Formal Independence Analysis*

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Abstract. In this paper we propose a new lens through which to observe the information contained in a formal context. Instead of focusing on the hierarchical relation between objects or attributes induced by their incidence, we focus on the "unrelatedness" of the objects with respect to those attributes with which they are not incident. The crucial order concept for this is that of maximal anti-chain and the corresponding representation capabilities are provided by Behrendt's theorem. With these tools we introduce the fundamental theorem of Formal Independence Analysis and use it to provide an example of what its affordances are for the analysis of data tables. We also discuss its relation to Formal Concept Analysis.

1 Introduction

The original intent of Wille on creating Formal Concept Analysis (FCA) was, in his own words, an "attempt to unfold lattice-theoretical concepts, results, and methods in a continuous relationship with their surrounding" [9, §1]. In an interesting *Final remarks* of that seminal work Wille renounces any attempt at exhaustiveness of the lattice restructuring program and recommends: "Besides the interpretation by hierarchies of concepts, other basic interpretations of lattices should be introduced; ..." In this light, we may wonder what other views of the information carried by a context might be.

In this paper we propose an alternative conceptualization for the information contained in a formal context. Since the intuitive interpretation of the analogues of formal concepts, which we have named *formal tomoi*, describe sets of objects and attributes that have nothing to do with each other, we call this conceptualization *Formal Independence Analysis (FIA)*.

We base it in terms of the anti-chains of a certain order related to the context, and its lattice of anti-chains (Section 2) using Behrendt's theorem [1] which has universal

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representation capabilities for finite complete lattices. In this paper, we study mathematical properties of the lattice of anti-chains, in relation to the possible extension of Behrendt's theorem to a more general framework.

The set of anti-chains of a given a poset can be ordered by using two natural approaches (${}_{\star}\preccurlyeq$ and $\preccurlyeq^{\star}$) which lead to isomorphic structures, namely, a distributive lattice. The lattice of anti-chains of a distributive lattice L turns out to be isomorphic to L; however, the lattice of *maximal* anti-chains with any of the previous orderings is also a lattice, but not necessarily distributive. Now Behrendt's theorem states that any lattice (distributive or not) is isomorphic to the lattice of maximal anti-chains of *certain* poset. The focus of this paper is to study the existence of a representation theorem by using maximal anti-chains *within* the original lattice. For this, we rephrase Behrendt's theorem in terms of FCA using *tomoi*, and analyze possible extensions.

2 Preliminaries

Adjunctions and Galois connections. Different notions of Galois connection or adjunction can be found in the literature; these notions are strongly related, but do not coincide. The transition between the two types of adjunctions (connections) relies on using the opposite ordering in *both* preordered sets, whereas the transition between adjunctions to connections and vice versa relies on using the opposite ordering in *just one* of the preordered sets. The four different types of Galois connections and adjunctions are summarized in Table 1.

Table 1: Galois connections and adjunctions: equivalent characterizations

Galois Connections	
Right-Galois Connection between $\mathbb{A}$ and $\mathbb{B}$ $(f,g)\colon \mathbb{A} \leftharpoondown\!\!\rightharpoondown \mathbb{B}$	Left-Galois Connection between $\mathbb{A}$ and $\mathbb{B}$ $(f,g)\colon \mathbb{A} \leftharpoonup\!\!\rightharpoonup \mathbb{B}$
$b \leq f(a) \Leftrightarrow a \leq g(b)$ for all $a \in A$ and $b \in B$	$f(a) \leq b \Leftrightarrow g(b) \leq a$ for all $a \in A$ and $b \in B$
f and g are antitone and $g \circ f$ and $f \circ g$ inflationary	f and g are antitone and $g \circ f$ and $f \circ g$ deflationary
Adjunctions	
Adjunction between $\mathbb{A}$ and $\mathbb{B}$ $(f,g)\colon \mathbb{A} \leftrightharpoons \mathbb{B}$	Co-adjunction between $\mathbb{A}$ and $\mathbb{B}$ $(f,g)\colon \mathbb{A} \rightleftharpoons \mathbb{B}$
$f(a) \leq b \Leftrightarrow a \leq g(b)$ for all $a \in A$ and $b \in B$	$b \leq f(a) \Leftrightarrow g(b) \leq a$ for all $a \in A$ and $b \in B$
f and g are isotone, $g \circ f$ inflationary and $f \circ g$ deflationary	f and g are isotone, $g \circ f$ deflationary and $f \circ g$ inflationary

See [4,6] for a revision of the genesis and importance of Galois Connections and adjunctions, as well as a discussion of the different notation and nomenclatures for these concepts. [5] is an early tutorial with mathematical applications in mind, and [8] deals fully on how to extend FCA with the different types of connections to provide different "flavors" of FCA, as well as extending it to non-binary incidences.

The analysis of (in)comparability in a poset. Let $\langle P, \leq \rangle$ be a poset. We say that $x, y \in P$ are *comparable* if $x \leq y$ or $y \leq x$, and *incomparable* otherwise, and write $x \parallel y$. The initial analysis of posets is made in terms of *comparability* [2].

Definition 1. *Let $\langle P, \leq \rangle$ be a poset, and $Q \subseteq P$. Then*
1. *Q is an* (order) ideal *if $x \in Q$ and $y \leq x$, then $y \in Q$.*
2. *Q is an* (order) filter *if $x \in Q$ and $y \geq x$, then $y \in Q$.*
3. *$\downarrow Q = \{y \in P :$ there exists $x \in Q$ with $x \leq y\}$ is called the ideal* generated by *Q*
4. *$\uparrow Q = \{y \in P :$ there exists $x \in Q$ with $y \leq x\}$ is called the* filter generated by *Q.*

We will write $O(P)$ and $F(P)$ to denote the sets of ideals and filters (respectively) of $\langle P, \leq \rangle$. If considered as posets, ordered by set inclusion, both are distributive lattices and are dually-isomorphic.

Definition 2. *For a poset $\langle P, \leq \rangle$ and $Q \subseteq P$ define:*
1. *$a \in Q$ is a* minimal element of *Q if $a \geq x$ and $x \in Q$ imply $a = x$.*
2. *$a \in Q$ is a* maximal element of *Q if $a \leq x$ and $x \in Q$ imply $a = x$.*

We will write $\mathrm{Minl}(Q)$ and $\mathrm{Maxl}(Q)$ to denote the set of minimal, respectively maximal, elements of Q.

Definition 3 ([2]). *For a poset $\mathbb{P} = \langle P, \leq \rangle$, a set of pairwise incomparable elements of P is called an* anti-chain. *We denote the set of anti-chains of a poset as $A(\mathbb{P})$.*

Definition 4. *Given $\langle P, \leq \rangle$ a poset, it is possible to lift the ordering structure to the powerset 2^P by defining*

$$X \;{}_{\star}\!\preccurlyeq Y \iff \textit{for all } x \in X \textit{ there exists } y \in Y \textit{ such that } x \leq y \tag{1}$$

$$X \preccurlyeq^{\star} Y \iff \textit{for all } y \in Y \textit{ there exists } x \in X \textit{ such that } x \leq y \tag{2}$$

$$X \preccurlyeq Y \iff X \;{}_{\star}\!\preccurlyeq Y \textit{ and } X \preccurlyeq^{\star} Y \tag{3}$$

In general ${}_{\star}\!\preccurlyeq$ and $\preccurlyeq^{\star}$ are both simply preordering relations in $\langle 2^P, \subseteq \rangle$. Observe that in the set of anti-chains $A(\mathbb{P})$, the relations ${}_{\star}\!\preccurlyeq$ and $\preccurlyeq^{\star}$ are also antisymmetric.

There exists a relationship with the inclusion ordering of ideals and filters since given $S, T \subseteq P$ then $S \;{}_{\star}\!\preccurlyeq T \iff \downarrow S \subseteq \downarrow T$ and $S \preccurlyeq^{\star} T \iff \uparrow T \subseteq \uparrow S$. Because of these equivalences, we will call ${}_{\star}\!\preccurlyeq$ as *ideal containment relation* and $\preccurlyeq^{\star}$ as *filter containment relation*.

Maximal anti-chains.

Definition 5. *For a poset $\langle P, \leq \rangle$, an anti-chain $\gamma \in A(\mathbb{P})$ is said to be* maximal *if every element of P is comparable to some element of γ.*

For any subset $Q \subseteq P$, the set of elements of P which are comparable to some element of Q is called the *neighborhood* of Q and it is denoted by $\updownarrow Q = \uparrow Q \cup \downarrow Q$. An anti-chain γ is maximal if and only if $\updownarrow \gamma = P$. The set of maximal anti-chains of a set is denoted as $MA(\mathbb{P})$,

$$MA(\mathbb{P}) = \{\gamma \in A(\mathbb{P}) \mid \updownarrow \gamma = P\}.$$

It is worth noting that the orderings ${}_\star\!\preccurlyeq$ and $\preccurlyeq^\star$ coincide in $MA(\mathbb{P})$.

Proposition 1 ([1]). *If $\mathbb{P}$ is a finite poset then $\langle MA(\mathbb{P}), {}_\star\!\preccurlyeq\rangle$ is a lattice.*

In [7] Reuter asserts, "Given an anti-chain A of P, the completion of A to a maximal anti-chain is not unique but there exists a unique lowest completion." This completion can be described in terms of the operators below:

Definition 6. *For a finite partial order $\langle P, \leq\rangle$, and $A, B \in 2^P$ we define the* highest anti-chain complement of A, denoted A^-, *and the* lowest anti-chain complement of B, denoted B_-, *as*

$$\begin{aligned} \cdot^- &: 2^P \to 2^P & \cdot_- &: 2^P \to 2^P \\ A &\mapsto A^- = \max(P \setminus \updownarrow A) & B &\mapsto B_- = \min(P \setminus \updownarrow B)\,. \end{aligned} \tag{4}$$

The analysis of incomparability by means of FCA. Due to the universal complete lattice representation capabilities of FCA, we must expect the lattices of anti-chains to be describable as the concept lattice of a context. The first result in this direction is due to Wille himself [10, Proposition 1, in our notation].

Proposition 2. *Let $\langle P, \leq\rangle$ be an ordered set. The concepts of the context $(P, P, \ngeq)$ are exactly the pairs $(A, P \setminus A)$ where A is an order ideal of P; especially*

$$\underline{\mathfrak{B}}(P, P, \ngeq) \cong \langle A(\mathbb{P}), {}_\star\!\preccurlyeq\rangle \cong \langle O(P), \subseteq\rangle \qquad \underline{\mathfrak{B}}(P, P, \ngeq)^d \cong \langle A(\mathbb{P}), \preccurlyeq^\star\rangle \cong \langle F(P), \subseteq\rangle$$

Moreover, when focusing on maximal anti-chains, we have the following isomorphism is credited by Reuter [7] to Behrendt [1] and Wille [10].

Proposition 3. *Let $\mathbb{P} = \langle P, \leq\rangle$ be a poset. Then $\langle MA(\mathbb{P}), \preccurlyeq\rangle \cong \underline{\mathfrak{B}}(P, P, \ngtr)$.*

The proposition above states that maximal anti-chains can be obtained as a concept lattice for a certain context. On the other hand, Behrendt's theorem [1] is a universal representation theorem for lattices in terms of maximal anti-chains.

Theorem 1 (Behrendt). *Let $\mathbb{L} = \langle L, \leq\rangle$ be a finite lattice. Then there exists a poset $\mathbb{P} = \langle P, \leq_P\rangle$ such that $|P| = 2|L|$, where any chain has at most 2 elements and such that $\mathbb{L} \cong MA(\mathbb{P})$, i.e., $\mathbb{L}$ is isomorphic to the lattice of maximal anti-chains of $(P, \leq_P)$.*

This is our starting point for the development of Formal Independence Analysis.

3 Formal Independence Analysis (FIA).

It is straightforward that the three following structures are equivalent: formal contexts, bipartite graphs, and posets without chains with length higher than 2. As an example of the interoperability of the three structures: given a formal context (G,M,I), a bipartite graph can be obtained as $\langle G \sqcup M, I' \rangle$ where $G \sqcup M = (G \times \{0\}) \cup (M \times \{1\})$ is the disjoint union of G and M, and I' is defined as $(g,0)\, I'\, (m,1)$ if and only if $g\, I\, m$. From such a bipartite graph, a poset can be obtained with simply considering the reflexive closure of I'. Finally, from a poset $(P,\leq)$ without chains of length higher than 2, a formal context can be obtained by setting G and M to be, respectively, the sets of minimals and maximals of P.

To begin with, it is worth analyzing the proof of Behrendt's Theorem, using the terminology of FCA. Given a finite lattice[3] $\mathbb{L} = \langle L, \leq \rangle$, the proof considers another poset (of twice the cardinality of L) whose set of maximal anti-chains is isomorphic to $\mathbb{L}$. Specifically, given $\mathbb{L} = \langle L, \leq \rangle$, the new poset is obtained as the disjoint union $L \sqcup L = L \times \{0,1\}$ with the ordering relation defined as the reflexive closure of

$$(z_1,0)\, I\, (z_2,1) \quad \text{if and only if} \quad z_1 < z_2 \text{ or } z_1 \parallel z_2\,.$$

The isomorphism is the following: given $z \in L$ is associated to the maximal anti-chain $\gamma_z = \alpha_z \cup \beta_z$ where

$$\alpha_z = \{(z',0) \in L \mid z \leq z'\} = \uparrow z \times \{0\} \qquad \beta_z = \{(z',1) \in L \mid z' \leq z\} = \downarrow z \times \{1\}$$

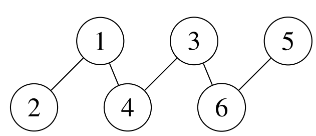
Instead of considering γ_z as the union $\alpha_z \cup \beta_z$, one might consider the pairs (α_z, β_z), which leads to the notion *formal tomoi*[4] in a formal context.

In order to formally introduce the definition, we will use one of the alternative interpretations above in order to apply the mappings $\cdot_-$ and $\cdot^-$ within a formal context (G,M,I).

Specifically, we will consider the 2-height poset $(G \sqcup M, \leq)$ and, given $\alpha \subseteq G$ we will define $\alpha^- = M \smallsetminus \updownarrow \alpha = M \smallsetminus \uparrow \alpha = M \smallsetminus \bigcup_{g\in\alpha} I(g,\cdot)$, and similarly for β_- given $\beta \subseteq M$. It is not difficult to see that there is a bijection between maximal anti-chains and pairs $(\alpha,\beta) \in 2^G \times 2^M$ satisfying $\alpha^- = \beta$ and $\beta_- = \alpha$.

The following example shows a subtle difference in the behaviour of the operators of highest and lowest complement depending on whether they are applied within a poset or within a formal context.

Example 1. Consider the following poset



On the one hand, considering the poset structure above, given $\alpha = \{4\}$ we would obtain $\alpha^- = \{2,5\}$; on the other hand, in the interpretation within a formal context, we would obtain $\alpha^- = \{5\}$. □

[3] But it is easy to see that the proof is also applicable to arbitrary complete lattices.

[4] From greek *tomos*, division pl. *tomoi*.

Notation. The situation above suggests to introduce a specific notation in order to avoid possible misunderstandings. Hence, we will use $\alpha^{\sim}$ and $\beta_{\sim}$ to indicate that we are assuming the construction within a formal context.

Now, the definition of a formal tomos is given as follows:

Definition 7. *Given a context (G,M,I), a* formal tomos *is a pair $(\alpha,\beta) \in 2^G \times 2^M$, such that $\alpha^{\sim} = \beta$ and $\beta_{\sim} = \alpha$. The set of formal tomoi of the context (G,M,I) will be denoted by $\mathfrak{A}(G,M,I)$.*

It is worth noting that the set of formal tomoi with the supset-subset hierarchical ordering, denoted $\underline{\mathfrak{A}}(G,M,I)$, is isomorphic to the corresponding lattice of maximal anti-chains. In fact, it turns out that $\underline{\mathfrak{A}}(G,M,I) = \underline{\mathfrak{B}}(G,M,\not I)^d$, since

$$\alpha^{\sim} = M \smallsetminus \bigcup_{g\in\alpha} I(g,\cdot) = \{m \in M \mid g \not I m \text{ for all } g \in \alpha\} \tag{5}$$

$$\beta_{\sim} = G \smallsetminus \bigcup_{m\in\beta} I(\cdot,m) = \{g \in G \mid g \not I m \text{ for all } m \in \beta\} \tag{6}$$

These operators adequately reflect the underlying philosophy of formal independence analysis and, in this terminology, we can obtain the following corollary of Theorem 1:

Corollary 1 (Behrendt's theorem in terms of tomoi). *Every finite lattice is isomorphic to a lattice of tomoi.*

Continuing this line of reasoning, we can state an analogue for tomoi of the basic theorem of FCA as follows:

Theorem 2 (Basic theorem of formal independence analysis).

1. *The context analysis phase: Given a formal context (G,M,I),*
   (a) *The operators $\cdot^{\sim} : 2^G \to 2^M$ and $\cdot_{\sim} : 2^M \to 2^G$ form a right-Galois connection $(\cdot^{\sim},\cdot_{\sim}) : (2^G,\subseteq) \rightharpoonup (2^M,\subseteq)$ whose* formal *tomoi are the pairs (α,β) such that $\alpha^{\sim} = \beta$ and $\alpha = \beta_{\sim}$.*
   (b) *The set of formal tomoi $\mathfrak{A}(G,M,I)$ with the relation*

   $$(\alpha_1,\beta_1) \le (\alpha_2,\beta_2) \;\; \textit{iff} \;\; \alpha_1 \supseteq \alpha_2 \;\; \textit{iff} \;\; \beta_1 \subseteq \beta_2$$

   is a complete lattice, which is called the tomoi lattice of (G,M,I) *and denoted $\underline{\mathfrak{A}}(G,M,I)$, where infima and suprema are given by:*

   $$\bigwedge_{t\in T}(\alpha_t,\beta_t) = \left(\bigcup_{t\in T}\alpha_t, \left(\bigcap_{t\in T}\beta_t\right)_{\sim}{}^{\sim}\right) \qquad \bigvee_{t\in T}(\alpha_t,\beta_t) = \left(\left(\bigcap_{t\in T}\alpha_t\right)^{\sim}{}_{\sim}, \bigcup_{t\in T}\beta_t\right)$$

   (c) *The mappings $\overline{\gamma} : G \to \underline{\mathfrak{A}}(G,M,I)$ and $\overline{\mu} : M \to \underline{\mathfrak{A}}(G,M,I)$*

   $$g \mapsto \overline{\gamma}(g) = (\{g\}^{\sim}{}_{\sim}, \{g\}^{\sim}) \qquad m \mapsto \overline{\mu}(m) = (\{m\}_{\sim}, \{m\}_{\sim}{}^{\sim})$$

   are such that $\overline{\gamma}(G)$ is infimum-dense in $\underline{\mathfrak{A}}(G,M,I)$, $\overline{\mu}(M)$ is supremum-dense in $\underline{\mathfrak{A}}(G,M,I)$.
2. *The context synthesis phase: Given a complete lattice $\mathbb{L} = \langle L,\le\rangle$*

(a) $\mathbb{L}$ *is isomorphic to*[5] $\underline{\mathfrak{A}}(G,M,I)$ *if and only if there are mappings* $\overline{\gamma}: G \to L$ *and* $\overline{\mu}: M \to L$ *such that*

- $\overline{\gamma}(G)$ *is infimum-dense in* $\mathbb{L}$, $\overline{\mu}(M)$ *is supremum-dense in* $\mathbb{L}$*, and*
- $g\,I\,m$ *is equivalent to* $\overline{\gamma}(g) \not\geq \overline{\mu}(m)$ *for all* $g \in G$ *and all* $m \in M$*.*

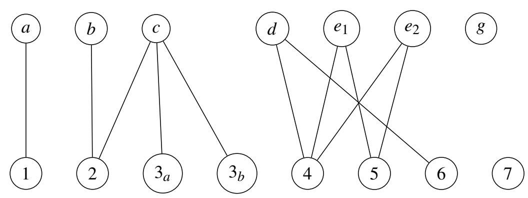
(b) In particular, $\mathbb{L} \cong \underline{\mathfrak{A}}(L,L,\not\geq)$ *and, if* L *is finite,* $\mathbb{L} \cong \underline{\mathfrak{A}}(M(\mathbb{L}),J(\mathbb{L}),\not\geq)$ *where* $M(\mathbb{L})$ *and* $J(\mathbb{L})$ *are the sets of meet- and join-irreducibles, respectively, of* $\mathbb{L}$*.*

Notice that the differences between the basic theorems of FIA and FCA are due to the fact that FCA focuses on the notion of "being related" which, in algebraic terms, leads to complete bipartite subgraphs and, in FCA terminology, to maximal rectangles, whereas FIA focuses on "unrelatedness", leading to completely independent subsets.

Example 2. Figure 1.a is the tabular representation of a context which admits a non-trivial block-diagonal form. Figure 1.b is the bipartite graph representation, where this block structure is also apparent. If we represent the concept lattice, as in Figure 2.a, these appear adjoined by top and bottom.

$\mathbb{K}_1$	a	b	c	d	e1	e2	g
1	×						
2		×	×				
3a			×				
3b			×				
4				×	×	×	
5					×	×	
6				×			
7							

(a) Tabular representation of $\mathbb{K}_1$



(b) Bipartite graph representation of $\mathbb{K}_1$

Fig. 1: **Equivalent representations of an example context** $\mathbb{K}_1 = (G,M,I)$**.** a) tabular representation. b) bipartite graph representation.

Consider the formal context of (G,M,I), e.g. that of Figure 1.a: we would like to find its independence lattice $\underline{\mathfrak{A}}(G,M,I)$. Note that there is an isolated object 7 and an isolated attribute g. These are both ignored and re-introduced later. To find the meet-irreducibles, we use the object-tomos mapping $\overline{\gamma}$ over the whole of G. The result of this operation can be seen in Table 2.a. Likewise, the result of the application of the attribute-tomos mapping $\overline{\mu}$ over the whole of M can be found in Table 2.b.

Since the object-tomoi are meet-irreducible and the attribute tomoi join-irreducible, the Dedekind-MacNeille completion using the lattice operations finds the lattice of tomoi $\underline{\mathfrak{A}}(G,M,I)$ as seen in Figure 2.b. Note that the illustration is actually built as $\underline{\mathfrak{B}}(M,G,I^{cd})$ since the SW used to represent the lattice only understand concept lattices.

[5] Read *can be built as*.

$g\setminus\overline{\gamma}(g)$	γ	μ
1	$\{1\}$	$\{b,c,d,e1,e2\}$
2	$\{2,3a,3b\}$	$\{a,d,e1,e2\}$
$3a$	$\{3a,3b\}$	$\{a,b,d,e1,e2\}$
$3b$	$\{3a,3b\}$	$\{a,b,e,e1,e2\}$
4	$\{4,5,6\}$	$\{a,b,c\}$
5	$\{5\}$	$\{a,b,c,d\}$
6	$\{6\}$	$\{a,b,c,e1,e2\}$

(a) Object tomoi of $\underline{\mathfrak{A}}(G,M,I)$

$m\setminus\overline{\mu}(m)$	γ	μ
a	$\{2,3a,3b,4,5,6\}$	$\{a\}$
b	$\{1,3a,3b,4,5,6\}$	$\{b\}$
c	$\{1,4,5,6\}$	$\{b,c\}$
d	$\{1,2,3a,3b,5\}$	$\{d\}$
$e1$	$\{1,2,3a,3b,6\}$	$\{e1,e2\}$
$e2$	$\{1,2,3a,3b,6\}$	$\{e1,e2\}$

(b) Attribute tomoi of $\underline{\mathfrak{A}}(G,M,I)$

Table 2: **Object and attribute tomoi for $\mathbb{K}_1$ and $\underline{\mathfrak{A}}(\mathbb{K}_1)$.** Not seen are object 7, appearing in each object-tomos, and attribute g, in every attribute-tomos.

4 Generalizing the construction of tomoi to posets

We have just seen that the operators $\cdot^{\sim}$ and $\cdot_{\sim}$ on a formal context, whose definition raised from a suitable modification of $\cdot^{-}$ and $\cdot_{-}$. In this section, we focus on the possible extensions of the notion of tomoi in a poset as general as possible. In the rest of the paper, we consider a poset $\mathbb{P}=\langle P,\leq\rangle$ **without infinite chains**.

Proposition 4. *For any $Q\subseteq P$, the following properties hold*

$$\mathrm{Minl}(Q),\ \mathrm{Maxl}(Q)\in A(\mathbb{P}) \tag{7}$$

$$\mathrm{Minl}(Q)\preccurlyeq^{\star} Q\ {}_{\star}\!\!\preccurlyeq \mathrm{Maxl}(Q) \tag{8}$$

We will now explore the properties of the operators of highest (resp. lowest) antichain complement $\cdot_{-}$ and $\cdot^{-}$.

Definition 8. *A set $Q\subseteq P$ is said to be* convex *if $a,b\in Q$ and $a\leq p\leq b$ imply $p\in Q$.*

Proposition 5.
1. *For any convex set $Q\subseteq P$, we have that $Q=\uparrow\mathrm{Minl}(Q)\cap\downarrow\mathrm{Maxl}(Q)$.*
2. *For any $Q\subseteq P$, the set $P\setminus\updownarrow Q$ is convex and, therefore,*

$$P\setminus\updownarrow Q=\{p\in P\mid\forall a\in Q,p\parallel a\}=\uparrow\mathrm{Minl}(P\setminus\updownarrow Q)\cap\downarrow\mathrm{Maxl}(P\setminus\updownarrow Q)$$

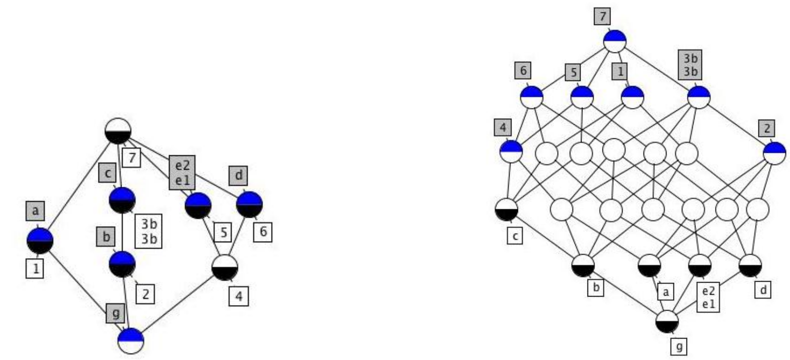
In addition, if we write $Q^{\parallel}=P\setminus\updownarrow Q$, then $(\cdot^{\parallel},\cdot^{\parallel})$ is a Galois connection in $(2^P,\subseteq)$.

Proof. Since $\mathbb{P}$ has not infinite chains, we have that $Q\subseteq\uparrow\mathrm{Minl}(Q)\cap\downarrow\mathrm{Maxl}(Q)$. By the other side, since Q is convex, any element $p\in\uparrow\mathrm{Minl}(Q)\cap\downarrow\mathrm{Maxl}(Q)$ belongs to Q.

The second item follows from the definition of $\updownarrow Q=\uparrow Q\cup\downarrow Q$ as the neighborhood of Q. Therefore $P\setminus\updownarrow Q$ are those elements *not* related to any element in Q. Note that $P\setminus(\uparrow Q\cup\downarrow Q)=(P\setminus\uparrow Q)\cap(P\setminus\downarrow Q)$ and it is convex.

Finally, for all $A,B\subseteq P$, it easy to see that $A\subseteq P\setminus\updownarrow B$ if and only if $B\subseteq P\setminus\updownarrow A$. □

The following result can be obtained as a consequence of the previous propositions:



(a) Concept lattice $\mathfrak{B}(G,M,I)$ (b) Tomoi lattice $\mathfrak{A}(G,M,I)$

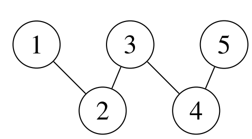
Fig. 2: **Two different lattices for context** $\mathbb{K}_1 = (G,M,I)$; in (a) the lattice of formal concepts *showing* three adjoint sublattices; in (b) the lattice of formal tomoi *describing* the three adjoint sublattices. Notice that object-concepts are join-irreducible in (a), but object-tomoi are meet-irreducible in (b), and likewise *mutatis mutandis* for attribute-concepts and attribute-tomoi.

Corollary 2. *For anti-chains $\alpha, \beta \in A(\mathbb{P})$, we have that:*
1. $\beta_- \preccurlyeq^\star \alpha \,{}_\star\!\preccurlyeq \beta^-$ *if and only if* $\alpha_- \preccurlyeq^\star \beta \,{}_\star\!\preccurlyeq \alpha^-$
2. $\alpha^-{}_- \preccurlyeq^\star \alpha \,{}_\star\!\preccurlyeq \alpha^{--}$
3. $\alpha_{--} \preccurlyeq^\star \alpha \,{}_\star\!\preccurlyeq \alpha_-{}^-$

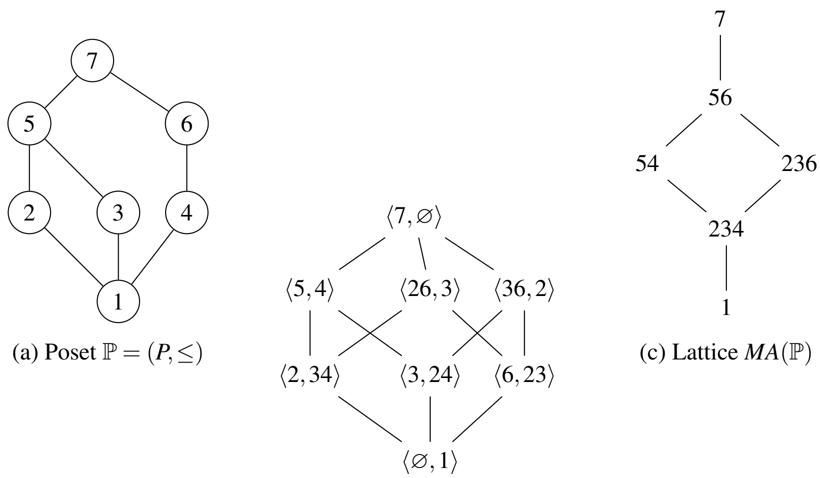
Proof.
1. We have that $\beta^{\parallel} = P \setminus \updownarrow\beta = \uparrow\beta_- \cap \downarrow\beta^-$. Therefore, $\alpha \subseteq \beta^{\parallel}$ if and only if $\beta_- \preccurlyeq^\star \alpha \,{}_\star\!\preccurlyeq \beta^-$. Then, the first item is a consequence of the fact trhat $(\cdot^{\parallel}, \cdot^{\parallel})$ is a Galois connection in $(2^P, \subseteq)$.
   Now, notice that (8) implies $\alpha_- \preccurlyeq^\star \alpha^-$ and $\alpha_- \,{}_\star\!\preccurlyeq \alpha^-$.
2. Since $\alpha_- \preccurlyeq^\star \alpha^- \,{}_\star\!\preccurlyeq \alpha^-$, by the first equivalence, one has $\alpha^-{}_- \preccurlyeq^\star \alpha \,{}_\star\!\preccurlyeq \alpha^{--}$.
3. Similar. □

Note that the first item of the corollary suggests the possible structure of Galois connection/adjunction of the pair operators $(\cdot_-, \cdot^-)$; moreover, from the second and third items, we have $\alpha^-{}_- \preccurlyeq^\star \alpha \,{}_\star\!\preccurlyeq \alpha_-{}^-$. This means that the only possibility for $(\cdot_-, \cdot^-)$ is to be an adjunction. Unfortunately, this is not the case, as shown in the example below.

Example 3. Given the poset $\mathbb{P}$ with the ordering depicted in the figure below



it is not the case that $(\cdot^-, \cdot_-)\colon \big(A(\mathbb{P}), \preccurlyeq^\star\big) \leftrightharpoons \big(A(\mathbb{P}), {}_\star\!\preccurlyeq\big)$, since $\{1,4\}^- \,{}_\star\!\preccurlyeq \{5\}$ holds but $\{1,4\} \preccurlyeq^\star \{5\}_-$ does not hold. □



(a) Poset $\mathbb{P} = (P, \leq)$

(b) Tomoi lattice $\underline{\mathfrak{A}}(P)$

(c) Lattice $MA(\mathbb{P})$

Fig. 3: Tomoi for an arbitrary poset and its maximal anti-chains do not always match.

Although, the operators do not form any kind of connection or adjunction, the notion of tomos still behaves properly, in the sense that it is strongly related to maximal anti-chains.

Proposition 6. *Let α, β be anti-chains such that $\alpha^- = \beta$ and $\beta_- = \alpha$. Then $\alpha \cup \beta$ is a maximal anti-chain.*

Proof. Since both α, β are anti-chains and $\alpha = \mathrm{Minl}\,(P \smallsetminus \updownarrow\beta) \subseteq P \smallsetminus \updownarrow\beta$, it is obvious that $\alpha \cup \beta$ is also an anti-chain. To show that it is maximal, assume that there exists $x \in P$ which is not related to any element of $\alpha \cup \beta$. If $x \in P \smallsetminus \updownarrow\alpha$, there exists $b \in \beta = \mathrm{Maxl}\,(P \smallsetminus \updownarrow\alpha)$ such that $x \leq b$, which contradicts that $x \in P \smallsetminus \updownarrow\beta$. Analogously, assuming that $x \in P \smallsetminus \updownarrow\beta$ also yields to a contradiction. $\square$

The relationship, however, is not one-one, as shown in the next example.

Example 4. For poset $\mathbb{P}$ in Fig. 3, the function which merges the two components of every tomos in order to obtain a maximal anti chain need not be either one-one nor onto. The three pairs $\langle 26,3\rangle, \langle 36,2\rangle, \langle 6,23\rangle$ are mapped to the same maximal anti-chain. On the other hand, no tomos leads to the maximal anti-chain 56. $\square$

Since operators $(\cdot^-, \cdot_-)$ do not behave properly because of the previous example, it is worth considering what happens when it is complemented with the original chain, i.e., consider the highest and lowest completions of a chain. In both cases, they are closure operators whose closures are, precisely, the maximal anti-chains.

Proposition 7. *Let α be a chain in $\mathbb{P}$*

1. *The mapping $\alpha \mapsto \alpha \cup \alpha^-$ is a closure operator in $(A(\mathbb{P}), {}_\star\!\preccurlyeq)$ whose set of closed elements is $MA(\mathbb{P})$.*

2. *The mapping* $\alpha \mapsto \alpha \cup \alpha_{-}$ *is a closure operator in* $(A(\mathbb{P}), \preccurlyeq^{\star})$ *whose set of closed elements is* $MA(\mathbb{P})$.

Proof. We only prove the first item, since the second is analogous.

The mapping is obviously inflationary and idempotent (the latter because for any anti-chain α, since $\alpha \cup \alpha^{-}$ is maximal, trivially holds that $(\alpha \cup \alpha^{-})^{-}$ is empty).

To show that the mapping is isotone, we first observe that, for any anti-chain α, the highest completion $\alpha \cup \alpha^{-}$ satisfies that

$$\downarrow(\alpha \cup \alpha^{-}) = P \smallsetminus \underline{\uparrow}\alpha \quad \text{where} \quad \underline{\uparrow}\alpha = \{x \in P : a < x \text{ for some } a \in \alpha\}.$$

As a consequence, $\alpha \cup \alpha^{-} \subseteq \operatorname{Maxl}(P \smallsetminus \underline{\uparrow}\alpha)$ which implies $\alpha \cup \alpha^{-} = \operatorname{Maxl}(P \smallsetminus \underline{\uparrow}\alpha)$ since $\alpha \cup \alpha^{-}$ is a maximal anti-chain.

Assume now that $\alpha_1 \;{}_{\star}\!\!\preccurlyeq \alpha_2$ and let us show that $\alpha_1 \cup \alpha_1{}^{-} \;{}_{\star}\!\!\preccurlyeq \alpha_2 \cup \alpha_2{}^{-}$. By Proposition 6, we know that the highest completion $\alpha \cup \alpha^{-}$ is maximal for all anti-chain α, then $\alpha_1 \cup \alpha_1{}^{-} \;{}_{\star}\!\!\preccurlyeq \alpha_2 \cup \alpha_2{}^{-}$ if and only if $\alpha_1 \cup \alpha_1{}^{-} \preccurlyeq^{\star} \alpha_2 \cup \alpha_2{}^{-}$. Hence, let us show that $\alpha_1 \cup \alpha_1{}^{-} \preccurlyeq^{\star} \alpha_2 \cup \alpha_2{}^{-}$ or equivalently $\alpha_1 \cup \alpha_1{}^{-} \subseteq \downarrow(\alpha_2 \cup \alpha_2{}^{-})$.

Since $\alpha_2 \subseteq \uparrow\alpha_1$, then $\underline{\uparrow}\alpha_2 \subseteq \uparrow\alpha_1 \subseteq \updownarrow\alpha_1$. Therefore, $P \smallsetminus \updownarrow\alpha_1 \subseteq P \smallsetminus \underline{\uparrow}\alpha_2$ whence

$$\alpha_1{}^{-} \subseteq P \smallsetminus \updownarrow\alpha_1 \subseteq P \smallsetminus \underline{\uparrow}\alpha_2 = \downarrow(\alpha_2 \cup \alpha_2{}^{-})$$

On the other hand, if we suppose that $x \in \alpha_1 \cap \underline{\uparrow}\alpha_2$, then there exists $y \in \alpha_2$ with $y < x$, but since $\alpha_2 \subseteq \uparrow\alpha_1$ there exists $z \in \alpha_2$ such that $y < x \leq z$, which is a contradiction because α_2 is an anti-chain. Thus, $\alpha_1 \subseteq P \smallsetminus \underline{\uparrow}\alpha_2 = \downarrow(\alpha_2 \cup \alpha_2{}^{-})$ □

5 Conclusions

In this paper we have tried to propose a new lens through which to observe the information contained in a formal context. Instead of focusing on the hierarchical relation between objects or attributes induced by their incidence we focus on the "unrelatedness" of the objects with respect to those attributes with which they are not incident.

We have named the framework that appears "Formal Independence Analysis" because it allows us to block-diagonalize formal contexts providing a means for decomposing them in terms of independent sub-contexts, in the sense that two independent sub-contexts do not share common attributes nor objects. Even if the formal context cannot be block-diagonalized the procedure still obtains joint sets of objects and attributes which are mutually unrelated, which we have named *formal tomoi* (that is "separations").

We have provided a fundamental theorem for Formal Independence Analysis and an example of use based on a formal context for which the context is effectively block-diagonalized to illustrate the possibilities of the technique.

The procedure seems to be specially interesting in data analysis where, dual to what formal concepts can glean from data describing the existence of hierarchy, formal tomoi would describe when data contexts—e.g. from genomics, contingency matrices, etc.—can be broken down into parts susceptible of independent analysis.

Further work is necessary to ascertain the relationship of lattices of formal tomoi to lattices of formal concepts, as well as to find out whether these are the only information lenses available for formal contexts.

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