) \sqsubseteq_S \{x\}$. Hence for $x \in \{x\}$ we have that there exists $y \in f(x)$ such that $x \preceq y \preceq x$, so $x = y \in f(x)$. The other implication holds trivially. □

It is not difficult to find examples which show that the inflationary requirement is essential.

Note that by the proposition above the existence of minimal fixed points of an inflationary multi-valued function is equivalent to the existence of minimal elements of $\Phi(f)$, therefore we will look for conditions to ensure the existence of minimal fixed points of $\Phi(f)$.

The previous proposition also holds when f is isotone, as a result under either isotone or inflationary f all the minimal elements of $\Phi(f)$ are minimal fixed points. This is established in the following theorem.

Theorem 2. *Let $f\colon M \longrightarrow 2^M$ be a isotone or inflationary multi-valued function. If $\Phi(f)$ has minimal elements then these minimal elements are minimal fixed points of f.*

Proof. Let f be isotone, and y a minimal element of $\Phi(f)$, so $\varnothing \neq f(y) \sqsubseteq_S \{y\}$ and there exists $y' \in f(y)$ such that $y' \preceq y$.

As f is isotone we have that $f(y') \sqsubseteq_S f(y)$, hence, since $y' \in f(y)$ there exists $y'' \in f(y')$ with $y'' \preceq y'$. Therefore, $f(y') \sqsubseteq_S \{y'\}$ and $y' \in \Phi(f)$ and $y' \preceq y$ but y is minimal in $\Phi(f)$, so $y = y' \in f(y)$ and y is a fixed point of f.

Let us see now that y is a minimal fixed point. Assume that x is a fixed point of f with $x \preceq y$. As, x is a fixed point we have that $x \in \Phi(f)$ but then, by the minimality of y in $\Phi(f)$, we would have $x = y$ and y is a minimal fixed point as well.

The case of f inflationary follows easily from Proposition 3. □

In order to give some conditions to ensure the existence of minimal elements of $\Phi(f)$, for multi-valued function on a multilattice, we will consider some kind of 'continuity' in our multi-valued functions. This continuity is understood in the sense of preservation of suprema and infima; but, obviously, we have to state formally what this preservation is meant since in complete multilattices we only have for granted the existence of *sets of multi-infima and sets of multi-suprema.*

In this context, it is convenient again to rely on coherent complete multilattices M without infinite antichains so that, at least, we have the existence of suprema and infima of chains.

Definition 7. *A multi-valued function $f\colon M \longrightarrow 2^M$ is said to be* sup-preserving *if and only if for every chain*[1] $X = (x_i)_{i\in I}$ *we have that:*

$$f(\sup\{x_i \mid i \in I\}) = \{y \mid \textit{there are } y_i \in f(x_i) \textit{ s.t. } y \in \text{multisup}\{y_i \mid i \in I\}\}$$

A multi-valued function $f\colon M \longrightarrow 2^M$ is inf-preserving *if and only if for every chain $X = (x_i)_{i\in I}$ we have that:*

$$f(\inf\{x_i \mid i \in I\}) = \{y \mid \textit{there are } y_i \in f(x_i) \textit{ s.t. } y \in \text{multinf}\{y_i \mid i \in I\}\}$$

The following theorem states that the property of being inf-preserving is a sufficient condition to ensure the existence of minimal fixed points of a multi-valued function.

Theorem 3. *Let $f\colon M \longrightarrow 2^M$ be an inf-preserving multi-valued function with $\Phi(f) \neq \varnothing$, then $\Phi(f)$ has minimal elements.*

[1] A chain X is a totally ordered subset and, for convenience, will be denoted as an indexed set $(x_i)_{i\in I}$.

Proof. We will apply Zorn's lemma, and prove that every chain of elements of $\Phi(f)$ has infimum in $\Phi(f)$.

By hypothesis $\Phi(f) \neq \varnothing$. Let $(x_i)_{i \in I}$ be a chain of elements of $\Phi(f)$ and consider $x = \inf\{x_i \mid i \in I\}$ (which exists by Lemma 1).

In order to prove that $x \in \Phi(f)$, we will prove the existence of a particular element of $f(x)$ which is smaller that x.

Firstly, as $x_i \in \Phi(f)$ we have that for all $i \in I$ there exists $y_i \in f(x_i)$ such that $y_i \preceq x_i$. Now, consider an element $y \in \text{multinf}\{y_i \mid i \in I\}$. It is straightforward to note that the inequality $y \preceq \inf\{x_i \mid i \in I\} = x$ holds. Now, as f is inf-preserving, we know that

$$f(x) = \{z \mid \text{there are } y_i \in f(x_i) \text{ s.t. } z \in \text{multinf}\{y_i \mid i \in I\}\}$$

hence, we have $y \in f(x)$ and, consequently $f(x) \sqsubseteq_S \{x\}$. We have proved that $\Phi(f)$ is closed for the infima of chains and, in particular, by Zorn's lemma, $\Phi(f)$ has minimal elements. □

Note that it is easy to check that an inf-preserving function is isotone, thus by a combination of Theorems 2 and 3 we obtain the existence of minimal fixed points of f.

It is worth to note that this result does not apply directly to the context of fuzzy logic programs on a multilattice, since minimal fixed-points are known to exist under the only assumptions of coherence and absence of infinite antichains of the underlying multilattice.

However, the obtained result follows the line of several versions of fixed point theorems for multi-valued functions on a lattice already present in the literature [14,15,16,17,18]. It is remarkable that most of these results were developed to be used in the context of the study of Nash equilibria of supermodular games, but extending the study in this direction is out of the scope of this paper.

4 On Fuzzy Logic Programs on a Multilattice

The previous results will be applied to the particular case of the immediate consequences operator for logic programs on a multilattice, as defined in [13].

To begin with we will recall the definition of the fuzzy logic programs based on a multilattice:

Definition 8. *A* logic program based on a multilattice $(M, \preceq)$ *is a set* $\mathbb{P}$ *of rules of the form* $A \leftarrow \mathcal{B}$ *such that:*

- A *is a propositional symbol, and*
- $\mathcal{B}$ *is a formula built from propositional symbols and elements of* M *by using isotone operators.*

In general, non-atomic formulae will be represented by $\mathcal{B} = @(B_1, \dots, B_n)$ *where* $@$ *denotes the composition of the isotone operators involved in the construction of* $\mathcal{B}$*, and* B_i *are either propositional symbols or elementos of* M*.*

The definition of interpretation and model of a program is given as follows:

Definition 9

- *An interpretation is a mapping I from the set of propositional symbols to M.*
- *We say that I satisfies a rule $A \leftarrow \mathcal{B}$ if and only if $\hat{I}(\mathcal{B}) \preceq I(A)$, where $\hat{I}$ is the homomorphic extension*[2] *of I to the set of all formulae.*
- *An interpretation I is said to be a* model *of a program $\mathbb{P}$ iff all rules in $\mathbb{P}$ are satisfied by I.*

A fixed point semantics was given by means of the following consequences operator.

Definition 10. *Consider a fuzzy logic program $\mathbb{P}$ based on a multilattice, an interpretation I, and a propositional symbol A; the* immediate consequences operator *is defined as follows:*

$$T_{\mathbb{P}}(I)(A) = \text{multisup}\left(\{I(A)\} \cup \{\hat{I}(\mathcal{B}) \mid A \leftarrow \mathcal{B} \in \mathbb{P}\}\right)$$

It is easy to see by the very definition that the immediate consequences operator is an inflationary multi-valued function defined on the set of interpretation of the program $\mathbb{P}$, which is a multilattice. Moreover, models of a program $\mathbb{P}$ are characterized as follows.

Proposition 4 (see [11]). *An interpretation I is a model of a program if and only if $I(A) \in T_{\mathbb{P}}(I)(A)$ for all propositional symbol A.*

The requirement that M is a coherent multilattice without infinite antichains was imposed in [11] in order to prove the existence of minimal fixed points. Then, a straightforward application of Proposition 2 generated the following result:

Theorem 4 (see [12]). *If $T_{\mathbb{P}}$ is sup-preserving, then ω steps are sufficient to reach a minimal model.*

In the rest of the section, we will concentrate on the condition of $T_{\mathbb{P}}$ being sup-preserving. To begin with, let us show that some part of the condition is always fulfilled:

Lemma 2. *Let $\{I_i\}_{i\in\Lambda}$ be a chain, then the following inequality holds*

$$\left\{J \mid \text{ there are } J_i \in T_{\mathbb{P}}(I_i) \text{ with } J \in \text{multisup}_{i\in\Lambda}\{J_i\}\right\} \sqsubseteq_S T_{\mathbb{P}}(\text{sup}_{i\in\Lambda}\{I_i\})$$

Proof. Consider $I = \text{sup}_{i\in\Lambda}\{I_i\}$ and $K \in T_{\mathbb{P}}(I)$, we have that $I_i \preceq I$ for every $i \in \Lambda$; as $T_{\mathbb{P}}$ is isotone we have that $T_{\mathbb{P}}(I_i) \sqsubseteq_S T_{\mathbb{P}}(I)$ then for this K there are $J_i \in T_{\mathbb{P}}(I_i)$ such that $J_i \preceq K$ for every $i \in \Lambda$. Therefore, $K \in UB\{J_i\}$ hence by coherence we have that there is $L \in \text{multisup}\{J_i\}$ such that $L \preceq K$ and by construction $L \in \{J \mid \text{ there are } J_i \in T_{\mathbb{P}}(I_i) \text{ with } J \in \text{multisup}\{J_i\}\}$ □

[2] The homomorphic extension $\hat{I}$ of I applied to a non-atomic formula $@(B_1, \dots, B_n)$, is defined as follows: $\hat{I}(@(B_1, \dots, B_n)) = @(I(B_1), \dots, I(B_n))$.

If we want to get the other inequality we need to assume that $T_{\mathbb{P}}(I)(A)$ is a singleton for all I and A (which we will call that $T_{\mathbb{P}}$ is a singleton) as the next theorem shows:

Theorem 5. *If $T_{\mathbb{P}}$ is a singleton for every interpretation I and the operators of the body of $\mathbb{P}$ are sup-preserving*[3]*, then $T_{\mathbb{P}}$ is sup-preserving.*

Proof. We have to prove that for every chain of interpretations $\{I_i\}_{i\in\Lambda}$ we have that

$$T_{\mathbb{P}}(\sup_{i\in\Lambda}\{I_i\}) = \{J \mid \text{ there are } J_i \in T_{\mathbb{P}}(I_i) \text{ with } J \in \text{multisup}_{i\in\Lambda}\{J_i\}\} \quad (1)$$

First of all, as $T_{\mathbb{P}}$ is a singleton for every interpretation we have that the left hand side of equality (1) is a singleton, so we have to see that the right part is a singleton too. By hypothesis, we have that $T_{\mathbb{P}}(I_i)$ is a singleton, so there is only one possible choice of J_i. Moreover, $T_{\mathbb{P}}$ is Smyth isotone, this, together with that $T_{\mathbb{P}}(I_i)$ are singletons lead us to $\{T_{\mathbb{P}}(I_i)\}_{i\in\Lambda}$ being a chain, so it has a supremum, namely J, and the right part of (1) is also a singleton. Therefore we have to prove that

$$T_{\mathbb{P}}(\sup_{i\in\Lambda}\{I_i\}) = \{J\}$$

Given $I = \sup_{i\in\Lambda}\{I_i\}$, by Lemma 2, we have that $\{J\} \subseteq T_{\mathbb{P}}(I)$ since both are singletons. To prove the other inequality we will prove that for every proporsitional symbol, A, we have that the element in $T_{\mathbb{P}}(I)(A)$ is less than or equal to $J(A)$, that $T_{\mathbb{P}}(I)(A) \preceq J(A)$.

By definition we have that[4]

$$T_{\mathbb{P}}(I)(A) = \sup\{I(A) \cup \{\hat{I}(\mathcal{B}) \text{ with } A \leftarrow \mathcal{B} \in \mathbb{P}\}\}$$

and we have that $J(A) = \sup\{J_i(A)\}$, where

$$J_i(A) = T_{\mathbb{P}}(I_i)(A) = \sup\{I_i(A) \cup \{I_i(\mathcal{B}) \text{ with } A \leftarrow \mathcal{B} \in \mathbb{P}\}\}$$

so $I_i(A) \preceq J(A)$ for every $i \in \Lambda$ and therefore $I(A) \preceq J(A)$

Now we will see that $I(\mathcal{B}) \preceq J(A)$ for every $A \leftarrow \mathcal{B} \in \mathbb{P}$. If $\mathcal{B}$ is a fact or is a propositional symbol then the inequality is trivial. Let us suppose that $\mathcal{B}$ is of the form $@[B_1, B_2]$, the case of n propositional symbols is proved in a similar way. We have that, $\hat{I}_i(\mathcal{B}) \preceq J(A)$ for every $i \in \Lambda$, so $@[I_i(B_1), I_i(B_2)]) \preceq J(A)$ for every $i \in \Lambda$, therefore we have that $\sup_{i\in\Lambda}\{@[I_i(B_1), I_i(B_2)]\} \preceq J(A)$. On the other hand, we have that: $\hat{I}(\mathcal{B}) = @[I(B_1), I(B_2)]$.

As @ is sup-preserving by hypothesis we have that:

$$\begin{aligned} @[I(B_1), I(B_2)] &= \sup_{j\in\Lambda}\{@[I_j(B_1), \sup_{l\in\Lambda}\{I_l\}(B_2)]\}\} \\ &= \sup_{j\in\Lambda}\{\sup_{l\in\Lambda}\{@[I_j(B_1), I_l(B_2)]\}\} \end{aligned}$$

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[3] This is Definition 7 for single-valued functions. That is $f(\sup_{i\in\Lambda}\{x_i\}) = \sup_{i\in\Lambda}\{f(x_i)\}$ for all chain $(x_i)_{i\in\Lambda}$.

[4] In the definition of $T_{\mathbb{P}}$ it is multisup instead of sup but as $T_{\mathbb{P}}$ is a singleton for all I and A we will write, abusing the notation, sup.

Now, as $\{I_i\}_{i\in\Lambda}$ is a chain we can suppose that $I_l \preceq I_j$, hence

$$\begin{aligned}\sup_{j\in\Lambda}\{\sup_{l\in\Lambda}\{@[I_j(B_1), I_l(B_2)]\}\} &\preceq \sup_{j\in\Lambda}\{\sup_{l\in\Lambda}\{@[I_j(B_1), I_j(B_2)]\}\} \\ &= \sup_{j\in\Lambda}\{@[I_j(B_1), I_j(B_2)]\}\end{aligned}$$

and we have that $\hat{I}(\mathcal{B}) \preceq J(A)$, so we have proved that for every A:

$$T_{\mathbb{P}}(I)(A) = \sup\{I(A) \cup \{\hat{I}(\mathcal{B}) \text{ with } A \leftarrow \mathcal{B} \in \mathbb{P}\}\} \preceq J(A) \qquad \square$$

Remark 1. Both conditions are necessary in the Theorem, as we can see in the following examples:

- Let us consider in $M6$ the program $A \leftarrow B$ (there is no conjunctors so they are sup-preserving), and the interpretations $I_1(A) = b$, $I_1(B) = a$ and $I_2(A) = c$, $I_2(B) = a$ we have that $I_1 \preceq I_2$ and that $T_{\mathbb{P}}(I_1)(A) = \{c, d\}$ ($T_{\mathbb{P}}$ is not a singleton) and $T_{\mathbb{P}}(I_2)(A) = \{c\}$.

  If $T_{\mathbb{P}}$ is sup-preserving, then we would have that

  $$\begin{aligned}T_{\mathbb{P}}(I_2)(B) &= T_{\mathbb{P}}(\sup\{I_1, I_2\})(B) \\ &= \{y \mid \text{ there are } y_i \in T_{\mathbb{P}}(I_i)(B) \text{ with } y \in \text{multisup}\{y_1, y_2\}\}\end{aligned}$$

  but $T_{\mathbb{P}}(I_2)(B) = \{c\}$ while the right part of the equality is $\{c, \top\}$.
- In the multilattice of Figure 2 let us consider the program with only one rule $A \leftarrow B * C$, where $*$ is commutative and defined as follows:

  $$\begin{array}{llll} x * x = x; & \bot * x = \bot; & \top * x = \top \text{ if } x \neq \bot; & c_i * c_j = c_{min(i,j)} \\ c * c_i = c_i; & c * d = \top; & d * x = x \text{ if } x \neq c & \end{array}$$

  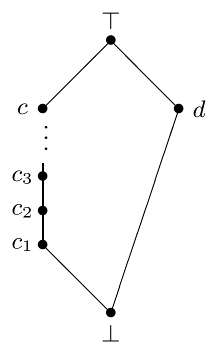

  Fig. 2.

  where x is an element of the multilattice of Figure 2. We have that $*$ is not sup-preserving because: $\sup\{c_i\} * d = c * d = \top \neq c = \sup\{c_i\} = \sup\{c_i * d\}$. However, it is easy to see that $T_{\mathbb{P}}$ is a singleton (we are in a lattice so multisup turns out to be sup). Now, if we consider the interpretations $\{I_i\}_{i\in\mathbb{N}}$ defined as $I_i(A) = c$; $I_i(B) = c_i$; $I_i(C) = d$ we have that $\{I_i\}_{i\in\mathbb{N}}$ is a chain whose supremum is the interpretation I defined as $I(A) = c$; $I(B) = c$; $I(C) = d$.

If $T_{\mathbb{P}}$ were sup-preserving then we would have that:

$$\begin{aligned} T_{\mathbb{P}}(I)(A) &= T_{\mathbb{P}}(\sup\{I_i\}_{i\in\mathbb{N}})(A) \\ &= \{y \mid \text{ there are } y_i \in T_{\mathbb{P}}(I_i)(A) \text{ with } y \in \text{multisup}\{y_i\}_{i\in\mathbb{N}}\} \end{aligned}$$

but $T_{\mathbb{P}}(I)(A) = \{\top\}$ while $T_{\mathbb{P}}(I_i)(A) = \{c\}$ for every $i \in \mathbb{N}$ so the right part of the equality is $\{c\}$. Thus, $T_{\mathbb{P}}$ is a singleton for every interpretation but not sup-preserving.

5 Conclusions

We have presented a prospective study of the theory of fixed points of multiple-valued functions defined on a multilattice, continuing the study of computational properties of multilattices initiated in [11, 13].

Some general results have been obtained regarding the existence of minimal fixed points for multiple-valued functions as well as their reachability in at most countably many steps by means of an iterative procedure finishing with the presentation of some conditions which ensure the hypotheses for that reachibility in countably many steps.

As an application of this theoretical result, we have shown that when the immediate consequences operator of a fuzzy logic program is sup-preserving (in the sense formally given above), then there is no need of transfinite iteration in order to attain any minimal fixed point.

The starting point has been to consider the different versions of fixed point theorems for multi-valued functions on a lattice already present in the literature [14, 15, 16, 17, 18]. It is remarkable that most of these results were developed to be used in the context of the study of Nash equilibria of supermodular games.

References

1. Damásio, C., Pereira, L.M.: Monotonic and residuated logic programs. In: Benferhat, S., Besnard, P. (eds.) ECSQARU 2001. LNCS (LNAI), vol. 2143, pp. 748–759. Springer, Heidelberg (2001)
2. Vojtáš, P.: Fuzzy logic programming. Fuzzy sets and systems 124(3), 361–370 (2001)
3. Medina, J., Ojeda-Aciego, M.: Multi-adjoint logic programming with continuous semantics. In: Eiter, T., Faber, W., Truszczyński, M. (eds.) LPNMR 2001. LNCS (LNAI), vol. 2173, pp. 351–364. Springer, Heidelberg (2001)
4. Fitting, M.: Bilattices and the semantics of logic programming. Journal of Logic Programming 11, 91–116 (1991)
5. Loyer, Y., Straccia, U.: Epistemic foundation of the well-founded semantics over bilattices. In: Fiala, J., Koubek, V., Kratochvíl, J. (eds.) MFCS 2004. LNCS, vol. 3153, pp. 513–524. Springer, Heidelberg (2004)
6. Lakshmanan, L.V.S., Sadri, F.: On a theory of probabilistic deductive databases. Theory and Practice of Logic Programming 1(1), 5–42 (2001)
7. Rounds, W., Zhang, G.-Q.: Clausal logic and logic programming in algebraic domains. Information and Computation 171, 183–200 (2001)

8. Benado, M.: Les ensembles partiellement ordonnés et le théorème de raffinement de Schreier, II. Théorie des multistructures. Czechoslovak Mathematical Journal 5(80), 308–344 (1955)
9. Hansen, D.: An axiomatic characterization of multilattices. Discrete Mathematics 1, 99–101 (1981)
10. Martínez, J., Gutiérrez, G., de Guzmán, I., Cordero, P.: Generalizations of lattices via non-deterministic operators. Discrete Mathematics 295, 107–141 (2005)
11. Medina, J., Ojeda-Aciego, M., Ruiz-Calviño, J.: Multi-lattices as a basis for generalized fuzzy logic programming. In: Bloch, I., Petrosino, A., Tettamanzi, A.G.B. (eds.) WILF 2005. LNCS (LNAI), vol. 3849, pp. 61–70. Springer, Heidelberg (2006)
12. Medina, J., Ojeda-Aciego, M., Ruiz-Calviño, J.: A fixed-point theorem for multi-valued functions with application to multilattice-based logic programming. Lect. Notes in Computer Science, vol. 4578, pp. 37–44 (2007)
13. Medina, J., Ojeda-Aciego, M., Ruiz-Calviño, J.: Fuzzy logic programming via multilattices. Fuzzy Sets and Systems 158(6), 674–688 (2007)
14. d'Orey, V.: Fixed point theorems for correspondences with values in a partially ordered set and extended supermodular games. Journal of Mathematical Economics 25, 345–354 (1996)
15. Echenique, F.: A short and constructive proof of Tarski's fixed-point theorem. International Journal of Game Theory 33, 215–218 (2005)
16. Stouti, A.: A generalized Amman's fixed point theorem and its application to Nash equilibrium. Acta Mathematica Academiae Paedagogicae Nyíregyháziensis 21, 107–112 (2005)
17. Zhou, L.: The set of Nash equilibria of a supermodular game is a complete lattice. Games and economic behavior 7, 295–300 (1994)
18. Khamsi, M.A., Misane, D.: Fixed point theorems in logic programming. Annals of Mathematics and Artificial Intelligence 21, 231–243 (1997)