On Galois Connections and Soft Computing*

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Abstract. After recalling the different interpretations usually assigned to the term Galois connection, both in the crisp and in the fuzzy case, we survey on several of their applications in Computer Science and, specifically, in Soft Computing.

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1 Introduction

Galois connections are ubiquitous; together with adjunctions, their close relatives, occur in a number of research areas, ranging from the most theoretical to the most applied. In a rather poetic tone, the preface of [8] reads, *Galois connections provide the structure-preserving passage between two worlds of our imagination*; and we should add that these two worlds can be so different that the slightest relationship could be seldom ever imagined.

The term *Galois connection* was coined by Øystein Ore [29] (originally, spelled *connexion*) as a general type of correspondence between structures, obviously named after the Galois theory of equations which is an example linking subgroups of automorphisms and subfields. Ore generalized to complete lattices the notion of *polarity*, introduced by Birkhoff [4] several years before, as a fundamental construction which yields from any binary relation two inverse dual isomorphisms. Later, when Kan introduced the *adjoint functors* [19] in a categorical setting, his construction was noticed to greatly resemble that of the Galois connection; actually, in some sense, both notions are interdefinable. The importance of Galois connections/adjunctions quickly increased to an extent that, for instance, the interest of category theorists moved from universal mapping properties and natural transformations to adjointness.

When examining the literature, one can notice a lack of uniformity in the use of the term Galois connection, mainly due to its close relation to adjunctions and that, furthermore, there are two versions of each one. In this paper, after recalling the different interpretations usually assigned to the term Galois connection, both in the crisp and in the fuzzy case, we briefly survey on several of their applications in Computer Science and, specifically, in Soft Computing.

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2 Galois Connections vs Adjunctions

This section is devoted to establish the different definitions of Galois connection, their characterization and the relation among them. For this purpose, the results will be formulated in the most general framework of preordered sets, that are sets endowed with a reflexive and transitive binary relation.

For a preordered set $\mathbb{A} = (A, \leq)$, its *dual* set is $\mathbb{A}^{op} = (A, \geq)$. We denote $a^{\downarrow} = \{x \in A : x \leq a\}$ and $a^{\uparrow} = \{x \in A : x \geq a\}$. Let $f\colon (A, \leq) \to (B, \leq)$ be a map between preordered sets.

- f is *isotone* if $a \leq b$ implies $f(a) \leq f(b)$, for all $a, b \in A$.
- f is *antitone* if $a \leq b$ implies $f(b) \leq f(a)$, for all $a, b \in A$. .

In the particular case in which $A = B$,

- f is *inflationary* (also called extensive) if $a \leq f(a)$ for all $a \in A$.
- f is *deflationary* if $f(a) \leq a$ for all $a \in A$.
- f is *idempotent* if $f \circ f = f$.
- f is a *closure operator* if it is inflationary, isotone and idempotent.
- f is a *kernel operator* if it is deflationary, isotone and idempotent.

For a more detailed study of closure and kernel operators we refer to [7].

Definition 1 (Galois Connections/Adjunctions). *Let $\mathbb{A} = (A, \leq)$ and $\mathbb{B} = (B, \leq)$ be preordered sets, $f : A \to B$ and $g : B \to A$ be two mappings. The pair (f, g) is called a*[1]

- Right Galois Connection between $\mathbb{A}$ and $\mathbb{B}$, *denoted by $(f, g) : \mathbb{A} \leftharpoondown\!\!\rightharpoondown \mathbb{B}$, if*

$$a \leq g(b) \textit{ if only if } b \leq f(a) \quad \textit{for all } a \in A \textit{ and } b \in B.$$

- Left Galois Connection between $\mathbb{A}$ and $\mathbb{B}$, *we write $(f, g) : \mathbb{A} \leftharpoonup\!\!\rightharpoonup \mathbb{B}$, if*

$$g(b) \leq a \textit{ if only if } f(a) \leq b \quad \textit{for all } a \in A \textit{ and } b \in B.$$

- Adjunction between $\mathbb{A}$ and $\mathbb{B}$, *denoted by $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$, if*

$$a \leq g(b) \textit{ if only if } f(a) \leq b \quad \textit{for all } a \in A \textit{ and } b \in B.$$

- Co-Adjunction between $\mathbb{A}$ and $\mathbb{B}$, *denoted by $(f, g) : \mathbb{A} \rightleftharpoons \mathbb{B}$, if*

$$g(b) \leq a \textit{ if only if } b \leq f(a) \quad \textit{for all } a \in A \textit{ and } b \in B.$$

All of the previous notions can be seen in the literature, in fact, one can even find the same term applied to different notions of connection/adjunction. Although it is true that the four definitions are strongly related, they do not have exactly the same properties; hence, it makes sense to specifically describe

[1] The arrow notation for the different versions is taken from [31].

what is the relation between the four notions stated above, together with their corresponding characterizations.

The following theorem states the existence of pairwise biunivocal correspondences between all the notions above. The transition between the two types of adjunctions (connections) relies on using the opposite ordering in *both* preordered sets, whereas the transition between adjunctions to connections and vice versa relies on using the opposite ordering in *just one* of the preordered sets.

Theorem 1. *Let $\mathbb{A} = (A, \leq)$ and $\mathbb{B} = (B, \leq)$ be preordered sets, $f : A \to B$ and $g : B \to A$ be two mappings. Then, the following conditions are equivalent*

1. $(f, g) : \mathbb{A} \leftharpoonup\!\!\rightharpoonup \mathbb{B}$
2. $(f, g) : \mathbb{A}^{op} \leftharpoondown\!\!\rightharpoondown \mathbb{B}^{op}$.
3. $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}^{op}$.
4. $(f, g) : \mathbb{A}^{op} \rightleftharpoons \mathbb{B}$

Observe that, as a direct consequence of this theorem, any property about Galois connections can be extended by duality to the other kind of connections.

Remark 1. Obviously, the ordering in which the mappings appear in the pair determines the kind of Galois connection or adjunction. Thus,

1. (f, g) is a right (left, resp.) Galois connection between $\mathbb{A}$ and $\mathbb{B}$ if and only if (g, f) is a right (left, resp.) Galois connection between $\mathbb{B}$ and $\mathbb{A}$.
2. (f, g) is an adjunction between $\mathbb{A}$ and $\mathbb{B}$ if and only if (g, f) is a co-adjunction between $\mathbb{B}$ and $\mathbb{A}$.

Any preordered set $(A, \leq)$ induces an equivalence relation in A defined as:

$$a_1 \approx a_2 \quad \text{if and only if} \quad a_1 \leq a_2 \text{ and } a_2 \leq a_1 \quad \text{for } a_1, a_2 \in A. \tag{1}$$

The notions of maximum and minimum in a poset can be extended to preordered sets as follows: an element $a \in A$ is a *p-maximum* (*p-minimum* resp.) for a set $X \subseteq A$ if $a \in X$ and $x \leq a$ ($a \leq x$, resp.) for all $x \in X$. The set of p-maximum (p-minimum) of X will be denoted as $\text{p-max}\, X$ ($\text{p-min}\, X$, resp.). Observe that, in a preordered set, different elements can be p-maximum for a set X, but, in this case, $a_1, a_2 \in \text{p-max}\, X$ implies $a_1 \approx a_2$.

Theorem 2. *Let $\mathbb{A} = (A, \leq), \mathbb{B} = (B, \leq)$ be two preordered sets, $f : A \to B$ and $g : B \to A$ be two mappings. The following conditions are equivalent:*

i) $(f, g) : \mathbb{A} \leftharpoonup\!\!\rightharpoonup \mathbb{B}$.
ii) *f and g are antitone maps, and $g \circ f$, $f \circ g$ are inflationary maps.*
iii) $f(a)^{\downarrow} = g^{-1}(a^{\uparrow})$ *for all* $a \in A$.
iv) $g(b)^{\downarrow} = f^{-1}(b^{\uparrow})$ *for all* $b \in B$.
v) *f is antitone and* $g(b) \in \text{p-max}\, f^{-1}(b^{\uparrow})$ *for all* $b \in B$.
vi) *g is antitone and* $f(a) \in \text{p-max}\, g^{-1}(a^{\uparrow})$ *for each* $a \in A$.

Theorem 1 and Theorem 2 provide characterizations for the different Galois connections/adjunctions that are summarized in Table 1.

Table 1. Summary of definitions and equivalent characterizations

Galois Connections	
Right Galois Connections between $\mathbb{A}$ and $\mathbb{B}$ $(f,g)\colon \mathbb{A} \leftharpoonup\!\!\!\rightharpoonup \mathbb{B}$	Left Galois Connections between $\mathbb{A}$ and $\mathbb{B}$ $(f,g)\colon \mathbb{A} \leftharpoondown\!\!\!\rightharpoondown \mathbb{B}$
$b \leq f(a) \Leftrightarrow a \leq g(b)$ for all $a \in A$ and $b \in B$	$f(a) \leq b \Leftrightarrow g(b) \leq a$ for all $a \in A$ and $b \in B$
f and g are antitone and $g \circ f$ and $f \circ g$ are inflationary	f and g are antitone and $g \circ f$ and $f \circ g$ are deflationary
$f(a)^{\downarrow} = g^{-1}(a^{\uparrow})$ for all $a \in A$	$f(a)^{\uparrow} = g^{-1}(a^{\downarrow})$ for all $a \in A$
$g(b)^{\downarrow} = f^{-1}(b^{\uparrow})$ for all $b \in B$	$g(b)^{\uparrow} = f^{-1}(b^{\downarrow})$ for all $b \in B$
f is antitone and $g(b) \in \text{p-max}\, f^{-1}(b^{\uparrow})$ for all $b \in B$	f is antitone and $g(b) \in \text{p-min}\, f^{-1}(b^{\downarrow})$ for all $b \in B$
g is antitone and $f(a) \in \text{p-max}\, g^{-1}(a^{\uparrow})$ for all $a \in A$	g is antitone and $f(a) \in \text{p-min}\, g^{-1}(a^{\downarrow})$ for all $a \in A$
Adjunctions	
Adjunction between $\mathbb{A}$ and $\mathbb{B}$ $(f,g)\colon \mathbb{A} \leftrightarrows \mathbb{B}$	co-Adjunction between $\mathbb{A}$ and $\mathbb{B}$ $(f,g)\colon \mathbb{A} \rightleftarrows \mathbb{B}$
$f(a) \leq b \Leftrightarrow a \leq g(b)$ for all $a \in A$ and $b \in B$	$b \leq f(a) \Leftrightarrow g(b) \leq a$ for all $a \in A$ and $b \in B$
f and g are isotone, $g \circ f$ is inflationary and $f \circ g$ is deflationary	f and g are isotone, $g \circ f$ is deflationary and $f \circ g$ is inflationary
$f(a) \uparrow = g^{-1}(a^{\uparrow})$ for all $a \in A$	$f(a)^{\downarrow} = g^{-1}(a^{\downarrow})$ for all $a \in A$
$g(b)^{\downarrow} = f^{-1}(b^{\downarrow})$ for all $b \in B$	$g(b)^{\uparrow} = f^{-1}(b^{\uparrow})$ for all $b \in B$
f is isotone and $g(b) \in \text{p-max}\, f^{-1}(b^{\downarrow})$ for all $b \in B$	f is isotone and $g(b) \in \text{p-min}\, f^{-1}(b^{\uparrow})$ for all $b \in B$
g isotone and $f(a) \in \text{p-min}\, g^{-1}(a^{\uparrow})$ for all $a \in A$	g is isotone and $f(a) \in \text{p-max}\, g^{-1}(a^{\downarrow})$ for all $a \in A$

Theorem 3. *Let $\mathbb{A} = (A, \leq), \mathbb{B} = (B, \leq)$ be two preordered sets, $f : A \to B$ and $g : B \to A$ be two mappings. If $(f,g)\colon \mathbb{A} \Leftrightarrow \mathbb{B}$, where $\Leftrightarrow \in \{\leftharpoonup\!\!\!\rightharpoonup, \leftharpoondown\!\!\!\rightharpoondown, \leftrightarrows, \rightleftarrows\}$, then, $(f \circ g \circ f)(a) \approx f(a)$, for all $a \in A$, and $(g \circ f \circ g)(b) \approx g(b)$ for all $b \in B$. Moreover,*

1. *If (f,g) is a left and right Galois connection (adjunction and co-adjunction resp.) then $(g \circ f)(a) \approx a$ for all $a \in A$ and $(f \circ g)(b) \approx b$ for all $b \in B$.*
2. *If (f,g) is a (left or right) Galois connection and a (co-) adjunction then $f(a_1) \approx f(a_2)$ for all $a_1, a_2 \in A$ with $a_1 \leq a_2$, and $g(b_1) \approx g(b_2)$ for all $b_1, b_2 \in B$ with $b_1 \leq b_2$.*

Moreover, for any preordered set $\mathbb{A} = (A, \leq)$, the quotient set $A/_{\approx}$ with the relation defined as "$[a_1] \leq [a_2]$ iff $a_1 \leq a_2$" is a partial ordered set (poset) denoted as $\mathbb{A}/_{\approx}$. Theorem 2 allows to translate Galois connections to the quotient posets as follows.

Theorem 4. *Let $\mathbb{A} = (A, \leq)$ and $\mathbb{B} = (B, \leq)$ be two preordered sets and let $\Leftrightarrow \in \{\leftharpoonup\!\!\!\rightharpoonup, \leftharpoondown\!\!\!\rightharpoondown, \leftrightarrows, \rightleftarrows\}$. If $(f,g) : \mathbb{A} \Leftrightarrow \mathbb{B}$ then $(f_{\approx}, g_{\approx})\colon \mathbb{A}/_{\approx} \Leftrightarrow \mathbb{B}/_{\approx}$ where $f_{\approx}([a]) = [f(a)]$ and $g_{\approx}([b]) = [g(b)]$ for all $a \in A$ and $b \in B$.*

Since any equivalence relation is a preorder, Galois connections between two sets endowed with equivalence relations can be considered. But the properties that we obtain on these cases are not significative as the following corollary shows.

Corollary 1. *Let $\mathbb{A} = (A, \leq), \mathbb{B} = (B, \leq)$ be two preordered sets, $f : A \to B$ and $g : B \to A$ be two mappings.*

1. (f, g) *is a left and right Galois connection (adjunction and co-adjunction, resp.) if and only if* $f_\approx$ *and* $g_\approx$ *are inverse mappings* ($g_\approx = f_\approx^{-1}$).
2. (f, g) *is a right and left Galois connection, and adjunction and a co-adjunction if and only if both relations* $\leq$ *are equivalence relations and* $f_\approx$ *and* $g_\approx$ *are inverse mappings.*

Example 1. Consider the set $\mathbb{P}$ of propositional logic programs on a finite set $\{p_1, p_2, \ldots, p_n\}$ of propositional symbols, and $\mathbb{B}^n$ the set of chains of n digits, which will be interpreted as assignments of truth-value to the corresponding propositional symbol. Define $\mathbb{P}_1 \leq \mathbb{P}_2$ if and only if their least models satisfy $lm(\mathbb{P}_1) \subseteq lm(\mathbb{P}_2)$. This relation is reflexive and transitive, but not antisymmetric.

We define a mapping $f\colon \mathbb{P} \to \mathbb{B}^*$ by assigning to each program P the chain of Boolean values corresponding to the least model of P. Conversely, a mapping $g\colon \mathbb{B}^* \to \mathbb{P}$ assigns to each chain in $\mathbb{B}^*$ the (trivial) program consisting just of the fact corresponding to the propositional symbols with truth-value 1. It is not difficult to check that (f, g) is both an adjunction and a co-adjunction. □

In the particular case of posets, from Theorem 3 and Theorem 2 , we obtain:

Theorem 5. *Let* $\mathbb{A} = (A, \leq)$ *and* $\mathbb{B} = (B, \leq)$ *be posets. If* $(f, g) : \mathbb{A} \leftrightarrows \mathbb{B}$ *where* $\leftrightarrows \in \{\leftharpoonup\!\rightharpoonup, \leftharpoondown\!\rightharpoondown, \leftrightharpoons, \rightleftharpoons\}$ *then* $g \circ f \circ g = g$ *and* $f \circ g \circ f = f$ *and therefore*

- *If* $(f, g)\colon \mathbb{A} \leftharpoonup\!\rightharpoonup \mathbb{B}$ *then* $g \circ f$ *and* $f \circ g$ *are closure operators.*
- *If* $(f, g)\colon \mathbb{A} \leftharpoondown\!\rightharpoondown \mathbb{B}$ *then* $g \circ f$ *and* $f \circ g$ *are kernel operators.*
- *If* $(f, g)\colon \mathbb{A} \leftrightharpoons \mathbb{B}$ *then* $g \circ f$ *is a closure operator and* $f \circ g$ *is a kernel operator.*
- *If* $(f, g)\colon \mathbb{A} \rightleftharpoons \mathbb{B}$ *then* $g \circ f$ *is a kernel operator and* $f \circ g$ *is a closure operator.*

3 Fuzzy Galois Connections

Since Lotfi Zadeh introduced Fuzzy Set Theory by considering the unit interval as truthfulness/membership degree structure, a wide range of algebraic structures has been used to this aim. The most usual structure in this context is that of residuated lattice, $\mathbb{L} = (L, \vee, \wedge, 1, 0, \otimes, \to)$, introduced in the 1930s by Dilworth [9] and used in the context of fuzzy logic by Goguen [14]. Thus, an $\mathbb{L}$-fuzzy set is a mapping from the universe set to the membership values structure $X\colon U \to L$ where $X(u)$ means the degree in which x belongs to X. Given X and Y two $\mathbb{L}$-fuzzy sets, X is said to *be included in* Y, denoted as $X \subseteq Y$, if $X(u) \leq Y(u)$ for all $u \in U$. An $\mathbb{L}$-fuzzy binary relation on U is an $\mathbb{L}$-fuzzy subset of $U \times U$, $\rho\colon U \times U \to L$, and it is said to be:

- *reflexive* if $\rho(a, a) = 1$ for all $a \in U$.
- *transitive* if $\rho(a, b) \otimes \rho(b, c) \leq \rho(a, c)$ for all $a, b, c \in U$.
- *symmetric* if $\rho(a, b) = \rho(b, a)$ for all $a, b \in U$.
- *antisymmetric* if $\rho(a, b) = \rho(b, a) = 1$ implies $a = b$, for all $a, b \in U$.

An $\mathbb{L}$-*fuzzy preordered set* is a pair (U, ρ_U) in which ρ_U is a reflexive and transitive $\mathbb{L}$-fuzzy relation. A (crisp) preordering can be given from (U, ρ_U) by considering its 1-cut ($a \leq_U b$ iff $\rho_U(a,b) = 1$). If ρ_U is antisymmetric then (U, ρ_U) is said to be an $\mathbb{L}$-*fuzzy poset* and, in this case, $(U, \leq_U)$ is a (crisp) poset. From now on, when no confusion arises, we will omit the prefix "$\mathbb{L}$-".

For any fuzzy preordered set $\mathbb{U} = (U, \rho_U)$, its *dual* fuzzy preordered set is defined as $\mathbb{U}^{op} = (U, \rho_U^{-1})$ where $\rho_U^{-1}(a,b) = \rho_U(b,a)$ for all $a, b \in U$.

For every element $a \in U$, the fuzzy extension of upper and lower closure $a^{\uparrow}$ and $a^{\downarrow} \colon U \to L$ are given by $a^{\downarrow}(u) = \rho_U(u,a)$ and $a^{\uparrow}(u) = \rho_U(a,u)$ for all $u \in U$.

A *p-maximum* (resp. p-minimum) for a fuzzy set X is an element a satisfying

- $X(a) = 1$ and
- $X \subseteq a^{\downarrow}$ (resp. $X \subseteq a^{\uparrow}$).

Observe that p-minimum and p-maximum elements are not necessarily unique. If the fuzzy relation is antisymmetric, then p-maximum (minimum, resp.) is a singleton.

Let $\mathbb{A} = (A, \rho_A)$ and $\mathbb{B} = (B, \rho_B)$ be fuzzy preordered sets. A mapping $f \colon A \to B$ is said to be

- *isotone* if $\rho_A(a_1, a_2) \leq \rho_B(f(a_1), f(a_2))$ for each $a_1, a_2 \in A$.
- *antitone* if $\rho_A(a_1, a_2) \leq \rho_B(f(a_2), f(a_1))$ for each $a_1, a_2 \in A$.

Moreover, a mapping $f \colon A \to A$ is said to be

- *inflationary* if $a \leq_A f(a)$ for all $a \in A$.
- *deflationary* if $f(a) \leq_A a$ for all $a \in A$.

The definition of *idempotent* mapping, *closure* operator and *kernel* operator follows in the same way as crisp case.

Definition 2 (Fuzzy Galois Connections/Adjunctions). *Let $\mathbb{A} = (A, \rho_A)$, $\mathbb{B} = (B, \rho_B)$ be fuzzy preordered sets, $f : A \to B$ and $g : B \to A$ be two mappings.*

- $(f,g) : \mathbb{A} \leftharpoonup\!\!\rightharpoonup \mathbb{B}$, *if* $\rho_A(a, g(b)) = \rho_B(b, f(a))$ *for all* $a \in A$ *and* $b \in B$.
- $(f,g) : \mathbb{A} \leftharpoondown\!\!\rightharpoondown \mathbb{B}$, *if* $\rho_A(g(b), a) = \rho_B(f(a), b)$ *for all* $a \in A$ *and* $b \in B$.
- $(f,g) : \mathbb{A} \leftrightharpoons \mathbb{B}$, *if* $\rho_A(a, g(b)) = \rho_B(f(a), b)$ *for all* $a \in A$ *and* $b \in B$.
- $(f,g) : \mathbb{A} \rightleftharpoons \mathbb{B}$, *if* $\rho_A(g(b), a) = \rho_B(b, f(a))$ *for all* $a \in A$ *and* $b \in B$.

Remark 2. Theorem 1 can be straightforwardly extended to the fuzzy case and, therefore, any property given for fuzzy Galois connection can be translated to the other cases.

As far as we know, all the fuzzy extension of the notions of Galois connections (and adjunctions) in the literature are given on the particular case of considering the fuzzy poset (with the properties induced by the truthfulness values structure) in which elements are fuzzy sets and the fuzzy relation is related to the assertion "to be a subset of" in a fuzzy setting (see Equation 2).

Example 2. Let $\mathbb{L} = (L, \vee, \wedge, 1, 0, \otimes, \to)$ be a complete residuated lattice and U be the universe set. Then $(\mathbb{L}^U, \cup, \cap, U, \varnothing, \otimes, \to)$ is a complete residuated lattice where $(X \cup Y)(u) = X(u) \vee Y(u)$, $(X \cap Y)(u) = X(u) \wedge Y(u)$, $\varnothing(u) = 0$, $U(u) = 1$, $(X \otimes Y)(u) = X(u) \otimes Y(u)$ and $(X \to Y)(u) = X(u) \to Y(u)$ for all $u \in U$.

The $\mathbb{L}$-fuzzy binary relation "to be a subset of" is defined as follows

$$S(X, Y) = \bigwedge_{u \in U} \big(X(u) \to Y(u)\big) \tag{2}$$

Thus, the inclusion of fuzzy sets $\subseteq$ can be obtained form S as follows: $X \subseteq Y$ if and only if $S(X, Y) = 1$. Observe that $(\mathbb{L}^U, \subseteq)$ is a complete lattice.

Let $f\colon A \to B$ be a mapping, $f\colon \mathbb{L}^A \to \mathbb{L}^B$ and $f^{-1}\colon \mathbb{L}^B \to \mathbb{L}^A$ defined as

$$f(X)(b) = \bigvee\{X(x) \mid f(x) = b\} \qquad f^{-1}(Y)(a) = Y(f(a)) \tag{3}$$

for all $b \in B, a \in A$. Then, $(f, f^{-1})\colon (\mathbb{L}^A, S) \leftrightharpoons (\mathbb{L}^B, S)$.

Notation 1 *From now on, we will use the following notation just introduced in (3): for a mapping $f\colon A \to B$ and a fuzzy subset Y of B, the fuzzy set $f^{-1}(Y)$ is defined as $f^{-1}(Y)(a) = Y(f(a))$, for all $a \in A$.*

Example 3 (See [2]). Let $\mathbb{L} = (L, \vee, \wedge, 1, 0, \otimes, \to)$ be a complete residuated lattice. Let A, B be two sets and $I\colon A \times B \to L$ be an $\mathbb{L}$-fuzzy relation. For all $X \in \mathbb{L}^A$ and $Y \in \mathbb{L}^B$, define

$$X^{\triangle}(b) = \bigwedge_{a \in A} \big(X(a) \to I(a, b)\big) \qquad Y^{\triangledown}(a) = \bigwedge_{b \in B} \big(Y(b) \to I(a, b)\big)$$

The pair $(^{\triangle}, ^{\triangledown})$ is a fuzzy Galois connection between $(\mathbb{L}^A, S_A)$ and $(\mathbb{L}^B, S_B)$. $\square$

Theorem 6. *Let $\mathbb{L} = (L, \vee, \wedge, 1, 0, \otimes, \to)$ be a complete residuated lattice, $\mathbb{A} = (A, \rho_A), \mathbb{B} = (B, \rho_B)$ be $\mathbb{L}$-fuzzy preordered sets and $f : A \to B$ and $g : B \to A$ be two mappings. The following conditions are equivalent:*

- *i)* $(f, g) : \mathbb{A} \leftharpoondown\!\rightharpoonup \mathbb{B}$.
- *ii)* *f and g are antitone maps, and $g \circ f$, $f \circ g$ are inflationary maps.*
- *iii)* $f(a)^{\downarrow} = g^{-1}(a^{\uparrow})$ *for all* $a \in A$.
- *iv)* $g(b)^{\downarrow} = f^{-1}(b^{\uparrow})$ *for all* $b \in B$.
- *v)* *f is antitone and* $g(b) \in \text{p-max}\, f^{-1}(b^{\uparrow})$ *for all* $b \in B$.
- *vi)* *g is antitone and* $f(a) \in \text{p-max}\, g^{-1}(a^{\uparrow})$ *for each* $a \in A$.

Proof. According to Remark 2, it suffices to prove that $i), ii), iii)$ and $v)$ are equivalent. Observe that $f(a)^{\downarrow}(b) = \rho_B(b, f(a))$ and $g^{-1}(a^{\uparrow})(b) = \rho_A(a, g(b))$ by the definition (see Notation 1) then $i)$ and $iii)$ are trivially equivalent.

i) $\Rightarrow$ ii) Let $a \in A$. As ρ_B is reflexive, $1 = \rho_B(f(a), f(a))$ and by hypothesis, $\rho_A(a, g(f(a))) = \rho_B(f(a), f(a)) = 1$, thus, $g \circ f$ is inflationary. Given $a_1, a_2 \in A$, it holds that $\rho_A(a_1, a_2) = \rho_A(a_1, a_2) \otimes 1 = \rho_A(a_1, a_2) \otimes \rho_A(a_2, g(f(a_2)))$. Being ρ_A transitive, $\rho_A(a_1, a_2) \le \rho_A(a_1, g(f(a_2))) = \rho_B(f(a_2), f(a_1))$. That is, f is antitone. Similarly, we prove that $f \circ g$ is inflationary and g is antitone.

ii) $\Rightarrow$ v) f is antitone, by the hypothesis. For all $b \in B$, since $f \circ g$ is inflationary, $1 = \rho_B(b, f(g(b))) = f^{-1}(b^{\uparrow})(g(b))$. On the other hand, since ρ_A is transitive, $g(b)^{\downarrow}(a) = \rho_A(a, g(b)) \leq \rho_A(a, g(f(a))) \otimes \rho_A(g(f(a)), g(b))$. As $g \circ f$ is inflationary and g antitone, $g(b)^{\downarrow}(a) \leq 1 \otimes \rho_A(g(f(a)), g(b)) = \rho_A(g(f(a)), g(b)) \geq \rho_B(b, f(a)) = f^{-1}(b^{\uparrow})(a)$.

v) $\Rightarrow$ i) It is sufficient to prove that $\rho_B(b, f(a) \geq \rho_A(a, g(b))$. Firstly note that $f^{-1}(b^{\uparrow})(g(b)) = 1$ is equivalent to $f \circ g$ being inflationary. Then,

$$\begin{aligned}\rho_B(b, f(a)) &\geq \rho_B(b, f(g(b))) \otimes \rho_B(f(g(b)), f(a)) \\ &= 1 \otimes \rho_B(f(g(b)), f(a)) \geq \rho_A(a, g(b))\end{aligned}$$

□

Remark 2 and Theorem 6 provide similar characterizations for the different fuzzy Galois connections/adjunctions that are summarized in Table 1.

As we have mentioned at the beginning of this section, any fuzzy preordered set $\mathbb{A} = (A, \rho_A)$, defines a (crisp) preordered set $\mathbb{A}_c = (A, \leq_A)$ where $a \leq_A b$ iff $\rho_A(a, b) = 1$. It is also straightforward that the pair $\mathbb{A}/_{\approx} = (A/_{\approx}, \rho_{A_{\approx}})$ where $\approx$ is the equivalence relation given by

$$a \approx b \text{ if and only if } \rho_A(a, b) = \rho_A(b, a) = 1 \tag{4}$$

and $\rho_{A_{\approx}}$ is defined as $\rho_{A_{\approx}}([a], [b]) = \rho_A(a, b)$ is a fuzzy poset. Moreover, any mapping f between fuzzy preordered sets defines a mapping $f_{\approx}$ between the quotient poset in the same way as in Theorem 4.

Theorem 7. *Let $\mathbb{A} = (A, \rho_A)$ and $\mathbb{B} = (B, \rho_B)$ be two fuzzy preordered sets and two mappings $f \colon A \to B$ and $g \colon B \to A$. Then, for $\leftrightarrows \in \{\hookleftarrow, \hookrightarrow, \leftleftarrows, \rightrightarrows\}$,*

1. *$(f, g) : \mathbb{A} \leftrightarrows \mathbb{B}$ implies $(f, g) \colon \mathbb{A}_c \leftrightarrows \mathbb{B}_c$.*
2. *$(f, g) : \mathbb{A} \leftrightarrows \mathbb{B}$ if and only if $(f_{\approx}, g_{\approx}) \colon \mathbb{A}/_{\approx} \leftrightarrows \mathbb{B}/_{\approx}$.*

The following example shows that the converse of item 1 above is not true.

Example 4. Let $\mathbb{L}$ be the (product) residuated lattice $([0, 1], \sup, \inf, 1, 0, \cdot, \to)$. Let $\mathbb{A} = (A, \rho_A)$ such that $A = \{a, b\}$, $\rho_A(a, a) = \rho_A(b, b) = 1$, $\rho_A(a, b) = 0.5$ and $\rho_A(b, a) = 0.2$ and let I be the identity mapping on A. Then $(I, I) \colon \mathbb{A}_c \hookleftarrow \mathbb{A}_c$ but (I, I) is not a Galois connection between $\mathbb{A}$ and $\mathbb{A}$ because $0.5 = \rho_A(a, I(b)) \neq \rho_A(b, I(a)) = 0.2$. □

Observe that, contrariwise to the crisp case, nontrivial fuzzy relations exist that are both equivalencies and partial orders. They are known as fuzzy equalities and are strongly reflexive ($\rho(a, b) = 1$ iff $a = b$), symmetric and transitive fuzzy relations. Obviously, in these cases, antitone and isotone maps coincide. But there exist nontrivial mappings between fuzzy posets (not fuzzy equalities) that are both antitone and isotone as the following example shows.

Example 5. Let $\mathbb{L}$ be the (product) residuated lattice $([0, 1], \sup, \inf, 1, 0, \cdot, \to)$. Consider $\mathbb{A} = (\{a, b, c\}, \rho)$ such that ρ is the $\mathbb{L}$-fuzzy preorder given by the table below, then the map $f\colon A \to A$ with $f(a) = f(c) = b$ and $f(b) = a$ is antitone and isotone.

ρ	a	b	c
a	1	0.5	0.3
b	0.5	1	0.2
c	0.2	0.1	1

□

Theorem 8. *Let* $\mathbb{A} = (A, \rho_A)$ *and* $\mathbb{B} = (B, \rho_B)$ *be fuzzy preordered sets and* $\leftrightarrows \in \{\leftharpoondown\!\!\rightharpoondown, \leftharpoonup\!\!\rightharpoonup, \leftrightharpoons, \rightleftharpoons\}$*. If* $(f, g)\colon \mathbb{A} \leftrightarrows \mathbb{B}$ *then, for all* $a \in A, b \in B$*, the following relations hold* $(f \circ g \circ f)(a) \approx f(a)$ *and* $(g \circ f \circ g)(b) \approx g(b)$*. Moreover,*

- *If* (f, g) *is both left and right Galois connection (resp., adjunction and co-adjunction) then* $(g \circ f)(a) \approx a$ *and* $(f \circ g)(b) \approx b$ *for all* $a \in A$ *and* $b \in B$*.*
- *If* (f, g) *is a (left or right) Galois connection and a (co-) adjunction then, for all* $a_1, a_2 \in A$*,* $\rho_A(a_1, a_2) = 1$ *implies* $f(a_1) \approx f(a_2)$ *and, for all* $b_1, b_2 \in B$*,* $\rho_B(b_1, b_2) = 1$ *implies* $g(b_1) \approx g(b_2)$ *.*

Corollary 2. *Let* $\mathbb{A} = (A, \rho_A)$ *and* $\mathbb{B} = (B, \rho_B)$ *be two fuzzy posets.*

- *If* $(f, g)\colon \mathbb{A} \leftharpoondown\!\!\rightharpoondown \mathbb{B}$ *then* $g \circ f$ *and* $f \circ g$ *are closure operators.*
- *If* $(f, g)\colon \mathbb{A} \leftharpoonup\!\!\rightharpoonup \mathbb{B}$ *then* $g \circ f$ *and* $f \circ g$ *are kernel operators.*
- *If* $(f, g)\colon \mathbb{A} \leftrightharpoons \mathbb{B}$ *then* $g \circ f$ *is closure operator and* $f \circ g$ *is kernel operator.*
- *If* $(f, g)\colon \mathbb{A} \rightleftharpoons \mathbb{B}$ *then* $g \circ f$ *is kernel operator and* $f \circ g$ *is closure operator.*

4 Applications to computer science and soft computing

Out of the plethora of applications of Galois connections that can be found in CS-related research topics, we have selected just a few to comment a little bit about them. Applications range from the mathematics of program construction to data analysis and formal concept analysis.

Program construction. Some authors propose the use of Galois connections (actually, adjunctions), as suitable calculational specifications: in [25] one can see how to simplify and easily handle data structures problems when using Galois connections. On the other hand, [27] considers them as program specifications, helping in the process of deriving and using an algebra of programming. Another common usage is based on that a Galois connection is a correspondence between two complete lattices that consists of an *abstraction* and *concretisation* functions.

Constraint Satisfaction Problems (CSPs). In this context, a well-known Galois connection between sets of relations and sets of (generalized) functions was used in [18] to investigate the algorithmic complexity of CSPs. Continuing this idea, [6] established a link between certain *soft* constraint languages and sets of certain sets determined by algebraic constructs called weighted polymorphisms, resulting in an alternative approach to the study of tractability in valued constraint languages. This approach was recently used in [20] generalizing the notions of polymorphisms and invariant relations to arbitrary categories.

Logic. There are strong links between Galois connections and logic. The first one arose in Lawvere's paper on the foundational value of category theory. This is made explicit in [30] which formally defines a Galois Connection (adjunction) between sets of sentences and classes of semantic structures.

In [17] the Information Logic of Galois Connections (ILGC) is introduced as a convenient tool for approximate reasoning about knowledge. Actually, they use adjunctions introduced as generalized rough approximation operations based on information relations. ILGC turns out to be the usual propositional logic together with two modal connectives. This natural inclusion of modal connectives in the framework, enables the study of such as intuitionistic logic [33, 11].

Mathematical morphology. Galois connections can be seen as well in the mathematical structure of morphological operations, in the research area known as mathematical morphology. In [15] the authors note the duality between the morphological operations of dilation and erosion, and identify them as the components of an adjunction (by the way, making the proper distinction between Galois connections and adjunctions). Originally, mathematical morphology was applied on binary images, but later extended to grayscale by applying techniques from the fuzzy realm, specifically, using fuzzy Galois connections (adjunctions); a survey on classical and fuzzy approaches of mathematical morphology can be found in [28]. Later, in [22] a further generalization based on lattices is given aimed at unifying morphological and fuzzy algebraic systems. The benefits of considering the different links, in terms of Galois connections, that can be established is exploited in [5], clarifying the links between different approaches in the literature and giving conditions for their equivalence. Recently, mathematical morphology is being studied using ideas and results from formal concept analysis [1].

Information processing and data analysis. Another use of Galois connections (adjunctions) can be seen in [12], where Galois connections are generalized from complete lattices to flow algebras (an algebraic structure more general than idempotent semirings, in which distributivity is replaced by monotonicity and the annihilation property is ignored).

Another interesting application can be found in the theory of relational systems: in [16] it was shown that every Galois connection between complete lattices determines an Armstrong system (a closed set of dependencies). More recently, [21] proved the converse, for any given Armstrong system it is possible to define a Galois connection which generates the original Armstrong system.

In [32] the author focuses on the different applications of Galois connections (both types, termed polarities and axialities) to different theories of qualitative data analysis. Specifically, the paper studies some links between axialities (adjunctions) and the area of data analysis.

Formal concept analysis (FCA). The notion of polarity, given by Birkhoff, is precisely that used by Ganter and Wille in order to define the derivation operators in crisp FCA. In this context, it is well-known that the concept lattices induced by Galois connections and by adjunctions are naturally isomorphic. Authors indistinctly use one or the other approach, for instance [26] considers the problem of attribute reduction by using axialities (adjunctions) inspired by the reduction method in rough set theory.

The introduction of fuzzy Galois connections [2] lead to the systematic development of fuzzy FCA. In this framework, the fuzzy concept lattices generated from Galois connections and from adjunction need not be isomorphic. In [3] it is shown that the expected isomorphism depends largely on the way the notion of a complement is defined, the usual based on a residuum w.r.t. 0, is replaced by one based on residua w.r.t. arbitrary truth degrees.

Different extensions of fuzzy FCA consider even more general frameworks, some of them use a sub-interval rather than a precise value of the lattice to refer to what extent an object satisfies an attribute [10, 23]. Usually, these extensions require to provide a sound minimal set of algebraic requirements for interval-valued implications in order to obtain a Galois connection. Anyway, there are approaches in which this vicissitude can be circumvented [24].

5 Conclusions

We have surveyed some applications of Galois connections and adjunctions in Computer Science and Soft Computing. The greater generality of the fuzzy case, which allows to see a single crisp notion from very different perspectives, suggests that it is worth to consider these notions either in a more general setting or by weakening its requirements [13].

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