On the ideal semantics of multilattice-based logic programs

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Abstract

We present several properties of the different semantics of ideals on an ordered multilattice in order to solve theoretical problems arisen from the use of multilattices as underlying set of truth-values for a generalized framework of logic programming.

1 Introduction

In the recent years there has been many studies treating to weaken the structure of the underlying set of truth-values for logic programming. Residuated lattices as set of truth values were proposed by [5] and, simultaneously, multi-adjoint lattices were introduced in [12]. Some papers use more restrictive structures, such as bilattices or trilattices [10], whereas in other papers, more general structures, such as algebraic domains [14], have been proposed. A common feature of all these approaches is that they are based on the structure of complete lattice, which imposes the existence of the least upper bound and greatest lower bound for every subset of the lattice.

Benado [1] and Hansen [7] proposed definitions of a structure so-called multilattice, which dropped the uniqueness condition regarding upper and lower bounds and replaced it by the "existence of *minimal* upper bounds and *maximal* lower bounds". This kind of structure also arises in the research area concerning fuzzy extensions of logic programming: for instance, one of the hypotheses of the main termination result for sorted multi-adjoint logic programs [3] can be weakened only when the underlying set of truth-values is a multi-lattice (the question of providing a counter-example on a lattice remains open). In [11] a first approach to logic programming based on multilattices was presented.

Our aim in this paper is to study the possibility of defining an ideal-based semantics for multilattice-based logic programs. To begin with, there are several proposals for the definition of ideal in a multilattice; the computational capabilities of each of these definitions have to be investigated, and the relationship between the corresponding general and restricted semantics have to be studied, in the style of [9].

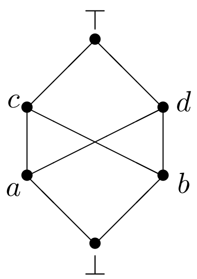
The structure of the paper is the following: in Sections 2 and 3 theoretical results about multilattices, the definition of extended logic programs, and its restricted and fixed-point semantics are recalled. Later in Section 4, the alternative notions of ideals for a multilattice are presented, and ideal-based semantics is introduced in Section 5. then, in Section 6 a comparison between the restricted and the different ideal-based semantics is presented. Finally, in the last section we draw some conclusions and prospects for future work.

2 Preliminary results

Recall that a lattice is a poset such that the set of upper (lower) bounds has a unique minimal (maximal) element, that is, a *minimum* (*maximum*). In a multilattice, this property is relaxed in the sense that minimal elements for the set of upper bounds should exist, but the uniqueness condition is dropped.

Definition 1 *A* complete multilattice *is a partially ordered set, $\langle M, \leq \rangle$, such that for every subset $X \subseteq M$, the set of upper (lower) bounds of X has minimal (maximal) elements, which are called* multi-suprema *(*multi-infima*).*

Example 1 *The first example of multi-lattice that one thinks is the following one which we call $M6$.*



Regarding computational properties of multilattices, it is interesting to impose certain conditions on the sets of upper (lower) bounds of a given set X.

Definition 2 *We say that a multilattice is* consistent *if any upper (lower) bound is greater (less) than a minimal (maximal).*

Another reasonable condition to require on a multilattice is that it should not contain infinite sets of mutually incomparable elements (also called *antichains*). Consistent multilattices without infinite antichains have interesting computational properties, such as the sets of multi-suprema or multi-infima for totally ordered subsets (also called *chains*) always have a supremum and an infimum.

We will assume in the rest of the paper that our underlying multilattices are complete, consistent and without infinite antichains.

3 Extended logic programs

In this section we recall a first approximation of the definition of an extended logic programming paradigm in which the underlying set of truth-values is assumed to have structure of multilattice which was presented in [11]. The proposed schema is an extension of the residuated logic programs of [5].

The definition of logic program is given, as usual, as a set of rules and facts.

Definition 3 *An* extended logic program *is a set $\mathbb{P}$ of rules of the form $A \leftarrow \mathcal{B}$ such that: (1) A is a propositional symbol of Π, and (2) $\mathcal{B}$ is a formula of $\mathfrak{F}$ built from propositional symbols and elements of M by using monotone operators.*

An interpretation is an assignment of truth-values to every propositional symbol in the language.

Definition 4 *An* interpretation *is a mapping $I \colon \Pi \to M$. The set of all interpretations is denoted $\mathcal{I}$.*

Note that by the unique homomorphic extension theorem, any interpretation I can be uniquely extended to the whole set of formulas (the extension will be denoted as $\hat{I}$). The ordering $\leq$ of the truth-values M can be extended point-wise to the set of interpretations as usual.

A rule of an extended logic program is satisfied whenever the truth-value of the head of the rule is greater or equal than the truth-value of its body. Formally:

Definition 5 *Given an interpretation I, a rule $A \leftarrow \mathcal{B}$ is* satisfied *by I iff $\hat{I}(\mathcal{B}) \leq I(A)$. An interpretation I is said to be a* model *of an extended logic program $\mathbb{P}$ iff all rules in $\mathbb{P}$ are satisfied by I, then we write $I \models \mathbb{P}$.*

Every extended program $\mathbb{P}$ has the top interpretation as a model; regarding minimal models, it is possible to prove the following lemma.

Lemma 1 *Given an extended logic program $\mathbb{P}$, there exist minimal models for $\mathbb{P}$.*

An interesting technical problem arises when trying to extend the definition of the immediate consequences operators to the framework of multilattice-based logic programs. Note that all the suprema involved in the usual definition of $T_{\mathbb{P}}$ do exist provided that we are assuming a *complete lattice* structure on the underlying set of truth-values; however, this needs not hold for a *multilattice*.

In order to work this problem out, we consider the following definition:

Definition 6 *Given an extended logic program $\mathbb{P}$, an interpretation I and a propositional symbol A; we can define $T_{\mathbb{P}}(I)(A)$ as:*

$$\text{multisup}\Big(\{I(A)\} \cup \{\hat{I}(\mathcal{B}) \mid A \leftarrow \mathcal{B} \in \mathbb{P}\}\Big)$$

The definition of $T_{\mathbb{P}}$ proposed above generates some coherence problems, in that the resulting 'value' is not an element, but a subset of the multilattice. A possible solution to this problem would be to consider a *choice function* $()^*$ which, given an interpretation, for any propositional symbol A selects an element in $T_{\mathbb{P}}(I)(A)$; this way, $T_{\mathbb{P}}(I)^*$ represents actually an interpretation. In the following lemma, the models of $\mathbb{P}$ are characterised in terms of $T_{\mathbb{P}}$.

Lemma 2 *The following statements are equivalent:*

1. *I is a model of $\mathbb{P}$.*
2. *$T_{\mathbb{P}}(I)(A) = \{I(A)\}$ for all $A \in \Pi$.*
3. *$T_{\mathbb{P}}(I)^* = I$ for all choice function.*
4. *$I \in T_{\mathbb{P}}(I)$, (abusing notation this means that $I(A) \in T_{\mathbb{P}}(I)(A)$ for all $A \in \Pi$).*

Note that item 4 above states that an interpretation I is a model of $\mathbb{P}$ if and only if it is a fixed point of $T_{\mathbb{P}}$, viewed as a non-deterministic operator.

Theorem 1 *Let I be a minimal model of $\mathbb{P}$, then we can construct a Kleene-like sequence of interpretations which converges to I.*

Once we have a semantics for our multilattices we can wonder whether it is possible to define an "ideal"-like semantics, in the style of [9]. The set of ideals of a lattice turns out to have good computational properties, for instance, it forms a complete lattice, we can define a fixed-point semantics for them where the immediate operator of consequences is continuous, and under suitable circumstances $I_r = \sup I_m$ where I_r is the least model of a program computed by its fixed-point semantics and I_m is its least model computed by the ideal fixed-point semantics.

4 Ideals of multilattices

As stated above, we are interested in studying the set of ideals of a multilattice, so that we can define an ideal-semantics as in [9]; however, the definition of ideal in a multilattice is not canonical. For instance, one can find the notion of s-ideals introduced by Rachůnek, or the l-ideals of Burgess, or the m-ideals given by Johnston [13, 8]. In this section, we study the differences between the various definitions and propose some new alternatives.

To begin with, let us recall the definition of ideal of a lattice:

Definition 7 *A nonempty subset D of a lattice L is said to be an ideal of L if it is downward closed and for all $a, b \in D$ we have that $a \vee b \in D$.*

For a multilattice, as stated above, at least the following extensions of the concept of ideal can be found:[1]

Definition 8 *Given a multilattice M and a non-void subset D of M, we say that D is:*

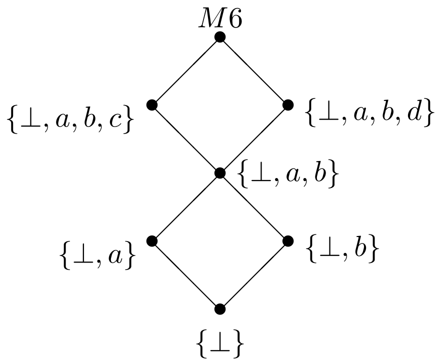
- *An* s-ideal *if and only if it is downward closed and for every $a, b \in D$ we have that $UB(\{a, b\}) \cap D \neq \varnothing$.*
- *An* l-ideal *if and only if for every $a, b \in D$ we have that $LB(UB(\{a, b\})) \subseteq D$.*
- *An* m-ideal *if and only if for every $a, b \in D$ such that* $\sup\{a, b\}$ *exists we have that $LB(\sup\{a, b\}) \subseteq D$.*

It is not difficult to check that, in the particular case of a lattice, all definitions above collapse to the usual definition of ideal of a lattice. Moreover, for the general case of a multilattice M, if $\mathfrak{I}_\alpha(M)$ denotes the set of α-ideals, where $\alpha \in \{l, m, s\}$, we clearly have

$$\mathfrak{I}_s(M) \subseteq \mathfrak{I}_l(M) \subseteq \mathfrak{I}_m(M). \qquad (1)$$

[1] We use UB (resp. LB) to denote the set of upper (resp. lower) bounds of a subset.

Example 2 *The lattice of l-ideal and m-ideals of $M6$ is the following:*



Recall the following interesting relationship between sublattices and ideals of a lattice, see [6, 2].

Lemma 3 *Let A be a sublattice of a lattice L, then A is an ideal of L if and only if for every $a \in A$ and $x \in L$ we have that $a \wedge x \in A$.*

In order to obtain a similar result for multilattices, we introduce the definition of submultilattice of a multilattice M.

Definition 9 *Let M be a multilattice and $\varnothing \neq B \subseteq M$, we say that B is a* submultilattice *of M if it inherits the multilattice structure when using the restriction of the multisuprema and multinfima to B.*

Notice that there are two reasonable possibilities of considering the restriction of the operators multisup and multinf:

- A submultilattice B of M is said to be *full* (or f-submultilattice) if for every $a, b \in B$ we have that all the multisuprema and multinfima of $\{a, b\}$ in M are in B.
- A submultilattice B of M is said to be *restricted* (or r-submultilattice) if for every $a, b \in B$ we have that at least one multisupremum and one multinfimum of $\{a, b\}$ in M is in B.

It is obvious that every f-submultilattice is a r-submultilattice, but not vice versa.

The following lemma gives us a relationship between (f- or) r-submultilattices and each type of ideal:

Lemma 4 *Let B be a (f- or) r-submultilattice of multilattice M then, B is an (s- or l- or) m-ideal if an only if for every $b \in B$ and $x \in M$ we have that* $\text{multinf}\{b, x\} \subseteq B$.

Proof. Let B be a r-submultilattice B, for the case of f-submultilattices the proof is equal.

Firstly, we suppose that B is an (s- or l- or) m-ideal then the property is clearly satisfied since every such ideal is downward closed.

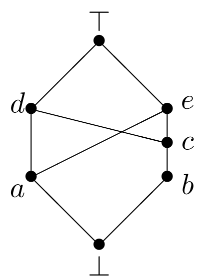
Reciprocally, by the chain of inclusions (1), it is enough to prove that B is an s-ideal, that is, B is downward closed and for every $a, b \in B$ we have that $UB(\{a, b\}) \cap B \neq \varnothing$.

It is straightforward that B is downward closed, since for $b \in B$, and every $x \in M$ such that $x \leq b$ we have $\text{multinf}\{b, x\} = \{x\} \subseteq B$ by our hypothesis.

Now, let $a, b \in B$, as B is a r-submultilattice, we have that at least an element c of $\text{multisup}\{a, b\}$ is in B, hence $c \in UB(\{a, b\}) \cap B \neq \varnothing$. □

From the previous result one could be tempted to say that all three types of ideals for multilattices coincide. This is not true, as the inclusions in (1) are, in general, strict:

- In $M6$ we have that $\{\bot, a, b\}$ is an l-ideal but it is not an s-ideal.
- In the following multi-lattice we have that $\{\bot, a, b\}$ is an m-ideal but it is not an l-ideal.



An interesting consequence of Lemma 4 is that, in difference with what happens with lattices, not every l-ideal or m-ideal is necessarily a (f- or) r-submultilattice, otherwise all three types of ideals would coincide. However, every s-ideal of a consistent multilattice is a r-submultilattice but not necessarily a f-submultilattice.

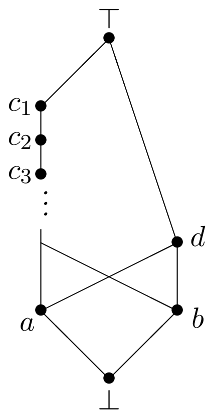
Lemma 5 *Let D be a s-ideal of a consistent multilattice M, then D is a r-submultilattice.*

Proof. Given $a, b \in D$, we must prove that $\text{multisup}\{a,b\} \cap D \neq \varnothing$ and $\text{multinf}\{a,b\} \cap D \neq \varnothing$.

The last one is trivial from the downward closed property. For the other one, as D is a s-ideal, we consider $c \in UB(\{a,b\}) \cap D$ then, from the consistent of M, there exists $d \in \text{multisup}\{a,b\}$ such that $d \leq c$ and, as D is downward closed and $c \in D$, we obtain that $d \in \text{multinf}\{a,b\} \cap D$. □

In the following example we show that the result above does not hold if the initial multilattice is not consistent.

Example 3 *In the figure we have a non-consistent multilattice, the subset $D = \{\bot, a, b, c_1, c_2, c_3, \ldots\}$ is an s-ideal, but we can check easily that $\text{multisup}\{a,b\} \cap D = \varnothing$, hence it is not a submultilattice.*



Example 4 *An s-ideal is not always a f-submultilattice, for example in the (consistent) multilattice $M6$ we have that $\{\bot, a, b, c\}$ is an s-ideal but not a f-submultilattice.*

Another interesting property on the framework of ideals of lattices is the following result, which relates the kernel of a join-homomorphism (the inverse image of $\bot$) with ideals.

Theorem 2 (Birkhoff [2]) *If $\Phi\colon L_1 \longrightarrow L_2$ is an join-homomorphism between lattices then $\ker(\Phi)$ is an ideal.*

In the statement, a join-homomorphism between lattices is an application which preserves joins. The extension to the framework of multilattices is the following:

Definition 10 *Let M_1 and M_2 be multilattices; a mapping $\Phi\colon M_1 \longrightarrow M_2$ is said to be a multisup-homomorphism if for every $B \in M_1$ and $b \in \text{multisup}\{B\}$, we have that $\Phi(b) \in \text{multisup}\{\Phi(B)\}$.*

With these definitions we can translate the theorem above to multilattices.

Theorem 3 *If $\Phi\colon M_1 \longrightarrow M_2$ is a multisup-homomorphism between multilattices, then $\ker(\Phi)$ is an (s- or l- or) m-ideal.*

Proof. By the chains of inclusions (1) we only have to prove the result for m-ideals.

Let us consider $a, b \in \ker(\Phi)$ such that $\sup\{a,b\}$ exists we have to prove that $LB(\sup\{a,b\}) \subseteq \ker(\Phi)$.

Firstly we will prove that $\sup\{a,b\} \in \ker(\Phi)$ and using this result we will prove that $c \in \ker(\Phi)$ for all $c \in LB(\sup\{a,b\})$.

Since $a, b \in \ker(\Phi)$ we have $\Phi(a) = \Phi(b) = \bot$ so $\text{multisup}\{\Phi(a), \Phi(b)\} = \bot$. As Φ is a multisup-homomorphism we have that[2] $\Phi(\sup\{a,b\}) \subseteq \text{multisup}\{\Phi(a), \Phi(b)\} = \bot$ so $\sup\{a,b\} \in \ker(\Phi)$.

Now, consider $c \in LB(\sup\{a,b\})$ we have that

$$\begin{aligned} &\Phi(\text{multisup}\{c, \sup\{a,b\}\}) = \\ &\quad = \Phi(\{\sup\{a,b\}\}) = \{\bot\} \end{aligned}$$

Since Φ is a multisup-homomorphism we obtain that

$$\begin{aligned} \{\bot\} &= \Phi(\text{multisup}\{c, \sup\{a,b\}\}) \subseteq \\ &\subseteq \text{multisup}\{\Phi(c), \Phi(\sup\{a,b\})\} \end{aligned}$$

then

$$\bot \in \text{multisup}\{\Phi(c), \Phi(\sup\{a,b\})\}$$

hence $\Phi(c) = \bot$ and $c \in \ker(\Phi)$. □

Now that we have shown that the proposed definitions of ideals for a multilattice are friendly to most of the properties of ideals of a lattice, let us concentrate now on the algebraic structure of the sets of every type of ideal.

[2]Notice that we write sup instead of multisup since $\sup\{a,b\}$ exists and hence is the only multisup.

5 Ideal semantics

Assume that the set of ideals $\mathfrak{I}_\alpha(M)$ is ordered by set-inclusion, then it is easy to check that $\mathfrak{I}_l(M)$ and $\mathfrak{I}_m(M)$ are complete lattices. On the other hand, $\mathfrak{I}_s(M)$ is a complete multilattice (provided that M is complete and infinite antichains do not exist) [8].

The structure of complete lattice of some sets of ideals enables us to provide an ideal-based fixpoint semantics for extended programs in terms of the Knaster-Tarski theorem.

In this section we provide an ideal-based semantics for the extended programs. To begin with, we will consider α-ideals (where α denotes either l or m, since $\mathfrak{I}_l(M)$ and $\mathfrak{I}_m(M)$ form a complete lattice), and our interpretations will attach an ideal to any propositional symbol:

Definition 11 *An* interpretation *is a mapping $I\colon \Pi \to \mathfrak{I}_\alpha(M)$. The set of all interpretations is denoted $\mathcal{I}$.*

The ordering $\leq$ of the truth-values M can be extended point-wise to the set of interpretations $\mathcal{I}$ as usual; and also endows $\mathcal{I}$ with a complete lattice structure.

Now, we cannot apply the homomorphic extension theorem as usual, since the connective operators are interpreted as functions in M, not between ideals. Thus, we have to explicitly define how to extend a given interpretation to any body-formula:

- Given I and $a \in M$, we define $\hat{I}(a)$ as the least α-ideal containing a.
- Given I, atoms $A_1, \dots, A_n$, and any isotone n-ary function @ we define $\hat{I}(@(A_1, \dots, A_n))$ as

$$\biguplus \Big\{ @(a_1, \dots, a_n) \mid a_i \in I(A_i), i \in \{1, \dots, n\} \Big\}$$

where $\biguplus$ computes the least α-ideal generated by its arguments.

Note that @ can be seen as an operator in $\mathfrak{I}_\alpha$ which sends the ideals $I(A_1), \dots, I(A_n)$ to the ideal defined above.

Remark 1 *Notice that $\biguplus$ is not defined for s-ideals: for instance, in M6 there is not a least ideal containing $\{a, b\}$, but two minimal ideals. This is a consequence of $\mathfrak{I}_s(M)$ not being a lattice.*

As in the restricted case, a rule of an extended logic program is satisfied whenever the truth-value of the head of the rule is greater or equal than the truth-value of its body. Formally:

Definition 12 *Given an interpretation I, a rule $A \leftarrow \mathcal{B}$ is* satisfied *by I if and only if $\hat{I}(\mathcal{B}) \subseteq I(A)$. An interpretation I is said to be a* model *of an extended logic program $\mathbb{P}$ if and only if all rules in $\mathbb{P}$ are satisfied by I, then we write $I \models \mathbb{P}$.*

After defining a semantics for the different ideals of a multilattice we are going to provide our ideal-based framework with a fixpoint semantics and study if it has similar properties to the ideal fixed semantics for lattices in order to compute the least ideal-model of our programs.

Taking the usual definition of immediate operator of consequences $T_\mathbb{P}$, we have that the transcription of this to our lattice of ideals is the following:

Definition 13 *Given an extended logic program $\mathbb{P}$, an interpretation I and a propositional symbol A; the immediate operator of consequences $T_\mathbb{P}$ is defined about I and A as*

$$T_\mathbb{P}(I)(A) = \biguplus \left(\bigcup_{A \leftarrow \mathcal{B} \in \mathbb{P}} \hat{I}(\mathcal{B}) \right)$$

As we are working on a lattice structure we can apply the results of [12], and obtain that:

1. $T_\mathbb{P}$ is an isotone operator.
2. An interpretation I is a model if and only if $T_\mathbb{P}(I) \leq I$.
3. $T_\mathbb{P}$ is a continuous operator if and only if all the operators @ involved in the bodies are continuous (as operators in $\mathfrak{I}_\alpha$).
4. If $T_\mathbb{P}$ is continuous, then it has a least fixed point that can be attained after at most ω iterations.

6 Restricted vs ideal semantics

In this section we study the relationship between the different semantics we have introduced. We are interested in whether we can obtain minimal models of the restricted semantic via the minimum model of the different ideals semantics.

The following theorem gives us an interesting relationship between the least model of the l-ideal semantics and the minimal models provided by the restricted semantics.

In the theorem below we will write $T_{\mathbb{P}}$ (resp. $L_{\mathbb{P}}$) for the immediate consequences operator of the restricted (resp. the l-ideal) semantics, and $\sqsubseteq_S$ for the Smyth ordering between subsets, i.e. $X \sqsubseteq_S Y$ if and only if for all $y \in Y$ there exists $x \in X$ such that $x \leq y$.

Theorem 4 *Consider a program $\mathbb{P}$ and a propositional symbol A, then for all ordinal α we have that ${L_{\mathbb{P}}}^{\uparrow\alpha}(A) \sqsubseteq_S {T_{\mathbb{P}}}^{\uparrow\alpha}(A)$.*

This result justifies considering the (multisuprema of the) least l-model in order to obtain a minimal model of the restricted semantics, but this in general is not possible, as the following example shows:

Example 5 *If we consider the multilattice $M6$ and the program $\mathbb{P}$:*

$$B \leftarrow a; \quad A \leftarrow B; \quad C \leftarrow B$$
$$B \leftarrow b; \quad A \leftarrow c; \quad C \leftarrow d$$

The minimal models in the restricted semantics are M_1^r and M_2^r defined as:

$$M_1^r(A) = c; \quad M_2^r(A) = \top$$
$$M_1^r(B) = c; \quad M_2^r(B) = d$$
$$M_1^r(C) = \top; \quad M_2^r(C) = d$$

But the least l-model M^l is defined as

$$M^l(A) = \{\bot, a, b, c\}$$
$$M^l(B) = \{\bot, a, b\}$$
$$M^l(C) = \{\bot, a, b, d\}$$

Clearly the least l-model satisfies the theorem above but neither of the two possible restricted interpretations computed in terms of the multisuprema of M^l is a model of the program.

The behaviour in the previous example is the same in the case of m-ideals, since in $M6$ the set of l-ideals and m-ideals coincide.

7 Conclusions

The study of the possible relationships between the restricted semantics and the ideal semantics provided by the different definition of ideals in the literature leads to two interesting problems to be in the future:

- Further investigate the possibility of providing a set-based fixpoint semantics (not necessarily based on ideals) as an alternative way to reach a minimal model of the restricted semantics.
- Consider alternative approaches, such as a procedural semantics following the ideas underlying the tabulation proof procedure introduced in [4].

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