Dual multi-adjoint concept lattices

J. Medina[a,1,3], M. Ojeda-Aciego[b,2,3]

[a] *Dept. Matemáticas, Universidad de Cádiz. Spain*
[b] *Dept. Matemática Aplicada, Universidad de Málaga. Spain*

Abstract

Several papers relate different alternative approaches to classical concept lattices: such as property-oriented and object-oriented concept lattices and the dual concept lattices. Whereas the usual approach to the latter is via a negation operator, this paper presents a fuzzy generalization of the dual concept lattice, the dual multi-adjoint concept lattice, in which the philosophy of the multi-adjoint paradigm is applied and no negation on the lattices is needed.

Key words: Concept lattices, Galois connection, implication triples, dual concept lattices

1. Introduction

The field of formal concept analysis (FCA), introduced by Wille in a crisp context some thirty years ago [36], has become an important and appealing research topic both from the theoretical perspective [26, 35] and from the applicative one [29, 30, 33]. This work is more focused on advances in the theory of fuzzy formal concept analysis than in practical applications.

Concerning generalizations of the initial approaches, based on classical boolean logic, many extensions have been developed [1–3, 7, 13, 17, 21, 31, 32]. In the framework of fuzzy FCA, multi-adjoint concept lattices, were introduced [27] as a new general approach to formal concept analysis, in which the philosophy of the multi-adjoint paradigm is applied (see [28] for more information). With the idea of providing a general framework in which different fuzzy approaches could be conveniently accommodated, the authors worked in a general non-commutative environment; and this naturally leads to the consideration of adjoint triples as the main building blocks of a multi-adjoint concept lattice.

In relation to knowledge representation and knowledge discovery in relational information systems, some extensions have related FCA to rough set theory, introducing

Email addresses: `jesus.medina@uca.es` (J. Medina), `aciego@uma.es` (M. Ojeda-Aciego)

[1] Partially supported by the UE through FEDER programme and Spanish Science Ministry under grant TIN2009-14562-C05-05.
[2] Partially supported by the UE through FEDER programme and Spanish Science Ministry under grant TIN2009-14562-C05-01.
[3] Partially supported by Junta de Andalucía under grant P09-FQM-5233.

different frameworks for formal concept analysis based on rough set theory instead of on classical set theory [9, 34].

Recently, the philosophy of the multi-adjoint paradigm has been used to present a fuzzy generalization of property-oriented concept lattices and of object-oriented concept lattices [24, 25].

This work focuses on another interesting extended framework for formal concept analysis: the *dual formal concept lattice* introduced in [9], which is built using the dual sufficiency modal operator. This paper presents a generalization of the dual formal concept lattice. The proposed approach is interesting in that, to the best of our knowledge, the construction of dual concept lattices is made by assuming a negation operator on the carrier set. The multi-adjoint paradigm is adapted to this new environment and provides the construction of dual multi-adjoint concept lattices without the need of negations operators and, as usual in the multi-adjoint framework, the carriers can be arbitrary complete lattices, different adjoint triples can be assumed, etc.

The proposed multi-adjoint environment provides a new point of view to obtain information from databases with incomplete information (in the sense of lack of information) and/or imprecise (in the sense of inaccurate) information, which will give more flexibility than the existing procedures. In particular, it is shown that a construction of concept lattices is possible in terms of interior operators, thus providing an approach to the construction based on an alternative type of operators (in difference to previous approaches in which at least one composition of the derivation operators leads to a closure). Applications and practical examples on this framework will be studied further.

This paper is structured as follows: a summary of formal concept analysis and derivation operators is introduced in Section 2. Later, Section 3 recalls the main computation operators, the adjoint triples, and a general and flexible fuzzy concept lattice structure, the multi-adjoint concept lattices; the "dual" of this structure that embeds the crisp definition given in [9] is presented in Section 4. Finally, the paper ends with some conclusions and prospects for future work.

2. Recalling derivation operators in the crisp case

Formal concept analysis considers a set of **a**ttributes A, a set of o**b**jects B and a crisp relation between them $R\colon A \times B \to \{0,1\}$, where, for each $a \in A$ and $b \in B$, we have that $R(a,b) = 1$, if a and b are related, or $R(a,b) = 0$, otherwise. We will also write aRb when $R(a,b) = 1$. The triple (A, B, R) is called a *formal context* and the classical derivation operators[4] ${}^{\triangle}\colon 2^B \to 2^A$, ${}^{\triangle}\colon 2^A \to 2^B$, are defined for each $X \subseteq B$ and $Y \subseteq A$ as follows:

$$X^{\triangle} = \{a \in A \mid \text{for all } b \in X, aRb\} = \{a \in A \mid \text{if } b \in X, \text{ then } aRb\} \tag{1}$$

$$Y^{\triangle} = \{b \in B \mid \text{for all } a \in Y, aRb\} = \{b \in B \mid \text{if } a \in Y, \text{ then } aRb\} \tag{2}$$

These operators are so-called *sufficiency operators* [16], although in order to distinguish on which carriers are defined, they are also called the *extent* and *intent* mappings, respectively.

[4]Ganter and Wille used originally the notation $'$ for this operator, hence the name. Note that we change the notation so that it fits that used in the generalizations.

Given a context (A, B, R), a *concept* in (A, B, R) is defined to be a pair (X, Y), where $X \subseteq B$, $Y \subseteq A$, satisfying that $X^{\triangle} = Y$ and $Y^{\triangle} = X$. The element X of the concept (X, Y) is the *extent* and Y the *intent*.

The set of concepts in a context (A, B, R) is denoted as $\mathcal{B}(A, B, R)$ and it is a complete lattice [15], with the inclusion ordering on the left argument or the opposite of the inclusion ordering on the right argument, that is, given $(X_1, Y_1), (X_2, Y_2) \in \mathcal{B}(A, B, R)$, we have that $(X_1, Y_1) \leq (X_2, Y_2)$ if $X_1 \subseteq X_2$ (or, equivalently, $Y_2 \subseteq Y_1$).

The most important feature of the mappings ${}^{\triangle} \colon 2^B \to 2^A$ and ${}^{\triangle} \colon 2^A \to 2^B$, is that they form a Galois connection.

Proposition 1. *Given a formal context (A, B, R) and the mappings ${}^{\triangle} \colon 2^B \to 2^A$ and ${}^{\triangle} \colon 2^A \to 2^B$, defined above, the pair $({}^{\triangle}, {}^{\triangle})$ is a Galois connection between P_1 and P_2, that is:*

1. ${}^{\triangle} \colon 2^B \to 2^A$ *and* ${}^{\triangle} \colon 2^A \to 2^B$ *are order-reversing.*
2. $X \subseteq X^{\triangle\triangle}$, *for all* $X \subseteq B$.
3. $Y \subseteq Y^{\triangle\triangle}$, *for all* $Y \subseteq A$.

These definitions of extent and intent operators are the original ones provided by Ganter and Wille, but other possibilities have been explored, still within a crisp framework, in areas such as qualitative data analysis [14, 16], crisp rough set theory [37], fuzzy rough set theory [8, 23]. Considering the sets A, B, and a crisp relation $R \colon A \times B \to \{0, 1\}$, the derivation operators ${}^{\pi} \colon 2^B \to 2^A$, ${}^{N} \colon 2^B \to 2^A$, ${}^{\nabla} \colon 2^B \to 2^A$ are defined, for each $X \subseteq B$, as:

$$
\begin{aligned}
X^{\pi} &= \{a \in A \mid \text{ there exists } b \in X, \text{ such that } aRb\} \\
X^{N} &= \{a \in A \mid \text{ for all } b \in B, \text{ if } aRb, \text{ then } b \in X\} \\
X^{\nabla} &= \{a \in A \mid \text{ there exists } b \in X^c, \text{ such that } aR^cb\}
\end{aligned}
$$

where X^c is the complement of X, and R^c is the complement relation of R. These operators are called *possibility*, *necessity* and *dual sufficiency* operators, respectively. Analogously, abusing of notation, we can define the mappings: ${}^{\pi} \colon 2^A \to 2^B$, ${}^{N} \colon 2^A \to 2^B$ and ${}^{\nabla} \colon 2^A \to 2^B$.

The derivation operators introduced above can be paired in several ways to form either new Galois connections or closure operators [9, 14–16, 36] so that different concept lattices are obtained: the *property-oriented concept lattice*, *object-oriented concept lattice* and *dual formal concept lattice*.

It is easy to see that the dual sufficiency operator satisfies $X^{\nabla} = ((X^c)^{\triangle})^c$, for each $X \subseteq B$, therefore it can be expected that formal concept lattices arising from these operators should be related. In the rest of the paper we will focus on this relation, but in the more general framework of multi-adjoint concept analysis.

3. Adjoint triples and multi-adjoint concept lattices

This section recalls the necessary definitions from multi-adjoint concept analysis, specifically, the multi-adjoint concept lattice as well as its main building blocks, the adjoint triples [27]. These triples are a generalization of the well-known t-norm and its residuated implication satisfying the adjointness property [18, 19]. A triple is obtained since we do not assume that the conjunctors verify the commutative property. This directly provides two different ways of applying the adjointness property, depending on which argument is fixed.

Definition 1. Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets and $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$ be mappings, then $(\&, \swarrow, \nwarrow)$ is an *adjoint triple* with respect to P_1, P_2, P_3 if:

1. $\&$ is order-preserving in both arguments.
2. $\swarrow$ and $\nwarrow$ are order-preserving on the first argument and order-reversing on the second argument.
3. Adjointness property: $x \leq_1 z \swarrow y$ iff $x \& y \leq_3 z$ iff $y \leq_2 z \nwarrow x$, where $x \in P_1$, $y \in P_2$ and $z \in P_3$.

Note that, in fact, properties (1) and (2) hold as a consequence of the adjointness property.

For example, the usual pairs formed by a t-norm and its residuated implication can be seen as "degenerate" examples of adjoint triples. As a t-norm is commutative, we have that $\swarrow = \nwarrow$ and (both) coincide with *the* residuated implication.

The approach based on adjoint triples is justified from the actual applications, in fact, when one learns a conjunctor from data given from examples [10, 11, 38] it is quite possible that the conjunctor so-obtained turns out to be non-commutative and, thus, two adjoint implications arise.

The following definition presents the basic structure which allows the existence of several adjoint triples for a given triplet of lattices.

Definition 2. A *multi-adjoint frame* $\mathcal{L}$ is a tuple

$$(L_1, L_2, P, \preceq_1, \preceq_2, \leq, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$$

where $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$ are complete lattices, $(P, \leq)$ is a poset and, for all $i = 1, \dots, n$, $(\&_i, \swarrow^i, \nwarrow_i)$ is an adjoint triple with respect to L_1, L_2, P.
Multi-adjoint frames are denoted $(L_1, L_2, L, \&_1, \dots, \&_n)$.

Considering a multi-adjoint frame, a *multi-adjoint context* is a tuple consisting of a set of objects, a set of attributes and a fuzzy relation among them; in addition, the multi-adjoint approach also includes a function which assigns an adjoint triple to each object (or attribute).

Definition 3. Let $(L_1, L_2, P, \&_1, \dots, \&_n)$ be a multi-adjoint frame, a *context* is a tuple (A, B, R, σ) such that A and B are non-empty sets (usually interpreted as attributes and objects, respectively), R is a P-fuzzy relation $R\colon A \times B \to P$ and $\sigma\colon B \to \{1, \dots, n\}$ is a mapping which associates any element in B with some particular adjoint triple in the frame.

Given a multi-adjoint frame and context, the mappings ${}^\uparrow\colon L_2^B \longrightarrow L_1^A$ and ${}^\downarrow\colon L_1^A \longrightarrow L_2^B$ are defined, for all $g \in L_2^B$ and $f \in L_1^A$, as:

$$g^\uparrow(a) = \inf\nolimits_1\{R(a, b) \swarrow^{\sigma(b)} g(b) \mid b \in B\} \quad (3)$$

$$f^\downarrow(b) = \inf\nolimits_2\{R(a, b) \nwarrow_{\sigma(b)} f(a) \mid a \in A\} \quad (4)$$

These definitions generalize the classical ones given in (1), (2), and can be seen as further extensions of the fuzzy ones given in [4, 20]. Moreover, these two arrows generate a Galois connection.

A *multi-adjoint concept* is a pair $\langle g, f\rangle$ such that $g \in L_2^B$, $f \in L_1^A$ and satisfying that $g^\uparrow = f$ and $f^\downarrow = g$; with $(^\uparrow, ^\downarrow)$ being the Galois connection defined above. The set of all multi-adjoint concepts is called multi-adjoint concept lattice.

Definition 4. The *multi-adjoint concept lattice* associated to a multi-adjoint frame $(L_1, L_2, P, \&_1, \dots, \&_n)$ and a context (A, B, R, σ) is the set

$$\mathcal{M} = \{\langle g, f\rangle \mid g \in L_2^B, f \in L_1^A \text{ and } g^\uparrow = f, f^\downarrow = g\}$$

in which the ordering is defined by $\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle$ if and only if $g_1 \preceq_2 g_2$ (equivalently $f_2 \preceq_1 f_1$).

The pair $(\mathcal{M}, \preceq)$ is indeed a complete lattice with supremum and infimum operators defined as follows

$$\inf\{\langle g_i, f_i\rangle \mid i \in I\} = \langle \inf_2\{g_i \mid i \in I\}, (\sup_1\{f_i \mid i \in I\})^{\downarrow\uparrow}\rangle \quad (5)$$

$$\sup\{\langle g_i, f_i\rangle \mid i \in I\} = \langle (\sup_2\{g_i \mid i \in I\})^{\uparrow\downarrow}, \inf_1\{f_i \mid i \in I\}\rangle \quad (6)$$

4. Dual multi-adjoint concept lattice

We stated in Section 2, that the operator ∇ can be obtained from the $\triangle$ operator, as $X^\nabla = ((X^c)^\triangle)^c$, for all $X \subseteq B$. In this section, our aim is to introduce an adequate fuzzy extension of the operators involved in the previous equality. As we will rely on our previous definition of fuzzy sufficiency operator [27], our main task is to choose a convenient fuzzy extension of the dual sufficiency operator. In [12] a fuzzy extension of the dual sufficiency operator is introduced, but they make explicit use of a negation operator (defined on an underlying residuated lattice). As negation is somehow connected to the notion of duality, our aim here is to found the construction on this connection in order to avoid as much syntactic sugar as possible.

First of all, we need to recall the definition and notation of dual ordering. Given a set P and an ordering relation $\leq$ on P, the *dual* ordering of $\leq$ is the relation $\leq^\partial$, defined as $x_1 \leq^\partial x_2$ if and only if $x_2 \leq x_1$, for all $x_1, x_2 \in P$. Usually, we will write P instead of the partially ordered set $(P, \leq)$, similarly we will write P^∂ instead of $(P, \leq^\partial)$, and we will say that P^∂ is the *dual* of P.

Now, the notions of frame and context in this new environment must be defined.

Definition 5. A *dual multi-adjoint frame*, denoted $(L_1, L_2, P, \&_1, \dots, \&_n)^\partial$, is defined in terms of two complete lattices $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$, a poset $(P, \leq)$, and adjoint triples $(\&_i, \swarrow^i, \nwarrow_i)$ with respect to $L_1^\partial, L_2^\partial, P$, for all $i = 1, \dots, n$. That is, a dual multi-adjoint frame is a (standard) frame $(L_1, L_2, P, \preceq_1^\partial, \preceq_2^\partial, \leq, \&_1, \dots, \&_n)$.

The notion of context is exactly that given in the previous section.

From now on, we will fix a dual multi-adjoint frame, $(L_1, L_2, P, \&_1, \dots, \&_n)^\partial$ and context, (A, B, R, σ), in order to introduce the mappings which will build the dual multi-adjoint formal concept lattice, $^{\uparrow_\nabla} : L_2^B \to L_1^A$ and $^{\downarrow^\nabla} : L_1^A \to L_2^B$.

The mappings ${}^{\uparrow_\triangledown}\colon L_2^B \to L_1^A$ and ${}^{\downarrow^\triangledown}\colon L_1^A \to L_2^B$ are defined, given $g\colon B \to L_2$, $f\colon A \to L_1$, as

$$\begin{aligned} g^{\uparrow_\triangledown}(a) &= \sup{}_1\{R(a,b) \swarrow^{\sigma(b)} g(b) \mid b \in B\} \\ f^{\downarrow^\triangledown}(b) &= \sup{}_2\{R(a,b) \nwarrow_{\sigma(b)} f(a) \mid a \in A\} \end{aligned}$$

where $\sup_1$, $\sup_2$ are the supremum operators on L_1 and L_2, respectively.

It is worth to note that the previous definition is not just a change of infima by suprema in (3) and (4), since in that framework the adjoint triple is considered with regard to L_1, L_2 and P and in this case the triple is considered with regard to $L_1^\partial, L_2^\partial$ and P. Hence, the implications in equalities (3) and (4) are not those used here.

The following result shows that the definitions just introduced provide an equality very similar to that of the classical case.

Proposition 2. *The mappings* ${}^{\uparrow_\triangledown}\colon L_2^B \to L_1^A$ *and* ${}^{\downarrow^\triangledown}\colon L_1^A \to L_2^B$ *satisfy that* $g^{\uparrow_\triangledown} = ((g^\partial)^\uparrow)^\partial$, $f^{\downarrow^\triangledown} = ((f^\partial)^\downarrow)^\partial$, *for all* $g \in L_2^B$, $f \in L_1^A$, *where* $g^\partial\colon B \to L_2^\partial$ *and* $f^\partial\colon A \to L_1^\partial$, *respectively, map elements exactly as g and f (but their codomain is the opposite lattice); and the mappings* ${}^\uparrow$ *and* ${}^\downarrow$ *are defined by Equations (3) and (4).*

Proof. By applying the definition of ${}^\uparrow\colon (L_2^\partial)^B \to (L_1^\partial)^A$, ${}^\downarrow\colon (L_1^\partial)^A \to (L_2^\partial)^B$ as in Equations (3) and (4), we get

$$\begin{aligned} (g^\partial)^\uparrow(a) &= \inf{}_1^\partial\{R(a,b) \swarrow^{\sigma(b)} g^\partial(b) \mid b \in B\} \\ (f^\partial)^\downarrow(b) &= \inf{}_2^\partial\{R(a,b) \nwarrow_{\sigma(b)} f^\partial(a) \mid a \in A\} \end{aligned}$$

where we recall that $(\&_i, \swarrow^i, \nwarrow_i)$ are adjoint triples on L_1^∂, L_2^∂ and P, and $\inf_1^\partial$, $\inf_2^\partial$ are the infimum operators on L_1^∂ and L_2^∂, respectively.

Finally, a new dualization step transforms the infima on L_i^∂ into suprema on L_i.

The next result shows some properties of the mappings ${}^{\uparrow_\triangledown}$, ${}^{\downarrow^\triangledown}$.

Proposition 3. 1. ${}^{\uparrow_\triangledown}\colon L_2^B \to L_1^A$ *and* ${}^{\downarrow^\triangledown}\colon L_1^A \to L_2^B$ *are order-reversing.*

2. ${}^{\uparrow_\triangledown\downarrow^\triangledown}\colon L_2^B \to L_2^B$, ${}^{\downarrow^\triangledown\uparrow_\triangledown}\colon L_1^A \to L_1^A$ *are interior operators.*
3. $g^{\uparrow_\triangledown\downarrow^\triangledown\uparrow_\triangledown} = g^{\uparrow_\triangledown}$, $f^{\downarrow^\triangledown\uparrow_\triangledown\downarrow^\triangledown} = f^{\downarrow^\triangledown}$, *for all* $g \in L_2^B$, $f \in L_1^A$.

Proof. Consider $g_1, g_2 \in L_2^B$ such that $g_1 \preceq_2 g_2$, then $g_2^\partial \preceq_2^\partial g_1^\partial$. Applying ${}^\uparrow$, we obtain $(g_1^\partial)^\uparrow \preceq_2^\partial (g_2^\partial)^\uparrow$, which is equivalent to $((g_2^\partial)^\uparrow)^\partial \preceq_2 ((g_1^\partial)^\uparrow)^\partial$, that is $g_2^{\uparrow_\triangledown} \preceq_2 g_1^{\uparrow_\triangledown}$. Thus, ${}^{\uparrow_\triangledown}$ is order-reversing. The proof for ${}^{\downarrow_\triangledown}$ follows similarly.

Now, we will prove that ${}^{\uparrow_\triangledown\downarrow^\triangledown}$ is an interior operator, the other composition can be proved analogously.

Given $g \in L_2^B$, as ${}^{\uparrow\downarrow}$ is a closure operator, we have that $g^\partial \preceq_2^\partial (g^\partial)^{\uparrow\downarrow}$, that is equivalent to $((g^\partial)^{\uparrow\downarrow})^\partial \preceq_2 g$, therefore

$$g^{\uparrow_\triangledown\downarrow^\triangledown} = (((g^{\uparrow_\triangledown})^\partial)^\downarrow)^\partial = (((((g^\partial)^\uparrow)^\partial)^\partial)^\downarrow)^\partial = ((g^\partial)^{\uparrow\downarrow})^\partial \preceq_2 g$$

Property (3) follows directly from Property (2).

As a consequence of the previous result, the pair $(^{\uparrow_\triangledown}, {}^{\downarrow^\triangledown})$ is not a Galois connection but satisfies that $^{\uparrow_\triangledown\downarrow^\triangledown}$ and $^{\downarrow^\triangledown\uparrow_\triangledown}$ are interior operators, and this is enough to form a concept lattice.

Definition 6. 1. A *dual concept* is a pair $\langle g, f\rangle$ such that $g \in L_2^B, f \in L_1^A$ and the equations $g^{\uparrow_\triangledown} = f$ and $f^{\downarrow^\triangledown} = g$ hold.

2. Given a dual multi-adjoint frame $(L_1, L_2, P, \&_1, \dots, \&_n)^\partial$ and a context (A, B, R, σ), a *dual multi-adjoint concept lattice* is the pair $(\mathcal{M}^\triangledown, \leq^\triangledown)$, where

$$\mathcal{M}^\triangledown = \{\langle g, f\rangle \mid g \in L_2^B, f \in L_1^A \text{ and } g^{\uparrow_\triangledown} = f, f^{\downarrow^\triangledown} = g\}$$

is the set of dual concepts, and $\leq^\triangledown$ is the order defined $\langle g_1, f_1\rangle \leq^\triangledown \langle g_2, f_2\rangle$ if and only if $g_1 \preceq_2 g_2$ (or, equivalently, $f_2 \preceq_1 f_1$).

As $(^{\uparrow_\triangledown}, {}^{\downarrow^\triangledown})$ is not a Galois connection, the proof that $(\mathcal{M}^\triangledown, \leq^\triangledown)$ is indeed a complete lattice does not follow the usual approach. In order to prove this fact, we will consider an auxiliary multi-adjoint concept lattice.

The dual multi-adjoint frame $(L_1, L_2, P, \&_1, \dots, \&_n)^\partial$ provides adjoint triples $(\&_i, \swarrow^i, \nwarrow_i)$ defined on L_1^∂, L_2^∂ and P. Therefore, we can consider a *standard* multi-adjoint concept with respect to the Galois connection $(^\uparrow, {}^\downarrow)$ as a pair $\langle g^\partial, f^\partial\rangle$, where $g^\partial \in (L_2^\partial)^B$ and $f^\partial \in (L_1^\partial)^A$, and verifying $(g^\partial)^\uparrow = f^\partial$ and $(f^\partial)^\downarrow = g^\partial$. Hence, the following set

$$\mathcal{M}' = \{\langle g^\partial, f^\partial\rangle \mid \text{ if } \langle g^\partial, f^\partial\rangle \text{ is a multi-adjoint concept}\}$$

endowed with the ordering relation, $(g_1^\partial, f_1^\partial) \leq (g_2^\partial, f_2^\partial)$ if and only if $g_1^\partial \preceq_2^\partial g_2^\partial$ (or, equivalently, $f_2^\partial \preceq_1^\partial f_1^\partial$) is a complete lattice.

The following proposition relates the concept lattice above to dual multi-adjoint concept lattices, justifying why the name of "dual multi-adjoint concept lattice" has been considered for this new construction.

Proposition 4. *Let $(L_1, L_2, P, \&_1, \dots, \&_n)^\partial$ be a dual multi-adjoint frame, (A, B, R, σ) a context and $(\mathcal{M}', \leq)$, $(\mathcal{M}^\triangledown, \leq^\triangledown)$ as defined above. Then, $\langle g, f\rangle \in (\mathcal{M}^\triangledown, \leq^\triangledown)$, if and only if $\langle g^\partial, f^\partial\rangle \in (\mathcal{M}', \leq)$. Moreover, given $\langle g_1, f_1\rangle, \langle g_2, f_2\rangle \in (\mathcal{M}^\triangledown, \leq^\triangledown)$ we obtain that $\langle g_1, f_1\rangle \leq^\triangledown \langle g_2, f_2\rangle$ if and only if $\langle g_2^\partial, f_2^\partial\rangle \leq \langle g_1^\partial, f_1^\partial\rangle$.*

Proof. Given $\langle g, f\rangle \in (\mathcal{M}^\triangledown, \leq^\triangledown)$, then $f = g^{\uparrow_\triangledown} = ((g^\partial)^\uparrow)^\partial$, and, by duality, we have $f^\partial = (((g^\partial)^\uparrow)^\partial)^\partial = (g^\partial)^\uparrow$. Consequently, $\langle g^\partial, f^\partial\rangle \in (\mathcal{M}', \leq)$.

Now, we assume that $\langle g^\partial, f^\partial\rangle \in (\mathcal{M}', \leq)$, then $f^\partial = (g^\partial)^\uparrow$, and, by duality, we have $f = (f^\partial)^\partial = ((g^\partial)^\uparrow)^\partial = g^{\uparrow_\triangledown}$. Therefore, $\langle g, f\rangle \in (\mathcal{M}^\triangledown, \leq^\triangledown)$.

Finally, given $\langle g_1, f_1\rangle, \langle g_2, f_2\rangle \in (\mathcal{M}^\triangledown, \leq^\triangledown)$, the following chain of equivalences are obtained:

$$\langle g_1, f_1\rangle \leq^\triangledown \langle g_2, f_2\rangle \quad \text{iff} \quad g_1 \preceq_2 g_2 \quad \text{iff} \quad g_2^\partial \preceq_2^\partial \; g_1^\partial \quad \text{iff} \quad \langle g_2^\partial, f_2^\partial\rangle \leq \langle g_1^\partial, f_1^\partial\rangle$$

which finishes the proof.

As a consequence of this result, we obtain that $(\mathcal{M}^\triangledown, \leq^\triangledown)$ is, indeed, a complete lattice.

Theorem 1. *Given a dual multi-adjoint frame $(L_1, L_2, P, \&_1, \dots, \&_n)^\partial$ and a context (A, B, R, σ), the dual multi-adjoint formal concept lattice $(\mathcal{M}^\nabla, \preceq^\nabla)$ is a complete lattice, where the infimum and supremum operators are defined as*

$$\begin{aligned} \inf\{\langle g_i, f_i\rangle \mid i \in I\} &= \langle (\inf_2\{g_i \mid i \in I\})^{\uparrow_\triangledown \downarrow^\nabla}, \sup_1\{f_i \mid i \in I\}\rangle \\ \sup\{\langle g_i, f_i\rangle \mid i \in I\} &= \langle \sup_2\{g_i \mid i \in I\}, (\inf_1\{f_i \mid i \in I\})^{\downarrow^\nabla \uparrow_\triangledown}\rangle \end{aligned}$$

Proof. Considering the multi-adjoint concept lattice $(\mathcal{M}', \preceq)$, defined above and used in Proposition 4, Equations (5) and (6) are written in this framework as:

$$\inf\{\langle g_i^\partial, f_i^\partial\rangle \mid i \in I\} = \langle \inf_2^\partial\{g_i^\partial \mid i \in I\}, (\sup_1^\partial\{f_i^\partial \mid i \in I\})^{\downarrow\uparrow}\rangle \tag{7}$$
$$\sup\{\langle g_i^\partial, f_i^\partial\rangle \mid i \in I\} = \langle (\sup_2^\partial\{g_i^\partial \mid i \in I\})^{\uparrow\downarrow}, \inf_1^\partial\{f_i^\partial \mid i \in I\}\rangle \tag{8}$$

where $\sup_j^\partial$ and $\inf_j^\partial$ are the supremum and infimum on L_j^∂, respectively, with $j \in \{1, 2\}$. Therefore, by Proposition 4, Expressions (7) and (8) are equivalent to:

$$\begin{aligned} \sup\{\langle g_i, f_i\rangle \mid i \in I\} &= \langle \sup_2\{g_i \mid i \in I\}, (\inf_1\{f_i \mid i \in I\})^{\downarrow^\nabla \uparrow_\triangledown}\rangle \\ \inf\{\langle g_i, f_i\rangle \mid i \in I\} &= \langle (\inf_2\{g_i \mid i \in I\})^{\uparrow_\triangledown \downarrow^\nabla}, \sup_1\{f_i \mid i \in I\}\rangle \end{aligned}$$

for each family of dual concepts $\langle g_i, f_i\rangle \in (\mathcal{M}^\nabla, \preceq^\nabla)$, with i in an index set I, which leads us to assure that the pair $(\mathcal{M}^\nabla, \preceq^\nabla)$ is a complete lattice.

The following proposition proves the consistency of our approach in relation to the classical crisp case. Specifically, considering $L_1 = L_2 = P = \{0, 1\}$, we obtain $g^{\uparrow_\triangledown} = g^\nabla$ and $f^{\downarrow^\nabla} = f^\nabla$, for all g and f crisp subsets of X and A, respectively. The formal statement and proof are given below:

Proposition 5. *Given $L_1 = L_2 = P = \{0, 1\}$, a dual multi-adjoint frame $(L_1, L_2, P, \&_1, \dots, \&_n)^\partial$ and a context (A, B, R, σ), we have that $g^{\uparrow_\triangledown} = g^\nabla$ and $f^{\downarrow^\nabla} = f^\nabla$, for all crisp subsets g and f of X and A, respectively.*

Proof. As $L_1 = L_2 = P = \{0, 1\}$, only one adjoint triple can be considered, which corresponds to the classical conjunction and implication (hence no need to specify the σ mapping), and only one implication (denoted $\swarrow$) is assumed since the classical conjunction is commutative. The difference is that now, they are considered to be defined on $\{0, 1\}^\partial$, specifically, $\&\colon \{0, 1\}^\partial \times \{0, 1\}^\partial \to \{0, 1\}$ and $\swarrow\colon \{0, 1\} \times \{0, 1\}^\partial \to \{0, 1\}^\partial$.

Given a subset g of X, that is, $g\colon X \to \{0, 1\}$, the equality $g^{\uparrow_\triangledown} = g^\nabla$ will be proved. The other equality follows similarly.

We will prove that, for all $a \in A$, $g^{\uparrow_\triangledown}(a) = 1$ if and only if $g^\nabla(a) = 1$, which finishes the proof.

Given $a \in A$, $g^{\uparrow_\triangledown}(a) = \sup_1\{R(a, b) \swarrow g^\partial(b) \mid b \in B\} = 1$, if and only if $\inf_1^\partial\{R(a, b) \swarrow g^\partial(b) \mid b \in B\} = 0$. Hence, from the definition of the classical implication $\swarrow$, the value 0 can only be obtained from $(0 \swarrow 1)$, that is, if $g^\partial(b) = 1$ and $R(a, b) = 0$, which is equivalent to the existence of $b \in X$ which is not an element of g, e.g. $g(b) = 0$, and that a and b are not related, therefore, $g^\nabla(a) = 1$. As the procedure explained is a chain of equivalences, then the equality holds.

A worked out example

In the following, we show how the new "dual" concepts are generated in this dual framework.

To begin with, let us fix the dual multi-adjoint frame and context of the example. We will assume that $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and P are the regular 5-partition $[0,1]_4 = \{0, 0.25, 0.50, 0.75, 1\}$ with the usual ordering. Let us consider the adjoint triple $(\&, \swarrow, \nwarrow)$, where $\&\colon [0,1]_4^\partial \times [0,1]_4^\partial \to [0,1]_4$, $\swarrow\colon [0,1]_4 \times [0,1]_4^\partial \to [0,1]_4^\partial$ and $\nwarrow\colon [0,1]_4 \times [0,1]_4^\partial \to [0,1]_4^\partial$ are defined as

&	1	0.75	0.50	0.25	0
1	0	0	0	0	0
0.75	0	0	0	0	0.25
0.50	0	0	0	0	0.25
0.25	0	0	0.25	0.50	0.75
0	0	0.25	0.50	0.75	1

$\swarrow$	1	0.75	0.50	0.25	0
0	0	0.25	0.50	0.50	1
0.25	0	0	0.25	0.50	0.50
0.50	0	0	0	0.25	0.50
0.75	0	0	0	0	0.25
1	0	0	0	0	0

$\nwarrow$	1	0.75	0.50	0.25	0
0	0	0.25	0.25	0.75	1
0.25	1	1	1	0.50	0.25
0.50	0	0	0	0.25	0.50
0.75	0	0	0	0	0.25
1	0	0	0	0	0

which might be obtained from examples [11].

We have that $(\&, \swarrow, \nwarrow)$ is an adjoint triple; notice that the operator $\&$ is not commutative because

$$0.25 \,\&\, 0.50 = 0.25 \neq 0 = 0.50 \,\&\, 0.25$$

and its two adjoint implications do not coincide.

Recall that the adjoint triple is defined with respect to $[0,1]_4^\partial$, $[0,1]_4^\partial$, $[0,1]_4$. As a result, if $\&$ is considered on the partitioned interval with the usual ordering, we have that $\&\colon [0,1]_4 \times [0,1]_4 \to [0,1]_4$ is decreasing in both arguments. Thus, although not explicitly, with this approach we are assuming that conjunctors have "negation" operators in both arguments, in a not restrictive way, in order to build a new concept lattice.

Hence, the dual multi-adjoint frame considered will be $([0,1]_4, [0,1]_4, [0,1]_4, \&)^\partial$ and the context (A, B, R, σ) will be given by the sets $A = \{a_1, a_2\}$ and $B = \{b_1, b_2, b_3\}$, the constant mapping σ and the fuzzy relation R, defined in Table 1.

Table 1: Fuzzy relation between objects and attributes

R	b_1	b_2	b_3
a_1	0.75	0.50	0
a_2	0.25	0.25	1

Now, we will compute a dual concept from an arbitrary fuzzy subset of B. Given the fuzzy subset of objects $g\colon B \to [0,1]_4$, defined as $g(b_1) = 0$, $g(b_2) = 0.5$ and $g(b_3) = 1$, the greatest concept that "embeds" g is $(g^{\uparrow_\triangledown \downarrow^\triangledown}, g^{\uparrow_\triangledown})$, is obtained below. Firstly,

$$\begin{aligned}
g^{\uparrow_\triangledown}(a_1) &= \sup\nolimits_1\{R(a_1, b_j) \swarrow^{\sigma(b)} g^\partial(b_j) \mid b_j \in B\} \\
&= \sup\nolimits_1\{0.75 \swarrow 0,\ 0.50 \swarrow 0.50,\ 0 \swarrow 1\} = \sup\nolimits_1\{0.25, 0, 0\} = 0.25 \\
g^{\uparrow_\triangledown}(a_2) &= \sup\nolimits_1\{0.25 \swarrow 0,\ 0.25 \swarrow 0.50,\ 1 \swarrow 1\} = \sup\nolimits_1\{0.50, 0.25, 0\} = 0.50
\end{aligned}$$

This mapping is used to compute $g^{\uparrow_\triangledown\downarrow^\triangledown}$:

$$\begin{aligned}
g^{\uparrow_\triangledown\downarrow^\triangledown}(b_1) &= \sup_1\{R(a_i, b_1) \nwarrow_{\sigma(b)} g^{\uparrow}(a_i) \mid a_i \in A\}\\
&= \sup_1\{0.75 \nwarrow 0.25, 0.25 \nwarrow 0.50\} = \sup_1\{0, 0\} = 0\\
g^{\uparrow_\triangledown\downarrow^\triangledown}(b_2) &= \sup_1\{0.50 \nwarrow 0.25, 0.25 \nwarrow 0.50\} = \sup_1\{1/4, 0\} = 0.25\\
g^{\uparrow_\triangledown\downarrow^\triangledown}(b_3) &= \sup_1\{0 \nwarrow 0.25, 1 \nwarrow 0.25\} = \sup_1\{0.75, 0\} = 0.75
\end{aligned}$$

We can check that $g^{\uparrow_\triangledown\downarrow^\triangledown} \preceq^\triangledown g$ since ${}^{\uparrow_\triangledown\downarrow^\triangledown}$ is an interior operator (the opposite character of ${}^{\uparrow\downarrow}$, which is a closure operator).

Using the provided relation in this paper, an adaptation of the algorithms given in the fuzzy formal concept analysis framework, such as the ones given in [5, 6, 22], can be applied to obtain the dual multi-adjoint concept lattice.

Notice that this concept lattice cannot be obtained via an easy transformation of operators in order to make them increasing in $[0,1]_4$ and to apply the multi-adjoint formal concept analysis theory to obtain the dual multi-adjoint concept lattice. The most natural approach to attain & increasing would be to consider $\&\colon [0,1]_4 \times [0,1]_4 \to [0,1]_4^\partial$, as it would be increasing in both arguments, but it would not be an adjoint conjunctor since, for instance, we have $0 \,\&\, 0.25 = 0.75$ but, by the adjoint property, the result should be the minimum element of the underlying lattice, namely, 1. Another possibility to make & increasing could be to consider $\&'\colon [0,1]_4 \times [0,1]_4 \to [0,1]_4$, defined as $x \,\&'\, y = (1-x) \,\&(1-y)$, but this requires the actual existence of an unary negation operator, which we try to avoid.

Our approach constructs directly concept lattices by using a novel approach in terms of "decreasing conjunctors". This approach is expected to have practical applications similarly to (multi-adjoint) property-oriented concept lattices and object-oriented concept lattices [24, 25], which are other variants of multi-adjoint formal concept analysis.

5. Conclusions and future work

We have generalized the classical dual concept lattices to a multi-adjoint environment in which we can use different adjoint triples defined on non-linear sets and no negation is needed.

The approach is interesting in that, up to now, the introduction of dual concept lattices has always been introduced in terms of an explicit negation operator on the underlying carrier. From the theoretical point of view, our approach is based on two derivation operators whose both compositions lead to an interior operator, in contrast to previous approaches in which at least one of these compositions should be a closure operator.

In addition, our proposal enables a new point of view to obtain information from databases with incomplete and/or imprecise information, which provide more flexibility to existing procedures. Indeed, some times it could be more efficient to compute the dual concept than the standard one. In the future, applications and practical examples of the approach introduced in this paper will be extensively studied.

References

[1] C. Alcalde, A. Burusco, R. Fuentes-González, I. Zubia, Treatment of L-fuzzy contexts with absent values, Information Sciences 179 (2009) 1–15.

[2] C. Alcalde, A. Burusco, R. Fuentes-González, I. Zubia, The use of linguistic variables and fuzzy propositions in the L-fuzzy concept theory, Computers & Mathematics with Applications 62 (8) (2011) 3111 – 3122.

[3] R. Bělohlávek, Fuzzy concepts and conceptual structures: induced similarities, in: Joint Conference on Information Sciences, 1998, pp. 179–182.

[4] R. Bělohlávek, Concept lattices and order in fuzzy logic, Annals of Pure and Applied Logic 128 (2004) 277–298.

[5] R. Bělohlávek, B. De Baets, J. Outrata, V. Vychodil, Lindig's algorithm for concept lattices over graded attributes, Lecture Notes in Computer Science 4617 (2007) 156–167.

[6] R. Bělohlávek, B. De Baets, J. Outrata, V. Vychodil, Computing the lattice of all fixpoints of a fuzzy closure operator, IEEE Transactions on Fuzzy Systems 18 (3) (2010) 546–557.

[7] A. Burusco, R. Fuentes-González, The study of L-fuzzy concept lattice, Mathware & Soft Computing 3 (1994) 209–218.

[8] X. Chen, Q. Li, Construction of rough approximations in fuzzy setting., Fuzzy Sets and Systems 158 (23) (2007) 2641–2653.

[9] Y. Chen, Y. Yao, A multiview approach for intelligent data analysis based on data operators, Information Sciences 178 (1) (2008) 1–20.

[10] A. Ciaramella, W. Pedrycz, R. Tagliaferri, The genetic development of ordinal sums, Fuzzy Sets and Systems 151 (2005) 303–325.

[11] A. Ciaramella, W. Pedrycz, R. Tagliaferri, Or/and neurons for fuzzy set connectives using ordinal sums and genetic algorithms, Lect. Notes in Artificial Intelligence 3849 (2006) 188–194.

[12] Y. Djouadi, H. Prade, Possibility-theoretic extension of derivation operators in formal concept analysis over fuzzy lattices, Fuzzy Optimization and Decision Making 10 (4) (2011) 287–309.

[13] D. Dubois, H. Prade, Putting fuzzy sets and rough sets together, in: R. Słowiński (ed.), Intelligent Decision Support, Kluwer Academic, Dordrecht, 2004, pp. 203–232.

[14] I. Düntsch, G. Gediga, Approximation operators in qualitative data analysis, in: Theory and Applications of Relational Structures as Knowledge Instruments, 2003, pp. 214–230.

[15] B. Ganter, R. Wille, Formal Concept Analysis: Mathematical Foundation, Springer Verlag, 1999.

[16] G. Gediga, I. Düntsch, Modal-style operators in qualitative data analysis, in: Proc. IEEE Int. Conf. on Data Mining, 2002, pp. 155–162.

[17] G. Georgescu, A. Popescu, Non-dual fuzzy connections, Arch. Math. Log. 43 (8) (2004) 1009–1039.

[18] P. Hájek, Metamathematics of Fuzzy Logic, Trends in Logic, Kluwer Academic, 1998.

[19] E. Klement, R. Mesiar, E. Pap, Triangular norms, Kluwer academic, 2000.

[20] S. Krajči, A generalized concept lattice, Logic Journal of IGPL 13 (5) (2005) 543–550.

[21] H. Lai, D. Zhang, Concept lattices of fuzzy contexts: Formal concept analysis vs. rough set theory, International Journal of Approximate Reasoning 50 (5) (2009) 695–707.

[22] C. Lindig, Fast concept analysis, in: G. Stumme (ed.), Working with Conceptual Structures-Contributions to ICCS 2000, 2000, pp. 152–161.

[23] G. L. Liu, Construction of rough approximations in fuzzy setting., Information Sciences 178 (6) (2008) 1651–1662.

[24] J. Medina, Towards multi-adjoint property-oriented concept lattices, Lect. Notes in Artificial Intelligence 6401 (2010) 159–166.

[25] J. Medina, Multi-adjoint property-oriented concept lattices, Information Sciences 190 (2012) 95–106.

[26] J. Medina, M. Ojeda-Aciego, Multi-adjoint t-concept lattices, Information Sciences 180 (5) (2010) 712–725.

[27] J. Medina, M. Ojeda-Aciego, J. Ruiz-Calviño, Formal concept analysis via multi-adjoint concept lattices, Fuzzy Sets and Systems 160 (2) (2009) 130–144.

[28] J. Medina, M. Ojeda-Aciego, P. Vojtáš, Similarity-based unification: a multi-adjoint approach, Fuzzy Sets and Systems 146 (2004) 43–62.

[29] E. Mephu Nguifo, V. Duquenne, M. Liquière, Introduction concept lattice-based theory, methods and tools for knowledge discovery in databases: Applications, Applied Artificial Intelligence 17 (3) (2003) 177–180.

[30] A. Mora, I. P. de Guzmán, M. Enciso, P. Cordero, Ideal non-deterministic operators as a formal

framework to reduce the key finding problem, International Journal of Computer Mathematics 88 (9) (2011) 1860–1868.
[31] S. Pollandt, Fuzzy Begriffe, Springer, Berlin, 1997.
[32] A. M. Radzikowska, E. E. Kerre, A comparative study of fuzzy rough sets, Fuzzy Sets and Systems 126 (2) (2002) 137–155.
[33] J. Villerd, S. Ranwez, M. Crampes, D. Carteret, Using concept lattices for visual navigation assistance in large databases, International Journal of General Systems 38 (4) (2009) 405–425.
[34] L. Wang, X. Liu, Concept analysis via rough set and AFS algebra, Information Sciences 178 (21) (2008) 4125–4137.
[35] X. Wang, W. Zhang, Relations of attribute reduction between object and property oriented concept lattices, Knowledge-Based Systems 21 (5) (2008) 398–403.
[36] R. Wille, Restructuring lattice theory: an approach based on hierarchies of concepts, in: I. Rival (ed.), Ordered Sets, Reidel, 1982, pp. 445–470.
[37] Y. Y. Yao, Y. Chen, Rough set approximations in formal concept analysis, in: Transactions on Rough Sets V, vol. 4100 of Lect. Notes in Computer Science, 2006, pp. 285–305.
[38] H. Zimmermann, P. Zysno, Latent connectives in human decision making, Fuzzy Sets and Systems 4 (1980) 37–51.