# A coalgebraic approach to non-determinism: applications to multilattices

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**Abstract**

Multilattices are a suitable generalization of lattices which enables to accommodate the formalization of non-deterministic computation; specifically, the algebraic characterization for multilattices provides a formal framework to develop tools in several fields of computer science. On the other hand, the usefulness of coalgebra theory has been increasing in the recent years, and its importance is undeniable. In this paper, somehow mimicking the use of universal algebra, we define a new kind of coalgebras (the ND-coalgebras) that allows to formalize non-determinism, and show that several concepts, widely used in computer science are, indeed, ND-coalgebras. Within this formal context, we study a minimal set of properties which provides a coalgebraic definition of multilattices.

*Key words:* Lattices, Coalgebras, Non-determinism, Multilattices

## 1. Introduction

The notion of multilattice was introduced by Benado [4], as an extension of the concept of lattice by means of multi-suprema (minimal upper bounds) and multi-infima (maximal lower bounds), and a suggestion to further develope the theory of multilattices appears in [36].

Although its original motivation was purely theoretical, multilattices (and relatives such as multisemilattices) have been identified in several disparate research areas: (1) in the field of automated deduction, specifically

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when devising a theory about implicates and implicants for certain temporal logics during the development of automated theorem provers for those logics [13]; (2) unification for logical systems, whose starting point was the existence of a most general unifier for any unifiable formula in Boolean logic: in 1999, Ghilardi [18] proved that there are no most general unifiers in intuitionistic propositional calculus but instead there is a finite set of maximal general unifiers; (3) sorted multi-adjoint logic programming: in [17] a tabulation algorithm was presented which terminates under very general conditions and could only fail provided that the underlying structure of the set of truth-values is at least as general as a multilattice. Apart form the mere identification of the structure of multilattice in existing research areas, the practical use of multilattices has already started by introducing a fuzzy logic programming framework which can be proved to embed several known approaches to logic programming with imperfect (that is, either uncertain, or imprecise, or absent) information [33].

The first applicable algebraic characterization is relatively recent [30, 31], and it reflects much better the corresponding classical theory about lattices than those given in [4, 5, 6, 20, 24, 25]. Since then, several works have been published about the mathematical theory of multilattices [10, 14, 42] and, in general, about hyperstructures and non-deterministic structures [9, 11, 16]. It is convenient to state that, in the meantime, several other generalizations of the notion of lattice have been developed so far: for instance, *nearlattices* [12], *near lattices* [39], *hyperlattices* [29], or *superlattices* [35].

Several lattice-like structures such as *chopped lattices* or *partial lattices*, or *nearlattices*, are also included in the notion of multilattice. On the other hand, alternative generalizations of lattices exist which do not follow our underlying motivation (see below): a *near lattice* is a set with two operations with weaker forms of associativity and commutativity; *superlattices* are closer to multilattices in that they can be considered generalizations of them, at the price of losing the one-to-one correspondence between the order-based and the algebraic formulations; finally, *hyperlattices* are a particular case of superlattice in which only the join is a hyperoperation.

We are focusing our attention on multilattices since we believe their computational properties are better suited to the aims stated as follows: The idea underlying the algebraic study of multilattices is the development of a new theory involving *non-deterministic operators* as a framework for formalizing key notions in computer science and artificial intelligence; For instance, [41] discusses how the study of non-determinism is useful for natural language processing; in [2] non-determinism is considered under the combination of modal and temporal logics to be used in communication systems; new results have been obtained in database theory [15]. Certainly, a lot of effort is being put in this area, as one can still see recent works dealing with non-determinism both from the theoretical and from the practical point of view [26, 28, 43].

Last but not least, this paper is concerned with coalgebras. In his paper [38], Rutten developed the theory of coalgebras which can be seen as a sort of dualization of universal algebra, when considered from a category theoretical standpoint. This theory is becoming an ideal framework for formalization in diverse branches of computer science. Specifically, concepts as important as Kripke structures, labeled transition systems, various types of automata (in particular, non-deterministic automata), reactive systems, causal maps, ambient calculus, services and contracts, have a coalgebraic explanation [7, 8, 19, 21, 23, 27, 34, 40, 44].

Certain abstract structures can be thought of both algebraically and coalgebraically. The context and the aims of the work usually indicates which framework one should consider; for instance, when non-deterministic behavior is assumed, the coalgebraic framework is generally preferred because it appears to fit more naturally. Following this trend, we started a research line consisting in developing a coalgebraic view of several mathematical structures of interest for the handling of non-determinism, in particular, for multilattices.

A particular case in which this situation arises is presented in the paragraphs below.

A typical example of coalgebra is the non-deterministic automaton in which, in its simplest version, we have a set of states $S$ and a transition function between states $S \to \mathcal{P}(S)$. Now, let us consider that such an automaton corresponds to an agent within a multiagent framework containing $n+1$ agents interacting. Each agent changes its state depending on its own state and the state of the rest of the agents. Thus, the transition function between states would be of type $S^{n+1} \to \mathcal{P}(S)$. However, the agent knows its own state whereas the rest of states have to be consulted, in such a way that the transition function can be considered of type $S \to \mathcal{P}(S)^{S^n}$. As a result, the properties of the transition function can be separated into two levels: those known to the agent, and those to be consulted. Note that the transformation from $S^{n+1} \to \mathcal{P}(S)$ to $S \to \mathcal{P}(S)^{S^n}$ is just an instance of the currying process (or partial application), which transforms a function that takes a tuple of arguments in such a way that it can be called as a chain of functions each with a single argument.

Let us consider now that these agents are working on DNA chains, and assume that interaction between agents relies on the obtention of common sub- or sup-chains. It is easy to check that the set of chains $\mathcal{W}$ with the relation "to be a substring of" has structure of multilattice where the multi-infima and multi-suprema determine functions of the type $\mathcal{W}\times\mathcal{W} \to \mathcal{P}(\mathcal{W})$. However, it will be interesting to be able to distinguish between the properties of multilattices which can be used by using the agent's knowledge, and those in which extra information is needed. Our contribution in this paper can be seen within this framework as a way to deal with operations which provide the multi-infima and multi-suprema as functions of the type $\mathcal{W} \to \mathcal{P}(\mathcal{W})^{\mathcal{W}}$.

In this work we introduce the *ND-coalgebras* as a class of coalgebras arising from non-determinism, and apply them to formalize the coalgebraic approach of multilattices. We shall use a particular collection of types, the *ND-functors*, that can be built inductively from the identity and the powerset functors.

## 2. Nd-groupoids, multisemilattices, multilattices

In order to introduce the notion of multilattice as an ordered structure, firstly it is necessary to define the concept of multi-supremum as an extension of supremum (resp. multi-infimum). In a partially ordered set (poset) a *multi-supremum* of a subset $B$ is a minimal element of the set of upper bounds of $B$ and $\mathrm{Msup}(B)$ denotes the set of multi-suprema of $B$; the notion of *multi-infima* is defined similarly.

The definition of multilattice will be based on that of multisemilattice, which is given below:

**Definition 1.** A *join-multisemilattice* is a poset $(M, \leq)$ in which, for all $a, b, x \in M$ with $a \leq x$ and $b \leq x$, there exists $z \in \mathrm{Msup}(\{a, b\})$ such that $z \leq x$. The dual concept of a join-multisemilattice is called *meet-multisemilattice*.

Similarly to what happens in the theory of lattices, a poset $(M, \leq)$ is said to be a *multilattice* if it is a join and meet-multisemilattice.

Note that the definition is consistent with the existence of two incomparable elements *without* any multi-supremum or multi-infimum. In other words, Msup($\{a, b\}$) and Minf($\{a, b\}$) can be empty. Moreover, if Msup($\{a, b\}$) and Minf($\{a, b\}$) are singletons for all $\{a, b\}$, then $(M, \leq)$ is a lattice, which implies that the multilattices are more general structures than lattices.

In the concept of ordered multisemilattice, minimal upper bounds (multi-suprema) play the role of least upper bounds in a lattice (analogously for the dual). The main difference that one notices is that the operators which compute multi-suprema are not single-valued, since there may be several multi-suprema or may be none.

By abstracting out the algebraic properties of the multi-suprema (or multi-infima) we obtain a more general hyperstructure of hypergroupoid, that is, a set $H$ together with a set-valued operation $H \times H \to \mathcal{P}(H) \smallsetminus \{\varnothing\}$. Notice, however, that the output of the Msup operator can be empty, therefore, it is convenient to drop the non-emptiness restriction in the codomain of the hyperoperation.

**Definition 2.** A *non-deterministic* (nd, for short) *groupoid* is a pair $(A, F)$ consisting of a non-empty set $A$ and a mapping $F \colon A \times A \to \mathcal{P}(A)$, where $F$ is called *nd-operation.*

Hyperstructure theory was initiated by Marty [32], who introduced hypergroups. Nowadays, a number of different hyperstructures are widely studied both from the theoretical point of view and for their use in applied mathematics and artificial intelligence. Hyperstructures generalize (total) algebraic structures and nd-structures provides a greater level of generality by allowing to include, as particular cases, *partial* algebraic structures (for instance, partial lattices).

**Notation:** Given an nd-groupoid $(A, F)$, we will use the following conventions:

- If $a \in A$ and $X \subseteq A$, then $F(a, X) = \{F(a, x) \mid x \in X\}$ and $F(X, a) = \{F(x, a) \mid x \in X\}$. In particular, $F(a, \varnothing) = F(\varnothing, a) = \varnothing$.
- When the result of the nd-operation is a singleton, we will often omit the braces, i.e., we just use $a$ instead of $\{a\}$, when no confusion can arise.

Again following classical lattice theory, it is possible to give an algebraic version of multisemilattices as pairs $(M, F)$, where $M$ is a non-empty set and $F$ satisfies suitable conditions.

In order to show this alternative definition, we start by lifting some properties to the non-deterministic case.

- **Idempotency**: $F(a,a) = a$ for all $a \in A$.
- **Commutativity**: $F(a,b) = F(b,a)$ for all $a, b \in A$.
- **Weak associativity** if $F(a,b)$ is a singleton, then

  $$F(F(a,b),c) \subseteq F(a,F(b,c)) \qquad \text{and} \qquad F(c,F(a,b)) \subseteq F(F(c,a),b)$$

  for all $a, b, c \in A$

We will focus our interest on the binary relation defined by

$$a \leq b \quad \text{if and only if} \quad F(a,b) = b$$

In general, this relation is not an ordering. However, it is reflexive if the nd-groupoid is idempotent; it is antisymmetric if the nd-groupoid is commutative; finally, it is transitive if the nd-groupoid is weakly associative.

The two following properties of nd-groupoids, called **comparability**, have been shown to have an important role in multilattice theory, since they allow to prove the equivalence between the order-based and the algebraic definitions of these new concepts:

- $\mathbf{C}_1$: $c \in F(a,b)$ implies that $a \leq c$ and $b \leq c$.
- $\mathbf{C}_2$: $c, d \in F(a,b)$ and $c \leq d$ imply that $c = d$.

Now, we can give the algebraic definition of a *join-multisemilattice* as an nd-groupoid that satisfies idempotency, commutativity, weak associativity and the comparability laws. Dually, one gives the definition of algebraic meet-multisemilattice.

Both algebraic and order-based definitions of multisemilattice have been shown to be equivalent (see [31, Theorem 2.11]). As in lattice theory, if $(A, \leq)$ is an ordered join-multisemilattice, the corresponding algebraic multisemilattice has $F(a,b) = \mathrm{Msup}\{a,b\}$ as the nd-operation. Conversely, if $(A, F)$ is an algebraic join-multisemilattice, the binary relation in $A$ given by $a \leq b$ if and only if $F(a,b) = b$ defines the corresponding ordered join-multisemilattice. Analogously, for $(A, \leq)$ an ordered meet-multisemilattice, $G(a,b) = \mathrm{Minf}\{a,b\}$ gives the nd-operation which satisfies the required properties. Otherwise, if $(A, G)$ is an algebraic meet-multisemilattice, the binary relation $a \leq b$ if and only if $G(a,b) = a$ defines an ordered meet-multisemilattice.

For the algebraic characterisation of a multilattice, we need to connect both multisemilattices via an extension of the absorption property.

Let $F$ and $G$ be nd-operations in $A$, the pair $(F, G)$ is said to have the property of **absorption** if for all $a, b \in A$ the following conditions hold:

(*i*) $G(a, c) = a$ for all $c \in F(a, b)$.
(*ii*) $F(a, c) = a$ for all $c \in G(a, b)$.

An *algebraic multilattice*, $(A, F, G)$, is a set $A$ with two nd-operations $F$ and $G$ satisfying the absorption property and such that $(A, F)$ and $(A, G)$ are multisemilattices.

## 3. ND-coalgebras

Along this section, we shall assume that the reader has some familiarity with standard notions of category theory: in a few words, it is sufficient to recall that categories and functors can be seen, respectively, as suitable extensions of the classes of sets of algebraic structures, and as functions between categories. Other basic concepts and further information on category theory can be consulted in [1, 3].

A *type* (or *signature*) is a non-trivial endofunctor in the category of sets, $\mathcal{T} : Set \to Set$. A *coalgebra of type* $\mathcal{T}$ is a pair $(A, \alpha)$ consisting of a set $A$ and a mapping $\alpha : A \to \mathcal{T}(A)$.

The set $A$ and the mapping $\alpha$ in a coalgebra have received different names in the literature:

- $A$ is called either *carrier*, or *base*, or the set of *states* (the latter is used when a coalgebra is viewed as a system).
- Likewise, $\alpha$ is the *structured mapping*, the *coalgebra mapping*, the *operation* or the $\mathcal{T}$*-transition system* (*dynamics*) of the coalgebra.

**Example 3.** In an nd-groupoid $(A, F)$, the operation $F : A \times A \to \mathcal{P}(A)$ can be represented as a mapping $\lambda F : A \to \mathcal{P}(A)^A$ defined by $\lambda F(a)(b) = F(a, b)$. This process is known as *currying*. Thus, any nd-groupoid can be seen as a coalgebra. □

In order to determine the type of the coalgebras that allows to formalize the notions of multisemilattice and multilattice, we have to take into account that the currying process applied to nd-operations with arity $n \geq 1$, yields a mapping from the set $X$ to $\mathcal{P}(X)^{X^{n-1}}$ and this codomain can be considered isomorphic to $\mathcal{P}(X^n)$.

Several endofunctors can be defined between $X$ and $\mathcal{P}(X^n)$. These functors differ on how they behave on morphisms. The type of the coalgebras which we are concerned with, is inspired by the (covariant) *powerset functor* also called *direct* or *existential image functor*.

**Definition 4.** Given $n \in \mathbb{N}$, the functor $\mathcal{T}_n : Set \to Set$ is defined by

- if $X$ is a set then $\mathcal{T}_n(X) = \mathcal{P}(X^n)$
- if $f : X \to Y$ is a morphism then $\mathcal{T}_n(f) : \mathcal{P}(X^n) \to \mathcal{P}(Y^n)$ is the morphism given, for all $\mathcal{X} \subseteq X^n$, by

$$\mathcal{T}_n(f)(\mathcal{X}) = \{(f(x_1), \dots, f(x_n)) \mid (x_1, \dots, x_n) \in \mathcal{X}\}$$

To formalize the notion of nd-algebra in the coalgebraic framework, we need to introduce a specific family of functors.

**Definition 5.** The collection of *ND-functors* in the category *Set* is defined as the least collection NDF containing $\mathcal{T}_n$, for all $n \in \mathbb{N}$, and closed for the product of functors, that is, if $\mathfrak{T}$ is a subset of NDF, then the product $\prod_{\mathcal{T} \in \mathfrak{T}} \mathcal{T}$ is in NDF.

An *ND-coalgebra* is a coalgebra of type $\mathcal{T}$ where $\mathcal{T}$ is an ND-functor, namely, a pair $\mathcal{A} = (A, \alpha)$ where $\alpha$ is a mapping $\alpha : A \to \mathcal{T}(A)$.

**Remark 6.** Notice that the ND-functors can be seen as an instance of *Kripke polynomial functors* [22]. When considering the coalgebraic treatment of multilattices, however, we will only need the functors $\mathcal{T}_n$, $n \leq 2$.

**Example 7.** If $(A, F)$ is a multisemilattice, then $(A, \alpha)$ is an ND-coalgebra where $\alpha : A \to \mathcal{T}_2(A)$ and $\alpha = \lambda F$. That is,

$$\begin{aligned}&\alpha : A \to \mathcal{P}(A^2)\\&\alpha(a) = \{(b, c) \mid c \in F(a, b)\}\end{aligned}$$

Equivalently, we can also define $\alpha$ as follows:

$$\begin{aligned}&\alpha : A \to \mathcal{P}(A)^A,\\&\alpha(a) = \alpha_a : A \to \mathcal{P}(A) \text{ for all } a \in A \text{ and}\\&\alpha_a(b) = F(a, b) \text{ for all } b \in A\end{aligned}$$

Notice that $(A, \alpha_a)$ is an ND-coalgebra as well, for all $a \in A$. $\square$

Similarly to the result of the previous example, any multilattice $(A, F, G)$ can be seen as an ND-coalgebra $(A, \alpha)$, whose type is $\mathcal{T}_2 \times \mathcal{T}_2$, where

$$\begin{aligned}&\alpha : A \to \mathcal{P}(A^2) \times \mathcal{P}(A^2),\\&\alpha(a) = \{(b, c) \mid c \in F(a, b)\} \times \{(b, c) \mid c \in G(a, b)\}\end{aligned}$$

or, equivalently,

$$\alpha : A \to \mathcal{P}(A)^A \times \mathcal{P}(A)^A,$$
$$\alpha(a) = \alpha_a : A \to \mathcal{P}(A) \times \mathcal{P}(A) \text{ for all } a \in A \text{ and}$$
$$\alpha_a(b) = (F(a,b), G(a,b)) \text{ for all } b \in A$$

Notice that the latter expression shows that $(A, \alpha_a)$ is an ND-coalgebra of type $\mathcal{T}_1 \times \mathcal{T}_1$.

**Example 8 (Non-deterministic automata).** Any non-deterministic automaton is an ND-coalgebra: Let $\mathcal{S} = (S, I, F, A, \delta)$ be an automaton, where $S$ is the set of states, $I \subseteq S$ is the set of initial states, $F \subseteq S$ is the set of final states, $A$ is the set of inputs and $\delta \subseteq S \times A \times S$ is the transition relation. Let $i$ and $f$ be the characteristic function of $I$ and $F$ respectively and for all input $a \in A$ and all state $s \in S$, $a(s)$ denotes the set of states available from $s$ via the input $a$. The automaton $\mathcal{S}$ is an ND-coalgebra of type $\mathcal{T}_0 \times \mathcal{T}_0 \times \prod_{a \in A} \mathcal{T}_1$ whose carrier set is $S$. □

**Example 9 (Functional Dependencies and Schema in Databases).** The concept of schema in databases has been formalized and generalized in [15]. This characterization has enabled to obtain new results in logics of functional dependencies and allows for applying artificial intelligence techniques in databases.

A schema is a triple $(A, \leq, F)$ where $A$ is a non-empty set, $\leq$ is an ordering relation in $A$ and $F$ is a non-deterministic ideal operator in $(A, \leq)$. That is, $F$ is a map from $A$ to $\mathcal{P}(A)$ satisfying reflexivity ($a \in F(a)$, for all $a \in A$), transitivity ($F^2 \subseteq F$) and that $F(a)$ is an ideal in $(A, \leq)$ (a lower closed and directed subset of $A$) for all $a \in A$.

Therefore, any schema is an ND-coalgebra of type $\mathcal{T}_1 \times \mathcal{T}_1$. □

Regarding to a categorical approach to ND-coalgebras, we provide below some remarks concerning the morphisms between ND-coalgebras.

Let us recall that, given an arbitrary type $\mathcal{T}$, the class of all $\mathcal{T}$-coalgebras forms a category in which the morphisms are defined as follows: given $\mathcal{A} = (A, \alpha)$ and $\mathcal{B} = (B, \beta)$ two $\mathcal{T}$-coalgebras, a map $f \colon A \to B$ is a standard *homomorphism* of coalgebras if $\mathcal{T} \circ \alpha = \beta \circ f$.

For the purposes of the study of nd-groupoids, a straightforward adoption of the previous definition does not lead to the intended behaviour, in that some important mappings, such as the inclusion map, do not fulfill the condition of being a morphism between coalgebras: consider the nd-groupoids $(A, \cdot)$ and $(B, \cdot)$ with the nd-operations defined as in the tables

$(A, \cdot)$

| $\cdot$ | $a$ |
|---|---|
| $a$ | $\{a\}$ |

$(B, \cdot)$

| $\cdot$ | $a$ | $b$ |
|---|---|---|
| $a$ | $\{a\}$ | $\{b\}$ |
| $b$ | $\{b\}$ | $\{b\}$ |

It is easy to check that the inclusion map $i\colon A \hookrightarrow B$ is a homomorphism of algebras, whereas its coalgebraic version is not: note that if they are viewed as coalgebras with type the ND-functor $\mathcal{T}_2$, the inclusion map $i\colon A \hookrightarrow B$ is not a homomorphism because $(\mathcal{T}_2(i) \circ \alpha)(a) = \{(a, a)\}$ but $(\beta \circ i)(a) = \{(a, a), (b, b)\}$.

The previous example suggests the modification of the standard definition in order to allow the inclusion maps as morphisms. Following Benado's initial ideas, we relax the condition required above by substituting the equality by an inclusion, and we still obtain a category which includes the category of coalgebras with the standard morphisms. For a fixed arbitrary ND-functor $\mathcal{T}$, a function $f\colon A \to B$ between two $\mathcal{T}$-coalgebras $(A, \alpha)$ and $(B, \beta)$ is said to be a *Benado-homomorphism* of coalgebras if $(\mathcal{T} \circ \alpha)(a) \subseteq (\beta \circ f)(a)$ for all $a \in A$. It is straightforward to show that the identity map is always a Benado-homomorphism and the composition of two Benado-homomorphisms is again a Benado homomorphism.

We end this section by providing some remarks about the existence of theterminal object and the induced behavioural equivalence.

Recall that we are considering concrete categories, in which the objects are sets with structure; therefore, it is reasonable to conjecture that the terminal objects are based on an underlying singleton. A straightforward application of the definition leads to two possible nd-structures on a singleton: one in which the operation is empty, and another one in which the operation is the trivial one.

We can consider two possible cases, depending on the type of homomorphisms we would like to consider:

1. In the cases of Benado-homomorphisms, *there exists a terminal object* which is a singleton with the trivial operation.
2. In the case of standard coalgebraic homomorphisms, it is easy to show that *no terminal object exists*, since it is not possible to define any homomorphism between the singleton with empty structure, and the singleton with trivial structure.

Once we have shown the existence of terminal object under suitable conditions, the construction of the induced behavioural equivalence, following Rutten [38], generates an improper equivalence in which every pair of states is related. This situation might have been expected, as we are dealing with structures under not too restrictive properties.

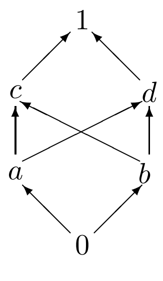
## 4. Some properties in binary ND-coalgebras

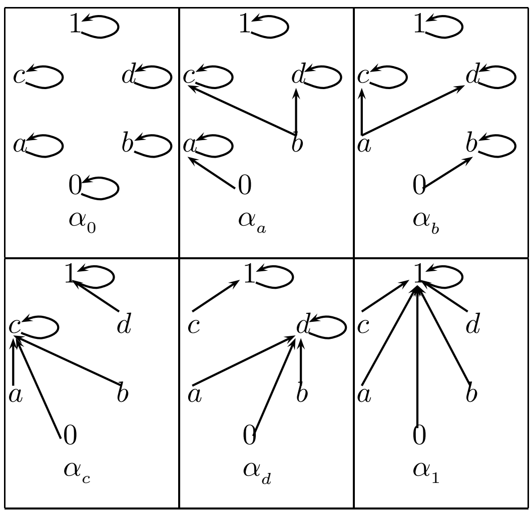
In this section we introduce some properties for ND-coalgebras that allow to characterize the structures of multisemilattice and multilattice. We will be especially concerned with a particular subclass of ND-coalgebras. Specifically, those coalgebras with type $\mathcal{T}_2$ and $\mathcal{T}_2 \times \mathcal{T}_2$ which will be called *binary* and *doubly binary* coalgebras, respectively.

Recalling Example 7, notice that in a binary coalgebra $(A, \alpha)$, for every $a \in A$, we have that $\alpha_a \in \mathcal{P}(A)^A \simeq \mathcal{P}(A^2)$, thus, $(A, \alpha_a)$ is an ND-coalgebra of type $\mathcal{T}_1$ and $\alpha_a$ can be treated as a binary relation. The following example works on this idea.

**Example 10.** The multilattice $(A, \leq)$ whose Hasse diagram, shown in the left of the picture below, defines two multisemilattices by considering the operations given by multi-suprema and multi-infima.

The first (resp. second) one can be interpreted as a binary ND-coalgebra $(A, \alpha)$ such that $\alpha_x(y) = \text{Msup}(\{x, y\})$, for all $x, y \in A$. In the right of the picture below, we can see directed graphs representing the binary relation $\alpha_x$, for each $x \in A$.



*4.1. Single-point properties*

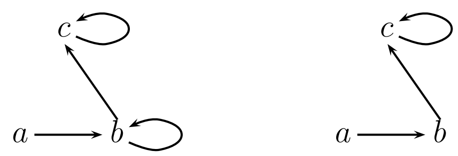
In this and the next sections, we will recall known properties and introduce new ones in order to provide the coalgebraic characterisation of multilattices.

We firstly start with some simple properties of a binary ND-coalgebra to ease reading. All of them share a common feature: they are focused on the behaviour of single elements of the carrier set.

**Definition 11.** A binary relation $R$ on a set $X$ is said to be

- *secondary reflexive* if it is reflexive in $R(X)$: That is, $x \in R(y)$ implies $x \in R(x)$, for all $x, y \in X$.
- *strongly secondary reflexive* if it is the identity relation in $R(X)$: That is, $x \in R(y)$ implies $R(x) = \{x\}$, for all $x, y \in X$.
- *collapsing* if whenever $x \in R(x)$ then $R(x) = \{x\}$, for all $x \in X$. That is, in a collapsing relation, if an element is related with itself, then it cannot be related to any other element.

**Example 12.** The relation $R_1$ depicted below is secondary reflexive and not collapsing. However, the relation $R_2$ is not secondary reflexive and it is collapsing.



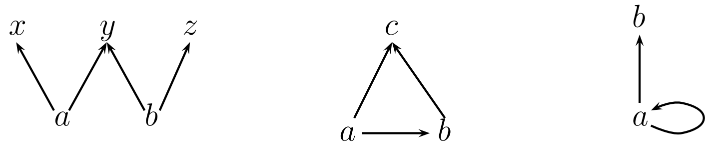

$R_1$ $R_2$

**Definition 13.** Let $R$ be a binary relation in a set $X$ and consider $x, y \in X$. We say that $x$ and $y$ are *siblings* if there exists an element $z \in X$ such that $x, y \in R(z)$. That is, $y \in R(R^{-1}(x))$.

The binary relation $R$ is called *uncoupling* if $x, y \in R(z)$ with $y \in R(x)$ implies $x = y$, for any $x, y, z \in X$. That is, two siblings cannot be related.

Note that the sibling relation is always symmetric, but does not have to be either reflexive (abusing the kinship analogy, for an element $x \in X$ to be a sibling a parent is needed, that is, $R^{-1}(x) \neq \varnothing$) or transitive.

**Example 14.** In the relation $R_1$ depicted below, the elements $x, y$ are siblings and so are $y, z$, however $x, z$ are not siblings. Moreover, neither $a$ nor $b$ are siblings of themselves. On the other hand, $R_1$ is uncoupling but $R_2$ and $R_3$ are not uncoupling.



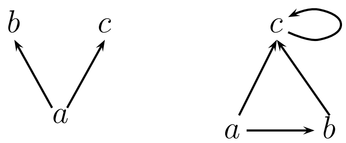

$R_1$ $R_2$ $R_3$

It is not difficult to observe that there should exist a strong connection between properties defined in this section. The following results follow easily from the definitions, but are explicitly stated since they will be used later in the paper.

**Lemma 15.**

1. *A strongly secondary reflexive binary relation is uncoupling.*
2. *An uncoupling binary relation is collapsing.*

The following relations $R_1$ and $R_2$ are respectively counterexamples for the converse results.




$R_1$ $R_2$

**Lemma 16.** *The following conditions are equivalent for a binary relation $R$:*

*i)* *$R$ is strongly secondary reflexive.*
*ii)* *$R$ is uncoupling and secondary reflexive.*
*iii)* *$R$ is collapsing and secondary reflexive.*

All the properties displayed above have been described for binary relations. Next, we will be concerned with binary ND-coalgebras where binary relations verifying those properties are assigned to the elements of the carrier.

**Definition 17.** A binary ND-coalgebra $\mathcal{A} = (A, \alpha)$ is said to be

- *(strongly) secondary reflexive* if $\alpha_a$ is a (strongly) secondary reflexive binary relation, for all $a \in A$.
- *collapsing* if $\alpha_a$ is a collapsing binary relation, for all $a \in A$.
- *uncoupling* if, for all $a \in A$, $\alpha_a$ is an uncoupling binary relation.

The fact that in a binary ND-coalgebra there are as many different binary relations as elements of the carrier set implies that we should state carefully the relationship between two elements of the carrier set; the following definition provides some useful terminology.

**Definition 18.** Let $\mathcal{A} = (A, \alpha)$ be a binary ND-coalgebra and $a \in A$.

1. The element $a$ is *self-conscious* if $a \in \alpha_a(a)$.
2. The element $a$ is *isolated* if $\alpha_a(a) = \{a\}$.

Obviously, an isolated element of a binary ND-coalgebra is self-conscious, and an element of a collapsing binary ND-coalgebra is self-conscious if and only if it is isolated. However, in general, both classes of elements need not coincide.

**Example 19.** Consider a poset $(A, \leq)$, then the elements of the binary ND-coalgebra $(A, \alpha)$, where $\alpha_a(b) = \mathrm{Msup}\{a, b\}$ for any $a, b \in A$, are isolated.

However, the elements of $(A, \beta)$, where $\beta_a(b)$ is the set of upper bounds of $\{a, b\}$ for any $a, b \in A$, are self-conscious but not isolated. □

*4.2. Combined properties*

Now we introduce properties in which more than one binary relation is involved. This is why they are called *combined* properties. Some of them can be deemed as generalizations of some well-known properties held in universal algebras, such as commutativity or associativity.

**Definition 20.** A binary ND-coalgebra $\mathcal{A} = (A, \alpha)$ is said to be

- *commutative* if $\alpha_a(b) = \alpha_b(a)$ for all $a, b \in A$.
- *separating* if whenever $c, d \in \alpha_a(b)$ and $c \in \alpha_c(d)$ then $c = d$, for all $a, b, c, d \in A$. Equivalently, for all $a \in A$ and $x \in \alpha_a(A)$ the following equality holds
$$(\alpha_a \circ \alpha_a^{-1})(x) \cap \alpha_x^{-1}(x) = \{x\}$$

**Example 21.**

a. Both binary ND-coalgebras given in Example 19 are commutative, but $(A, \alpha)$ is separating and $(A, \beta)$ is not.
b. Let us consider the ring $(\mathbb{Z}, +, \cdot)$ and the congruence relations $\equiv_{(\text{mod } a)}$, for each $a \in \mathbb{Z}$. That is, for all $n, m \in \mathbb{Z}$,

$$n \equiv m \,(\text{mod } a) \text{ if and only if } m - n = az \text{ with } z \in \mathbb{Z}$$

The binary ND-coalgebra $(\mathbb{Z}, \alpha)$ given by

$$\alpha_a(n) = \{m \in \mathbb{Z} \mid n \equiv m \,(\text{mod } a)\}$$

is not commutative because, for example,

$$\alpha_2(3) = \{2z + 1 \mid z \in \mathbb{Z}\} \;\neq\; \alpha_3(2) = \{3z + 2 \mid z \in \mathbb{Z}\}$$

It is also not separating, because $0, 4 \in \alpha_2(0)$ and $4 \in \alpha_4(0)$. □

Two alternative forms to generalize associativity in a non-deterministic framework are presented below.

**Definition 22.** A binary ND-coalgebra $\mathcal{A} = (A, \alpha)$ is

- *weakly associative* if, for elements $a, b \in A$ such that $\alpha_a(b) = \{c\}$, we have that $\alpha_c \subseteq \alpha_a \circ \alpha_b$.
- *m-associative* if for elements $a, b \in A$ such that $\alpha_a(b) = \{a\}$, it is satisfied $\alpha_a \subseteq \alpha_a \circ \alpha_b$.

Obviously a weakly associative ND-coalgebra is m-associative, but the reciprocal is not true as we can see in the following example.

**Example 23.** Let $(\mathbb{Z}^+, \alpha)$ be the binary ND-coalgebra defined by

$$\alpha_a(b) = \{ab^2, a^2b\}$$

It is m-associative, because if $\alpha_a(b) = \{a\}$, which is fulfilled only for $a = b = 1$, then for all $x \in \mathbb{Z}^+$:

$$\{x, x^2\} = \alpha_1(x) \subseteq (\alpha_1 \circ \alpha_1)(x) = \alpha_1(\{x, x^2\}) = \{x, x^2, x^4\}$$

Nevertheless, it is not weakly associative: for example $\alpha_2(2) = \{8\}$ but

$$\{8, 64\} = \alpha_8(1) \;\not\subseteq\; (\alpha_2 \circ \alpha_2)(1) = \alpha_2(\{2, 4\}) = \{8, 16, 32\}$$

□

This section is concluded with an illustrative example which collects many of the properties introduced in this section.

**Example 24.** Let us consider the ring $(\mathbb{Z}, +, \cdot)$ with the divisibility relation. The binary ND-coalgebra $(\mathbb{Z}, \alpha)$ given by

$$\alpha_a(b) = \{p \in \mathbb{Z} \mid p \text{ is prime and common factor of } a \text{ and } b\ \}$$

is strongly secondary reflexive, commutative, separating and weakly associative. Just the prime elements are self-conscious, moreover these elements are isolated. □

## 5. Multisemilattices are binary ND-coalgebras

We start this section by studying some relationships among the properties which have been defined in Section 4, in order to determine which ones are needed to define a coalgebraic theory of multisemilattices.

To begin with, let us recall that strongly secondary reflexivity is quite near to secondary reflexivity. As Lemma 16 reveals, this leads to an equivalence between both properties, under other additional conditions, particularly, it occurs when the binary ND-coalgebra is uncoupling. Now, we are also interested in how the m-associativity and weak associativity of a binary ND-coalgebra influence other properties.

**Lemma 25.** *Let $\mathcal{A} = (A, \alpha)$ be an m-associative and uncoupling binary ND-coalgebra whose elements are isolated, then it is strongly secondary reflexive.*

Proof. As every element $x \in A$ is isolated, $\alpha_x(x) = \{x\}$, so, due to the m-associativity, $\alpha_x \subseteq \alpha_x \circ \alpha_x$, in particular, for all $y$ we have $\alpha_x(y) \subseteq \alpha_x(\alpha_x(y))$.

Now, given $z \in \alpha_x(y)$, there exists $z' \in \alpha_x(y)$ such that $z \in \alpha_x(z')$. The binary relation $\alpha_x$ is uncoupling, therefore, $z' = z$. Thus, we have $z \in \alpha_x(z)$, which implies $\alpha_x(z) = \{z\}$, by Lemma 15. □

**Corollary 26.** *Let $\mathcal{A} = (A, \alpha)$ be an m-associative binary ND-coalgebra whose elements are isolated. Then, $\mathcal{A}$ is strongly secondary reflexive if and only if $\mathcal{A}$ is uncoupling.*

A particular class of binary ND-coalgebras enjoys the property that its relations $\alpha_a$ are idempotent.

**Lemma 27.** *Let $\mathcal{A} = (A, \alpha)$ be an m-associative and strongly secondary reflexive (uncoupling) binary ND-coalgebra whose elements are isolated. Then, $\alpha_a$ is idempotent, for all $a \in A$, that is, $\alpha_a \circ \alpha_a = \alpha_a$.*

Proof. As $\mathcal{A}$ is strongly secondary reflexive, we have $\alpha_a \circ \alpha_a \subseteq \alpha_a$. The other inclusion comes from $\mathcal{A}$ being m-associative and all elements being isolated. □

We now provide the coalgebraic version of the natural ordering for nd-groupoids that was introduced in Section 2.

**Definition 28.** Let $\mathcal{A} = (A, \alpha)$ be a binary ND-coalgebra and $a, b \in A$. It is said that *$a$ is upper bounded by $b$*, denoted by $a \leq b$, if $\alpha_b(a) = \{b\}$.

In the same way that nd-groupoids, some properties ensure that this relation is really an ordering relation.

**Proposition 29.** *Let $\mathcal{A} = (A, \alpha)$ be a commutative and m-associative binary ND-coalgebra whose elements are isolated. Then, $(A, \leq)$ is a poset.*

Proof. The binary relation defined above is reflexive because all elements in $\mathcal{A}$ are isolated and is antisymmetric due to the commutativity. On the other hand, given $x \leq y$ and $y \leq z$, as $\{z\} = \alpha_z(y)$ and $\mathcal{A}$ is m-associative, it holds that $\alpha_z(x) \subseteq \alpha_z(\alpha_y(x)) = \alpha_z(y) = \{z\}$. Then $x \leq z$ and so the relation is transitive. □

The following technical lemma is extracted and stated independently in order to simplify the presentation of some proofs in the rest of the paper.

**Lemma 30.** *Let $\mathcal{A} = (A, \alpha)$ be an m-associative binary ND-coalgebra. If $a, b \in A$ are upper bounded by an element $x \in A$, then there exists $c \in \alpha_a(b)$ such that $x \in \alpha_x(c)$.*

*Furthermore, if $\mathcal{A}$ is commutative and strongly secondary reflexive, then $c$ is also upper bounded by $x$.*

Proof. Let $a, b \in A$ be upper bounded by $x$, that is, $\alpha_x(a) = \{x\} = \alpha_x(b)$. As $\mathcal{A}$ is m-associative, $\{x\} = \alpha_x(b) \subseteq (\alpha_x \circ \alpha_a)(b) = (\alpha_x(\alpha_a(b)))$ whence, there exists $c \in \alpha_a(b)$ such that $x \in \alpha_x(c)$.

If $\mathcal{A}$ is also commutative, then $x \in \alpha_c(x)$, which implies $\{x\} = \alpha_c(x) = \alpha_x(c)$, if $\mathcal{A}$ is also strongly secondary reflexive. □

Definition 22 gives two different forms to generalize associativity. The following proposition proves that both properties can be used for our aim because, under certain conditions, they turn out to be equivalent.

**Proposition 31.** *Let $\mathcal{A} = (A, \alpha)$ be a commutative, strongly secondary reflexive and separating binary ND-coalgebra. Then, $\mathcal{A}$ is weakly associative if and only if $\mathcal{A}$ is m-associative.*

PROOF. Clearly, weak associativity implies m-associativity. Thus, we have to prove the converse. Following the same scheme of the proof of transitivity in Proposition 29, it is easy to check that

$$\alpha_x(a) = \{x\} \quad \text{and} \quad \alpha_a(y) = \{a\} \quad \text{imply} \quad \alpha_x(y) = \{x\} \tag{1}$$

Let $a, b$ be elements such that $\alpha_a(b) = \{c\}$ and consider $z \in \alpha_c(x)$, for an arbitrary element $x \in A$. Our aim is to show that $z \in \alpha_a(\alpha_b(x))$. As $\mathcal{A}$ is strongly secondary reflexive and $z \in \alpha_c(x) = \alpha_x(c)$, it holds that $\alpha_c(z) = \{z\} = \alpha_x(z)$. Likewise, $\alpha_a(c) = \{c\} = \alpha_b(c)$. Then, by commutativity and (1), we have $\alpha_z(a) = \{z\} = \alpha_z(b)$. Now, as $x$ and $b$ are upper bounded by $z$, there exists $t \in \alpha_b(x)$ such that $\{z\} = \alpha_t(z)$, by Lemma 30. We follow the same scheme again to consider and element $y \in \alpha_a(t)$ such that $\{z\} = \alpha_z(y)$. Therefore, we have just to show that $y = z$ to complete the proof.

As $y \in \alpha_t(a)$, then $\alpha_a(y) = \{y\} = \alpha_t(y)$. On the other hand, $\alpha_t(x) = \{t\} = \alpha_t(b)$, which implies $\alpha_y(x) = \{y\} = \alpha_y(b)$. Hence, $a$ and $b$ are upper bounded by $y$. Applying again Lemma 30 and taking into account that $\alpha_a(b) = \{c\}$, we obtain that $\{y\} = \alpha_y(c)$. Then, as $x$ and $c$ are upper bounded by $y$, there exists $u \in \alpha_x(c)$ such that $\{y\} = \alpha_y(u)$. From (1), we obtain that $z$ maps $u$ to $z$ itself. Since $\mathcal{A}$ is separating and $u$, $z$ are siblings (both of them belong to $\alpha_x(c)$), we have that $z = u$. As a result, $y = z$. □

All the requirements to characterize a multisemilatice are already available.

**Definition 32.** A commutative, m-associative, separating and uncoupling binary ND-coalgebra where all the elements are self-conscious is said to be a *coalgebraic multisemilattice.*

The previous definition states precisely those properties which define an algebraic multisemilattice within the theory of ND-coalgebras.

**Theorem 33.** *Algebraic and coalgebraic multisemilattices are equivalent structures.*

Proof. Firstly, we prove that, if $(A, F)$ is an algebraic multisemilattice, then the binary ND-coalgebra $\mathcal{A} = (A, \lambda F)$, where $(\lambda F)_x(y) = F(x, y)$, for all $x, y \in A$, is a coalgebraic multisemilattice:

Commutativity, the comparability property $C_2$ and idempotency guarantee that the ND-coalgebra is commutative, separating and all the elements are self-conscious. The comparability property $C_1$ ensures that the ND-coalgebra is strongly secondary reflexive and, by Lemma 15, it is uncoupling. In this context, self-consciousness and isolation are equivalent, therefore by using Proposition 31, one obtains the m-associativity.

Conversely, given $(A, \alpha)$ a coalgebraic multisemilattice, we are going to prove that $(A, \lambda^{-1}\alpha)$, where $(\lambda^{-1}\alpha)(x, y) = \alpha_x(y)$, for all $x, y \in A$, is an algebraic multisemilattice:

As the coalgebra is uncoupling and the elements are self-conscious, by Corollary 26, the coalgebra is strongly secondary reflexive. This, together with commutativity, implies $C_1$ of comparability. Finally, $C_2$ and weak associativity are trivially obtained as the coalgebra is separating and m-associative. □

## 6. Multilattices

In this section we apply the previous results to obtain a coalgebraic characterization of the notion of multilattice as a doubly binary ND-coalgebra consisting of two properly assembled binary ND-coalgebras.

In some sense, a doubly binary ND-coalgebra, $\mathcal{A} = (A, \gamma)$, where $\gamma_a = (\alpha_a, \beta_a)$ for an arbitrary element $a \in A$, supplies two different binary ND-coalgebras, namely $(A, \alpha)$ and $(A, \beta)$. In order to obtain a multilattice structure on $A$, it is necessary that these two binary ND-coalgebras satisfy the properties displayed in Definition 32 and, in addition, both operations should be adequately linked.

**Theorem 34.** *Let $\mathcal{A} = (A, \gamma)$ be a doubly binary ND-coalgebra, where $\gamma_a = (\alpha_a, \beta_a)$ for every $a \in A$, such that*

1. *$(A, \alpha)$ and $(A, \beta)$ are coalgebraic multisemilattices.*
2. *$\alpha_x(y) = \{y\}$ if and only if $\beta_x(y) = \{x\}$, for all $x, y \in A$.* (Duality)

*Then, $(A, \leq)$ is a multilattice where $x \leq y$ if and only if $\alpha_x(y) = \{y\}$.*

In the rest of the section, we will concentrate on an alternative formulation of the duality condition as an internal property of the ND-coalgebra in the sense of the absorption identities which arise in lattice theory.

**Definition 35.** Two binary ND-coalgebras, $(A, \alpha)$ and $(A, \beta)$, are said to be *assembled* if, for all $a \in A$, the following conditions hold:

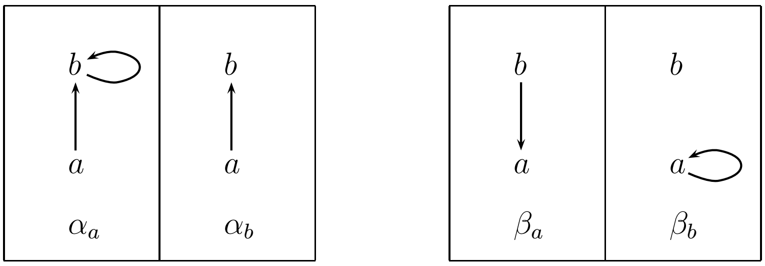
1. If $\beta_a(x) \neq \varnothing$ then $(\alpha_a \circ \beta_a)(x) = \{a\}$.
2. If $\alpha_a(x) \neq \varnothing$ then $(\beta_a \circ \alpha_a)(x) = \{a\}$.

A doubly binary ND-coalgebra $(A, \gamma)$, where $\gamma = (\alpha, \beta)$, satisfies the *assembly* property if $(A, \alpha)$ and $(A, \beta)$ are assembled.

**Lemma 36.** *Consider two commutative binary ND-coalgebras* $(A, \alpha)$ *and* $(A, \beta)$*. If they are assembled, then the duality condition holds.*

The following example shows that the converse result is not true.

**Example 37.** Given the set $A = \{a, b\}$, consider the commutative binary ND-coalgebras $(A, \alpha)$ and $(A, \beta)$ whose corresponding binary relations $\alpha_x$ and $\beta_x$ are depicted in the following figure, for each $x \in A$:



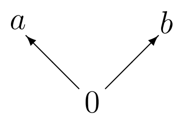

It is not difficult to prove that the duality condition holds. Nevertheless, $(A, \alpha)$ and $(A, \beta)$ are not assembled, because

$$\beta_a(b) = \{a\} \text{ and } (\alpha_a \circ \beta_a)(b) = \alpha_a(\{a\}) = \{b\} \neq \{a\}$$

$\square$

**Remark 38.** Notice that Lemma 36 would hold even when the conditions $\beta_a(x) \neq \varnothing$ and $\alpha_a(x) \neq \varnothing$ were removed in Definition 35. However, that modified assembly condition would be too restrictive, in the sense that it would not be satisfied in multilattices, as the following example demonstrates:

**Example 39.** The poset $(A, \leq)$ whose Hasse diagram is




is an ordered multilattice. From the coalgebraic point of view, it can be deemed as a doubly binary ND-coalgebra, $\mathcal{A} = (A, \gamma)$ where $\gamma_x = (\alpha_x, \beta_x)$ for every $x \in A$, and

$$\alpha_x(y) = \mathrm{Msup}\{x, y\} \quad \text{and} \quad \beta_x(y) = \mathrm{Minf}\{x, y\}$$

for any $y \in A$. Notice that $\alpha_a(b) = \varnothing$ and $(\beta_a \circ \alpha_a)(b) = \varnothing$ □

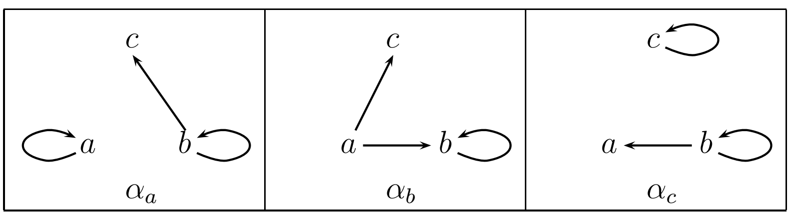
We next present the coalgebraic interpretation of absorption property, which was recalled in Section 2, to analyze its relation with assembly property.

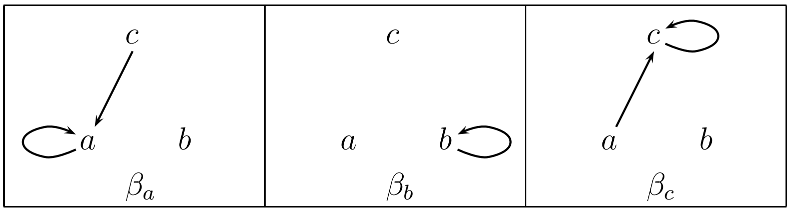
**Definition 40.** A doubly binary ND-coalgebra $(A, \gamma)$ with $\gamma = (\alpha, \beta)$ satisfies the *absorption* property if, for all $a \in A$, the following holds:

1. If $z \in \beta_a(A)$ then $\alpha_a(z) = \{a\}$.
2. If $z \in \alpha_a(A)$ then $\beta_a(z) = \{a\}$.

Obviously, the absorption property implies the assembly property, but the converse is not true.

**Example 41.** Consider $A = \{a, b, c\}$. The binary ND-coalgebras $(A, \alpha)$ and $(A, \beta)$ stated explicitly in the figure below are assembled, as can be checked routinely. However, the doubly binary ND-coalgebra $(A, \gamma)$, with $\gamma = (\alpha, \beta)$, does not satisfy the absorption property because $b \in \alpha_a(b)$ but $\beta_a(b) = \varnothing$.






Once again, strong secondary reflexivity is the key to get the equivalence aimed at between assembly and absorption.

**Proposition 42.** *Let $(A, \gamma)$ be a commutative doubly binary ND-coalgebra, where $\gamma = (\alpha, \beta)$.*[1] *Then, the following conditions are equivalent:*

1. *$(A, \gamma)$ satisfies the absorption property.*
2. *$(A, \alpha)$ and $(A, \beta)$ are strongly secondary reflexive and assembled.*
3. *$(A, \alpha)$ and $(A, \beta)$ are strongly secondary reflexive and duality holds.*

Proof. (1) $\Rightarrow$ (2) As we mentioned before, it is clear that absorption implies duality. On the other hand, given an element $a \in \alpha_x(y)$, by absorption, we have $\beta_x(a) = \{x\}$ which is equivalent to $\alpha_x(a) = \{a\}$. This proves the strongly secondary reflexivity.

(2) $\Rightarrow$ (3) This was noticed in Lemma 36.

(3) $\Rightarrow$ (1) Let $z$ be an element of $\beta_x(y)$. As $(A, \beta)$ is strongly secondary reflexive, it holds that $\beta_x(z) = \{z\}$ which is equivalent to $\alpha_x(z) = \{x\}$. Analogously, one proves the other absorption identity. $\square$

Now, we are in position to state the main theorem of this section.

**Theorem 43.** *Let $\mathcal{A} = (A, \gamma)$ be a doubly binary ND-coalgebra, where $\gamma_a = (\alpha_a, \beta_a)$, for every $a \in A$, such that*

*i) both $(A, \alpha)$ and $(A, \beta)$ are coalgebraic multisemilattices and*
*ii) $(A, \gamma)$ satisfies the assembly property.*

*Then, $(A, \leq)$ is a multilattice where $x \leq y$ if and only if $\alpha_x(y) = \{y\}$.*

Proof. Being $(A, \alpha)$ and $(A, \beta)$ coalgebraic multisemilattices, they both are commutative and strongly secondary reflexive. Hence, the assembly property implies the duality condition. Finally, suffice it to apply Theorem 34 to obtain that $(A, \leq)$ is a multilattice. $\square$

[1] $(A, \alpha)$ and $(A, \beta)$ are commutative.

## 7. Conclusions and further research

Following the trend of providing a coalgebraic approach for several non-deterministic structures, we have defined a suitable class of coalgebras, the ND-coalgebras, and developed a thorough analysis of the required properties in order to achieve a convenient coalgebraic characterization of multilattices which complements the algebraic one given in [13].

The class of ND-coalgebras can be regarded as a collection of coalgebras underlying non-deterministic situations, and creates a setting in which many other structures could be suitably described. A possible issue to be tackled in the future might be the coalgebraic explanation of a more general type of multisemilattices and multilattices which were thoroughly studied in [30]. For this purpose, it would be necessary to extend the definitions and properties introduced for binary and doubly binary ND-coalgebras.

Taking into account that the induced behavioural equivalence in the general case is the improper one, we will provide a coalgebraic study of several important classes of multilattices, such as those which are modular and/or distributive (some approaches to this have already been published [24, 37]). In such a modified context, it will make sense to further develop the coalgebraic theory of multilattices by studying the resulting axioms in relation to suitable coalgebraic modal logics.

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