International Journal of General Systems
Vol. 00, No. 00, October 2008, 1–13

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On basic conditions to generate multi-adjoint concept lattices via Galois connections

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(*November 22, 2012*)

This paper introduces sufficient and necessary conditions with respect to the fuzzy operators considered in a multi-adjoint frame under which the standard combinations of multi-adjoint sufficiency, possibility and necessity operators form (antitone or isotone) Galois connections. The underlying idea is to study the *minimal* algebraic requirements so that the concept-forming operators (defined using the same syntactical form than the extension and intension operators of multi-adjoint concept lattices) form a Galois connection. As a consequence, given a relational database, we have much more possibilities to construct concept lattices associated with it, so that we can choose the specific version which better suits the situation.

Keywords: Galois connection; formal concept analysis; multi-adjoint concept lattices.

1. Introduction

Different generalizations of formal concept analysis have been presented in the recent years, ranging from possibility-theoretic approaches to rough-set based approaches, and from interval valued contexts to similarity measures (Alcalde *et al.* 2011, Alqadah and Bhatnagar 2011, Chen and Yao 2008, Dubois and Prade 2011, 2012, Düntsch and Gediga 2003, Formica 2012, Yao 2004). Moreover, new types of incidence relations are been taken into account (Guo *et al.* 2011) together with alternative definitions of the concept-forming operators in order to obtain different interpretations for them. These operators can be combined by pairs in order to form (antitone or isotone) Galois connections, and, as a consequence, their compositions can be interpreted topologically as either closure or opening operators.

In this paper we are concerned with fuzzy extensions of the previous approaches, particularly, the multi-adjoint framework (Medina 2012, Medina and Ojeda-Aciego 2010, 2012, Medina *et al.* 2009) since, somehow, it embeds several other approaches (Bělohlávek 1998, Burusco and Fuentes-González 1994, Georgescu and Popescu 2004, Lai and Zhang 2009, Pollandt 1997) which are based on residuated structures in order to build the concept-forming operators.

Recently, algebraic requirements under which the composition of four different versions of concept-forming operators (fuzzy sufficiency, possibility, necessity, and dual sufficiency) are either closure or opening operators were introduced in (Djouadi and Prade 2011).

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ISSN: 0308-1079 print/ISSN 1563-5104 online

DOI: 10.1080/0308107YYxxxxxxxx
http://www.informaworld.com

Multi-adjoint concept lattices (Medina *et al.* 2009) were presented as a suitable generalization of several existing approaches to fuzzy concept analysis which use pairs of antitone Galois connections to build the concepts. More recently, multi-adjoint property-oriented and object-oriented concept lattices were introduced as a generalization of fuzzy rough sets (Medina 2012); in these settings, the main theoretical tool is that of isotone Galois connections.

Adjoint triples are the basic operators on which the multi-adjoint frameworks cited above are founded. These triples can be seen as adequate generalizations of the pair of classical conjunctor and implication connectives, as well as t-norms and their residua and enable us to define the different forms of concept-forming operators in an L-fuzzy environment.

Note, however, that there might be situations which suggest considering even softer restrictions on the conjunctors and implications. For instance, consider a relational database about which a subset of concepts is known, and our aim is to find concept-forming operators which generate a concept lattice which includes the whole subset of known concepts. Therefore, it would be desirable to have as much more possibilities to build a concept lattice associated with a relational database.

In this paper, we follow the spirit of (Djouadi and Prade 2011) and filter out some conditions imposed by adjoint triples which are not fundamental for the development of multi-adjoint formal concept analysis, in the sense that the Galois connections needed to build multi-adjoint concepts can be obtained by a smaller set of requirements. This applies both to antitone Galois connections for multi-adjoint concept lattices and isotone Galois connections for multi-adjoint object-oriented and property-oriented concept lattices. We introduce sufficient and necessary conditions under which the corresponding compositions of the concept-forming operators are either closure or opening operators and obtain new interesting consequences: in some sense, a set of minimal requirements to built the different versions of multi-adjoint concept lattices are given.

2. Basic operators for multi-adjoint concept lattices

A common result concerning different generalizations of formal concept analysis is that the pair of fuzzy extensions of the crisp concept-forming operators forms a Galois connection (Ganter and Wille 1999). There are two dual versions of this notion. The version we adopt in this section is the standard one in which the involved maps are antitone (order-reversing); these will be properly called *Galois connections*. The other version is that in which the maps are order-preserving, the so-called *isotone Galois connections* which will be studied in the next section. There are arguments for considering either version, although, at a theoretical level, the difference is not significant since we can pass from one to another simply by swapping a lattice by its dual.

In order to make this contribution self-contained, we recall now the formal definition of (antitone) Galois connection.

Definition 2.1: Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be posets, and ${}^{\downarrow}\colon P_1 \to P_2$, ${}^{\uparrow}\colon P_2 \to P_1$ mappings, the pair $({}^{\uparrow}, {}^{\downarrow})$ forms an *antitone Galois connection* between P_1 and P_2 if and only if:

(1) ${}^{\uparrow}$ and ${}^{\downarrow}$ are order-reversing.
(2) $x \leq_1 x^{\downarrow\uparrow}$ for all $x \in P_1$, that is, ${}^{\downarrow\uparrow}$ is extensive (wrt P_1)
(3) $y \leq_2 y^{\uparrow\downarrow}$ for all $y \in P_2$, that is, ${}^{\uparrow\downarrow}$ is extensive (wrt P_2)

From now on, we will fix two complete lattices $(L_1, \preceq_1)$, $(L_2, \preceq_2)$, a poset $(P, \leq)$

and two families of mappings $\swarrow^i: P \times L_2 \to L_1$, $\nwarrow_i: P \times L_1 \to L_2$, where i belongs to an index set Λ.

In this environment we consider two sets A and B, which usually represent a set of attributes and a set of objects, respectively; and a fuzzy relation between them, $R: A \times B \to P$. Moreover, a mapping $\sigma: A \times B \to \Lambda$ is assumed, which relates each pair (a, b) to a pair of implications $(\swarrow^{\sigma(a,b)}, \nwarrow_{\sigma(a,b)})$, similar to the point of view introduced in (Medina *et al.* 2009). The tuple (A, B, R, σ) will be called *(formal) context*.

Given a context (A, B, R, σ), we define the operators ${}^\uparrow: L_2^B \to L_1^A$, ${}^\downarrow: L_1^A \to L_2^B$, as

$$g^\uparrow(a) = \inf\{R(a, b) \swarrow^{\sigma(a,b)} g(b) \mid b \in B\} \tag{1}$$

$$f^\downarrow(b) = \inf\{R(a, b) \nwarrow_{\sigma(a,b)} f(a) \mid a \in A\} \tag{2}$$

for all $a \in A$ and $b \in B$.

Note that we are slightly abusing notation in that $({}^\uparrow, {}^\downarrow)$ depends on σ; furthermore, hereafter we will write $\swarrow^{a,b}$, $\nwarrow_{a,b}$ instead of $\swarrow^{\sigma(a,b)}$, $\nwarrow_{\sigma(a,b)}$.

The aim of this section is to find a weaker framework under which the fuzzy generalizations of the crisp concept-forming operators still form an antitone Galois connection; specifically, we will introduce a sufficient and necessary condition in terms of the mappings $\swarrow^i$, $\nwarrow_i$ considered with respect to L_1, L_2 and P, in order to prove whether the operators ${}^\uparrow: L_2^B \to L_1^A$, ${}^\downarrow: L_1^A \to L_2^B$ form an antitone Galois connection.

In order to obtain that the pair $({}^\uparrow, {}^\downarrow)$ is a Galois connection, we will firstly prove that these operators are order-reversing (antitone). The next result shows that this property is associated with the monotony of the operators ${}_z\swarrow: L_2 \to L_1$, ${}^z\nwarrow: L_1 \to L_2$, for $z \in P$, which are defined as ${}_z\swarrow(y) = z \swarrow y$, ${}^z\nwarrow(x) = z \nwarrow x$, for all $y \in L_2$, $x \in L_1$.

Proposition 2.2: *The mapping* ${}^\uparrow: L_2^B \to L_1^A$ *is antitone for all formal context* (A, B, R, σ) *if and only if* ${}_z\swarrow^i: L_2 \to L_1$ *is antitone for all* $z \in P$ *and* $i \in \Lambda$.

Analogously, ${}^\downarrow: L_1^A \to L_2^B$ *is antitone for all formal context* (A, B, R, σ) *if and only if* ${}^z\nwarrow_i: L_1 \to L_2$ *is antitone for all* $z \in P$ *and* $i \in \Lambda$.

Proof: On the one hand, given $z \in P$, $i \in \Lambda$, and $y_1, y_2 \in L_2$, such that $y_1 \preceq_2 y_2$, we consider $B = \{b\}$, $A = \{a\}$, $R(a, b) = z$, $\sigma(a, b) = i$ and $g_1, g_2 \in L_2^B$, defined as $g_1(b) = y_1$, $g_2(b) = y_2$, and, as a consequence of the hypothesis, we obtain

$$\begin{aligned}
z \swarrow^i y_2 &= R(a, b) \swarrow^i g_2(b) \\
&\overset{(1)}{=} \inf\{R(a, b) \swarrow^{a,b} g_2(b) \mid b \in B\} \\
&= g_2^\uparrow(b) \\
&\overset{(2)}{\preceq_1} g_1^\uparrow(b) \\
&= \inf\{R(a, b) \swarrow^{a,b} g_1(b) \mid b \in B\} \\
&\overset{(3)}{=} z \swarrow^i y_1
\end{aligned}$$

where (1) and (3) hold because B has only one element and (2) holds by hypothesis.

On the other hand, given $g_1, g_2 \in L_2^B$, with $g_1 \preceq_2 g_2$. As ${}_z\swarrow^i$ is antitone for all $z \in P$ and $i \in \Lambda$, then $(R(a, b) \swarrow^i \_)(g_2(b)) \preceq_1 (R(a, b) \swarrow^i \_)(g_1(b))$, for all $b \in B$,

$a \in A$, $i \in \Lambda$, and so

$$\begin{aligned} g_2^{\uparrow}(a) &= \inf\{R(a,b) \swarrow^{a,b} g_2(b) \mid b \in B\} \\ &\preceq_1 \inf\{R(a,b) \swarrow^{a,b} g_1(b) \mid b \in B\} \\ &= g_1^{\uparrow}(a) \end{aligned}$$

for all $a \in A$. Thus, $g_2^{\uparrow} \preceq_2 g_1^{\uparrow}$.

The other monotonicity follows similarly. □

Remark 1 : As a result of the previous proposition, we will assume hereafter that the arrows $\nwarrow_i$, $\swarrow^i$ are antitone in their second component for all $i \in \Lambda$.

The second result characterizes the extensity property of the composition $^{\uparrow\downarrow}$ in terms of the mappings $\nwarrow_i$, $\swarrow^i$.

Theorem 2.3 : *The inequality $g \preceq_2 g^{\uparrow\downarrow}$ holds for all context (A, B, R, σ) and $g \in L_2^B$ if and only if the following property is satisfied:*
$(F1)$: $y \preceq_2 z \nwarrow_i (z \swarrow^i y)$, *for all $y \in L_2$, $z \in P$ and $i \in \Lambda$.*

Proof: Firstly, given $y \in L_2$, $z \in P$, assume that the inequality $g \preceq_2 g^{\uparrow\downarrow}$ holds, for all context (A, B, R, σ) and $g \in L_2^B$. Hence, we can consider $B = \{b\}$, $A = \{a\}$, $R(a, b) = z$ and $g \in L_2^B$, defined as $g(b) = y$. Therefore,

$$\begin{aligned} y &= g(b) \\ &\preceq_2 g^{\uparrow\downarrow}(b) \\ &= \inf\{R(a',b) \nwarrow_{a',b} g^{\uparrow}(a') \mid a' \in A\} \\ &= R(a,b) \nwarrow_{a,b} g^{\uparrow}(a) \\ &= R(a,b) \nwarrow_{a,b} \inf\{R(a,b') \swarrow^{a,b'} g(b') \mid b' \in B\} \\ &= R(a,b) \nwarrow_{a,b} \big(R(a,b) \swarrow^{a,b} g(b)\big) \\ &= z \nwarrow_{a,b} (z \swarrow^{a,b} y) \end{aligned}$$

Thus, Condition $(F1)$ holds.

Conversely, assume property $(F1)$, and consider an arbitrary formal context (A, B, R, σ) and $g \in L_2^B$. Hence, by $(F1)$, the inequality $g(b) \preceq_2 R(a, b) \nwarrow_{a,b} \big(R(a, b) \swarrow^{a,b} g(b)\big)$ is satisfied, for all $b \in B$, $a \in A$, because $g(b) \in L_2$ and $R(a, b) \in P$. Consequently, we obtain:

$$\begin{aligned} g(b) &\preceq_2 R(a,b) \nwarrow_{a,b} \big(R(a,b) \swarrow^{a,b} g(b)\big) \\ &\preceq_2 R(a,b) \nwarrow_{a,b} \inf\{R(a,b') \swarrow^{a,b'} g(b') \mid b' \in B\} \\ &= R(a,b) \nwarrow_{a,b} g^{\uparrow}(a) \end{aligned}$$

for all $b \in B$, $a \in A$. Therefore, by the properties of infimum, we have:

$$g(b) \preceq_2 \inf\{R(a',b) \nwarrow_{a',b} g^{\uparrow}(a') \mid a' \in A\} = g^{\uparrow\downarrow}(b)$$

for all $b \in B$. Thus, $g \preceq_2 g^{\uparrow\downarrow}$. □

It is not difficult to check that the other composition can be characterized in a similar way.

Theorem 2.4: *The inequality $f \preceq_1 f^{\downarrow\uparrow}$ holds for all context (A, B, R, σ) and $f \in L_1^A$ if and only if the following property is verified:*
$(F2)$: $x \preceq_1 z \swarrow^i (z \nwarrow_i x)$*, for all* $x \in L_1$*,* $z \in P$ *and* $i \in \Lambda$*.*

Note that Theorems 2.3 and 2.4 together with Proposition 2.2 entail the following result.

Corollary 2.5: *The pair $(^\uparrow, ^\downarrow)$ is an antitone Galois connection for all formal context (A, B, R, σ) if and only if Properties $(F1)$ and $(F2)$ are verified, and $^z\nwarrow_i$, $_z\swarrow^i$ are antitone for all $z \in P$ and $i \in \Lambda$.*

If we consider just one pair of identical mappings, then we see that our approach generalizes that of (Djouadi and Prade 2011, Theorem 2), recalled in the following corollary.

Corollary 2.6: *Considering $(L, \preceq) = (L_1, \preceq_1) = (L_2, \preceq_2)$ and a (degenerate) pair of identical arrows $(\swarrow, \swarrow)$, then the compositions $^{\uparrow\downarrow}$, $^{\downarrow\uparrow}$ are closure operators if $\swarrow$ is antitone on its right argument and the following holds:*
$(R1)$: $y \preceq z \swarrow (z \swarrow y)$*, for all* $y \in L$*,* $z \in P$*.*

As a consequence of Proposition 2.2, Theorems 2.3 and 2.4, we obtain the following characterization concerning pairs that form Galois connections.

Theorem 2.7: *The pair $(^\uparrow, ^\downarrow)$ is an antitone Galois connection, for all formal context (A, B, R, σ), if and only if the pair $(_z\swarrow^i, {}^z\nwarrow_i)$ is an antitone Galois connection, for all $z \in P$ and $i \in \Lambda$.*

Proof: Assume $(^\uparrow, ^\downarrow)$ is an antitone Galois connection. By Proposition 2.2 we have that $_z\swarrow^i$, $^z\nwarrow_i$ are antitone, therefore we have just to prove

- $y \preceq_2 {}^z\nwarrow_i(_z\swarrow^i(y))$
- $x \preceq_1 {}_z\swarrow^i(^z\nwarrow_i(x))$

and these statements hold as a consequence of Theorems 2.3 and 2.4, since they are exactly $(F1)$ and $(F2)$, respectively.
The converse follows similarly. □

As a result of the previous theorem, we can transform a problem between $^\uparrow\colon L_2^B \to L_1^A$, $^\downarrow\colon L_1^A \to L_2^B$, whose definitions are complex, to a problem related to the more basic operators $(_z\swarrow^i, {}^z\nwarrow_i)$.

It is worth to notice that the previous result does not imply a biunivocal correspondence between antitone Galois connections and pairs of implications associated with an adjoint triple, albeit we keep on using the implication symbols as a relic from our inspirational examples. We introduce below two operators ϕ and ψ which allow to define an antitone Galois connection $(^\uparrow, ^\downarrow)$; however, (ϕ, ψ) is not a pair of residuated implications of an adjoint triple, particularly, because they are not fuzzy implications. For instance, it might be interesting to consider an operator $\psi\colon P \times L_1 \to L_2$, antitone in both arguments, and satisfying that the smaller the arguments in P, the less decrease the function $\psi(z, \_)\colon L_1 \to L_2$. This kind of operators can be useful, for instance, to avoid possible noise or errors in the approximation of attributes with little relation to an object.

Example 2.8 Let us assume $(L_1, \preceq_1) = (L_2, \preceq_2) = (P, \leq) = ([0, 1], \leq)$, where $\leq$ is the usual ordering in the unit interval, and the pair (ϕ, ψ), where $\phi\colon [0, 1] \times [0, 1] \to$

$[0,1]$, $\psi\colon [0,1]\times[0,1]\to[0,1]$ are defined as follows

$$\phi(z,y)=\begin{cases}1 & \text{if } z\le 1-y\\ \sqrt{\dfrac{1-y}{z}} & \text{if } 1-y<z<\dfrac{1}{2}\\ \dfrac{1-y}{z} & \text{if } 1-y<z,\quad z\ge\dfrac{1}{2}\end{cases}$$

$$\psi(z,x)=\begin{cases}1-x^2z & \text{if } z<\dfrac{1}{2}\\ 1-xz & \text{if } z\ge\dfrac{1}{2}\end{cases}$$

We will prove that, given $z\in[0,1]$, the pair of mappings $\phi_z\colon[0,1]\to[0,1]$, $\psi_z\colon[0,1]\to[0,1]$ defined as $\phi_z(y)=\phi(z,y)$, $\psi_z(x)=\psi(z,x)$, for all $x,y\in[0,1]$, form an antitone Galois connection.

To begin with, clearly ϕ and ψ are antitone in both arguments (hence they are not fuzzy implications) and, therefore, ϕ_z and ψ_z are antitone.

A straightforward computation suffices to prove that $x\le\phi_z(\psi_z(x))$ and $y\le\psi_z(\phi_z(y))$, for all $x,y\in[0,1]$. Consequently, given $z\in[0,1]$, the pair (ϕ_z,ψ_z) is an antitone Galois connection and so, by Theorem 2.7, the mappings ${}^{\uparrow_\phi}\colon[0,1]^B\to[0,1]^A$, ${}^{\downarrow^\psi}\colon[0,1]^A\to[0,1]^B$, defined as

$$g^{\uparrow_\phi}(a)=\inf\{\phi(R(a,b),g(b))\mid b\in B\}$$
$$f^{\downarrow^\psi}(b)=\inf\{\psi(R(a,b),f(a))\mid a\in A\}$$

for all $a\in A$ and $b\in B$, form an antitone Galois connection.

Remark 2 : As a consequence of the example above, we have that the framework considered in this paper is more general than the one given in (Medina *et al.* 2009), in which we considered adjoint triples $(\&_i,\swarrow^i,\nwarrow_i)$ and in the current setting we only need pairs $(\swarrow^i,\nwarrow_i)$, such that $({}_z\swarrow^i,{}^z\nwarrow_i)$ are antitone Galois connections, for all $z\in P$.

The following example reinforces the previous situation by showing two fuzzy implications as those used in fuzzy formal concept analysis which form an antitone Galois connection but, however, do not arise from an adjoint triple.

Example 2.9 Consider the pair $(\swarrow,\nwarrow)$ where $\swarrow\colon[0,1]\times[0,1]\to[0,1]$, and $\nwarrow\colon[0,1]\times[0,1]\to[0,1]$ are defined by

$$z\nwarrow x=\begin{cases}1-x(1-z) & \text{if } z\le\dfrac{1}{2}\\ 1-x^2(1-z) & \text{if } z>\dfrac{1}{2}\end{cases}$$

$$z \swarrow y = \begin{cases} \dfrac{1-y}{1-z} & \text{if } z \leq \dfrac{1}{2}, \quad z < y \\ \sqrt{\dfrac{1-y}{1-z}} & \text{if } z > \dfrac{1}{2}, \quad z < y \\ 1 & \text{if } z \geq y \end{cases}$$

Note that these operators are similar to the ones given in Example 2.8, in which we have replaced z by $1 - z$ and obtained, as a consequence, the monotonicity needed to obtain a fuzzy implication.

In this case, the pair $({}_z\swarrow, {}^z\nwarrow)$ is an antitone Galois connection, for all $z \in [0, 1]$, and it can be shown that there does not exist any conjunctor & such that $(\&, \swarrow, \nwarrow)$ is an adjoint triple.

The previous examples show that we are working in a more general framework than (Medina *et al.* 2009), but still we can define the set of all *formal concepts* $\langle g, f\rangle$, that is, the set

$$\mathcal{M} = \{\langle g, f\rangle \mid g \in L_2^B, f \in L_1^A \text{ and } g^{\uparrow} = f, f^{\downarrow} = g\}$$

with the ordering defined by $\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle$ if and only if $g_1 \preceq_2 g_2$ (equivalently $f_2 \preceq_1 f_1$), can be proven to be a complete lattice (the proof is similar to the one given in (Medina *et al.* 2009)).

Since we only have considered a more general framework and the definitions are analogous, we keep the same names and still call the previous construction to be the *multi-adjoint concept lattice* associated with a given formal context.

3. Multi-adjoint concept lattices from isotone Galois connections

Inspired by the properties considered in (Djouadi and Prade 2011), this section introduces sufficient and necessary conditions on the considered fuzzy conjuntor and implication, in order to provide that the combinations of possibility and necessity operators will be isotone Galois connections. As a consequence, two kinds of concept lattices arise, which generalize the multi-adjoint object-oriented and property-oriented concept lattices given in (Medina 2012).

Definition 3.1: Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be posets, and ${}^{\downarrow}\colon P_1 \to P_2$, ${}^{\uparrow}\colon P_2 \to P_1$ mappings, the pair $({}^{\uparrow}, {}^{\downarrow})$ forms an *isotone Galois connection* between P_1 and P_2 if and only if:

(1) ${}^{\uparrow}$ and ${}^{\downarrow}$ are order-preserving.
(2) $x^{\downarrow\uparrow} \leq_1 x$, for all $x \in P_1$.
(3) $y \leq_2 y^{\uparrow\downarrow}$, for all $y \in P_2$.

For linking the results concerning isotone connections with those related to antitone connections it is worth to recall the definition of dual order. Given a set P and an order relation, $\leq$, on P, the *dual* order of $\leq$ is the relation $\leq^{\partial}$, defined as $x_1 \leq^{\partial} x_2$ if and only if $x_2 \leq x_1$, for all $x_1, x_2 \in P$. Usually, we will write P instead of the partially ordered set $(P, \leq)$, P^{∂} instead of $(P, \leq^{\partial})$, and we will say that P^{∂} is the *dual* of P.

It is well known that the definition of isotone Galois connection follows from the original one simply writing P_2^{∂} instead of P_2. Hence, an isotone Galois connection

$(^{\uparrow}, ^{\downarrow})$ on P_1 and P_2 is an antitone Galois connection on P_1 and P_2^{∂}, and vice versa, and we can translate the properties from antitone Galois connections to isotone Galois connections and vice versa.

Recently, this idea has been used in (Medina 2012) in order to relate several kinds of concept lattices. A similar transformation will be applied in the next sections in order to use the properties proved for the previous antitone Galois connection $(^{\uparrow}, ^{\downarrow})$ in the framework of object-oriented and property-oriented concept lattices.

3.1 *Multi-adjoint property-oriented concept lattices*

From now on, we will fix two complete lattices $(L_2, \preceq_2)$, $(L_3, \preceq_3)$, a poset $(P, \leq)$ and two mappings $\& \colon P \times L_2 \to L_3$, $\nwarrow \colon L_3 \times P \to L_2$.

Given a context (A, B, R, σ), we define two mappings $^{\uparrow_\pi} \colon L_2^B \to L_3^A$, $^{\downarrow^N} \colon L_3^A \to L_2^B$ as

$$g^{\uparrow_\pi}(a) = \sup\{R(a, b) \&_{a,b} g(b) \mid b \in B\} \tag{3}$$

$$f^{\downarrow^N}(b) = \inf\{f(a) \nwarrow_{a,b} R(a, b) \mid a \in A\} \tag{4}$$

for each $g \in L_2^B$, $f \in L_3^A$ and $a \in A$, $b \in B$. Clearly, these definitions are generalizations of the classical possibility and necessity operators.

This section establishes similar results to Section 2 but now, with respect to the operators $^{\uparrow_\pi}$, $^{\downarrow_N}$ and the underlying fuzzy conjunctors $\&_i$ and implications $\nwarrow_i$, with i in the index set Λ.

Firstly, we will focus on the monotonicity-related properties of these operators. Specifically, we will consider the operators $_x\& \colon L_2 \to L_3$, $\nwarrow_x \colon L_3 \to L_2$, where $x \in P$ and which are defined as $_x\&(y) = x \& y$, $\nwarrow_x (z) = z \nwarrow x$, for all $y \in L_2$, $z \in L_3$.

Lemma 3.2: *If the lattice L_3^{∂} is considered instead of L_3, then the operators $^{\uparrow_\pi}$ and $^{\downarrow^N}$ satisfy Equations (1) and (2), with respect to $\&$, $\nwarrow_{op}$, respectively, where $\nwarrow_{op} \colon P \times L_3 \to L_2$ is defined as $x \nwarrow_{op} z = z \nwarrow x$, for all $x \in P$ and $z \in L_3$.*

Proof: Considering L_3^{∂} instead of L_3, we have that

$$g^{\uparrow_\pi}(a) = \inf_{3,\partial}\{R(a, b) \&_{a,b} g(b) \mid b \in B\}$$

for all $g \in L_2^B$ and where $\inf_{3,\partial}$ is the infimum in (L_3^{∂}). Hence, the operator $^{\uparrow_\pi}$, which is associated with $\&$, is a mapping defined as in Equation (1). [1] Moreover, it is clear that $^{\downarrow^N}$, which is associated with $\nwarrow_{op}$, is a mapping defined as in Equation(2). □

Therefore, the results given in the previous section can be transformed into this new approach. First of all, results similar to those stated in Proposition 2.2 can be obtained in this framework as follows:

Proposition 3.3: *The mapping $^{\uparrow_\pi} \colon L_2^B \to L_3^A$ is isotone for all formal context (A, B, R, σ) if and only if $_x\& \colon L_2 \to L_3$ is isotone for all $x \in P$.*

Analogously, $^{\downarrow^N} \colon L_3^A \to L_2^B$ is isotone for all formal context (A, B, R, σ) if and only if $\nwarrow_x \colon L_3 \to L_2$ is isotone for all $x \in P$.

Proof: Considering L_3^{∂} instead of L_3, the first statement is equivalent to: The mapping $^{\uparrow_\pi} \colon L_2^B \to (L_3^{\partial})^A$ is antitone for all formal context (A, B, R, σ) if and only

[1] By abuse of notation, note that some of the required subscripts of $\&$ and $\nwarrow$ will be omitted wherever possible.

if ${}_x\&\colon L_2 \to (L_3^\partial)$ is antitone.

Now, by Lemma 3.2, ${}^{\uparrow_\pi}\colon L_2^B \to (L_3^\partial)^A$ is an operator similar to the one given by Equation (1), but now associated with a mapping ${}_x\&$. Thus, Proposition 2.2 can be applied and the result is obtained simply by undoing the change, that is, considering L_3 instead of L_3^∂.

The other claim follows analogously.

□

Remark 1: As a result of the previous proposition, we will assume hereafter in this section that the conjunctors $\&_i$ are isotone in the right argument and the arrows $\nwarrow_i$ are isotone in the left argument, for all $i \in \Lambda$.

A characterization of the closure property of the composition ${}^{\uparrow_\pi\downarrow^N}\colon L_2^B \to L_2^B$ is given by using Lemma 3.2 and Theorem 2.3, which turns out to be a generalization of (Djouadi and Prade 2011, Theorem 4).

Theorem 3.4: *The inequality $g \preceq_2 g^{\uparrow_\pi\downarrow^N}$ holds for all context (A, B, R, σ) and $g \in L_2^B$ if and only if the following property holds.*
$(P1)$: $\quad y \preceq_2 (x \,\&_i\, y) \nwarrow_i x$, *for all $x \in P$, $y \in L_2$, $i \in \Lambda$.*

Similarly, a characterization of the interior property of the composition ${}^{\downarrow^N\uparrow_\pi}$ can be obtained, by using Theorem 2.4 and Lemma 3.2. This result is similar to (Djouadi and Prade 2011, Theorem 5), but considering ${}^{\uparrow_\pi}\colon L_2^B \to L_3^A$, ${}^{\downarrow^N}\colon L_3^A \to L_2^B$ instead.

Theorem 3.5: *The inequality $f^{\downarrow^N\uparrow_\pi} \preceq_1 f$ holds for all context (A, B, R, σ) and $f \in L_1^A$ if and only if the following property holds:*
$(P2)$: $\quad x \,\&_i (z \nwarrow_i x) \preceq_3 z$, *for all $x \in P$, $z \in L_3$, $i \in \Lambda$.*

Proposition 3.3, and Theorems 3.4 and 3.5, provide a similar set of consequences to those introduced in the previous section, but now in the framework of property-oriented concept lattices.

Corollary 3.6:

(1) The composition ${}^{\uparrow_\pi\downarrow^N}\colon L_2^B \to L_2^B$ is a fuzzy closure operator for all context (A, B, R, σ) if and only if Property $(P1)$ is satisfied.
(2) The composition ${}^{\downarrow^N\uparrow_\pi}\colon L_1^A \to L_1^A$ is a fuzzy interior operator for all context (A, B, R, σ) if and only if Property $(P2)$ is satisfied.

Corollary 3.7: *The pair $({}^{\uparrow_\pi}, {}^{\downarrow^N})$ is an isotone Galois connection for all formal context (A, B, R, σ) if and only if Properties $(P1)$ and $(P2)$ hold, and the operators ${}_x\&$ and $\nwarrow_x$ are isotone operators for all $x \in P$.*

As the pair $({}_x\&, \nwarrow_x)$ is an isotone Galois connection, for all $x \in P$ if and only if the mappings $\&$ and $\nwarrow$ verify $(P1)$ and $(P2)$ and are isotone, we obtain the following characterisation:

Proposition 3.8: *The pair $({}^{\uparrow_\pi}, {}^{\downarrow^N})$ is an isotone Galois connection for all context (A, B, R, σ) if and only if the pair $({}_x\&, \nwarrow_x)$ is an isotone Galois connection for all $x \in P$.*

Therefore, the properties of the pair $({}^{\uparrow_\pi}, {}^{\downarrow^N})$ only depend on $\&$ and $\nwarrow$. As a consequence, if we consider two mappings $\&\colon P \times L_2 \to L_3$ and $\nwarrow\colon L_3 \times P \to L_2$, such that $({}_x\&, \nwarrow_x)$ is an isotone Galois connection, for all $x \in P$, then the mappings ${}^{\uparrow_\pi}\colon L_2^B \to L_1^A$, ${}^{\downarrow^N}\colon L_2^A \to L_1^B$ lead us to build a lattice which generalize the concept lattice introduced in (Medina 2012). Specifically, the set

$$\mathcal{M}_{\pi N} = \{\langle g, f\rangle \mid g \in L^B, f \in L^A, \text{ with } g^{\uparrow_\pi} = f, f^{\downarrow^N} = g\}$$

is a complete lattice, called *multi-adjoint property-oriented concept lattice*, where the pairs of mappings $\langle g, f\rangle \in \mathcal{M}_{\pi N}$ are called *multi-adjoint property-oriented formal concepts*.

Note that the names considered here for the concept lattice and for the concepts are equal to the ones given in (Medina 2012), this is because, in spirit, they are equal. The main difference is that the considered operators can be more general, since the pair $(\&, \nwarrow)$ *need not* arise from an adjoint triple.

In the example below we introduce a pair $({}_x\&, \nwarrow_x)$ which is an isotone Galois connection, although there not exists $\swarrow$ such that $(\&, \swarrow, \nwarrow)$ is an adjoint triple. Moreover, we will obtain, by Proposition 3.8, that $^{\uparrow_\pi}$ and $^{\downarrow^N}$, defined as in Equations (3) and (4), form an isotone Galois connection and a property-oriented concept lattice can be considered.

Example 3.9 Let us consider $(L_1, \preceq_1) = (L_2, \preceq_2) = (P, \leq) = ([0,1], \leq)$, where $\leq$ is the usual ordering in the unit interval, and the mapping $\& \colon [0,1]\times[0,1] \to [0,1]$, defined, for all $x, y \in [0,1]$, as

$$x \,\&\, y = \begin{cases} xy^2 & \text{if } x < \dfrac{1}{2} \\ xy & \text{if } \dfrac{1}{2} \leq x \end{cases}$$

Clearly, $\&$ is isotone in both arguments.

Now, we consider the mappings $\swarrow \colon [0,1]\times[0,1] \to [0,1]$ and $\nwarrow \colon [0,1]\times[0,1] \to [0,1]$, defined, for all $x, y, z \in [0,1]$, as

$$z \swarrow y = \sup\{x \mid x \,\&\, y \leq z\}$$
$$z \nwarrow x = \sup\{y \mid x \,\&\, y \leq z\}$$

As $\&$ is continuous in its second argument, clearly $\nwarrow$ is a residuated implication wrt $\&$.

Concerning $\swarrow$, the conjunctor is not continuous in $x = 1/2$ and this entails that the adjoint property fails. Specifically, we will use the known fact (Medina *et al.* 2004) that given a conjunctor, isotone in both arguments, we have that the considered supremum in the definition of the previous mappings is a maximum if and only if the mappings are residuated implications associated with $\&$.

In this case, it is easy to show a situation in which the maximum of $X = \{x \mid x \,\&\, y \leq z\}$ for fixed $y, z \in [0,1]$ fails to exist.

If $2z < y \leq \sqrt{2z}$, then for all $x < \frac{1}{2}$, $x \,\&\, y = xy^2 < \frac{1}{2}2z = z$ and $z < \frac{y}{2} = \frac{1}{2}y = \frac{1}{2} \,\&\, y$. Hence,

$$\sup X = \sup\{x \mid x \,\&\, y \leq z\} = \sup\{x \in [0, 1/2)\} = \frac{1}{2}$$

however, $\frac{1}{2} \,\&\, y \not\leq z$, that is $\frac{1}{2} \notin X$ and the supremum is not a maximum.

Consequently, $(\&, \swarrow, \nwarrow)$ is not an adjoint triple, although $({}_x\&, \nwarrow_x)$ is an isotone Galois connection, for all $x \in [0,1]$.

It is worth to note that operators as those introduced above are not just pathological operators, but could be useful to make a distinction between the values provided by the relation and those given by the fuzzy subset of object, which is interesting for applications. For instance, the conjunctor defined in the example only weights similarly the values entered by the relation and the fuzzy subset when the value of the relation is greater or equal than the threshold 1/2.

3.2 *Multi-adjoint object-oriented concept lattices*

Considering the possibility and necessity operators on different domains ${}^{\uparrow_N}\colon L_3^B \to L_1^A$, ${}^{\downarrow^\pi}\colon L_1^A \to L_3^B$, new operators and a new concept lattices arise.

Although this approach is similar to the previous one, we believe it useful to make explicit all the new results in order to provide a complete presentation of this framework.

In this section, we will fix a poset $(P, \leq)$, two complete lattices $(L_1, \preceq_1)$, $(L_3, \preceq_3)$, and two mappings $\&\colon L_1 \times P \to L_3$, $\swarrow\colon L_3 \times P \to L_1$.

Hence, given a context (A, B, R, σ), two mappings ${}^{\uparrow_N}\colon L_3^B \to L_1^A$, ${}^{\downarrow_\pi}\colon L_1^A \to L_3^B$ can be defined as

$$g^{\uparrow_N}(a) = \inf\{g(b) \swarrow^{a,b} R(a,b) \mid b \in B\} \tag{5}$$

$$f^{\downarrow^\pi}(b) = \sup\{f(a) \,\&_{a,b}\, R(a,b) \mid a \in A\} \tag{6}$$

for each $g \in L_3^B$, $f \in L_1^A$ and $a \in A$, $b \in B$.

In order to relate this framework to the one given in Section 2, the operators $\&^{\text{op}}\colon P \times L_1 \to L_3$, $\swarrow^{\text{op}}\colon P \times L_3 \to L_1$, will be considered, which are defined as $y \,\&^{\text{op}}\, x = x \,\&\, y$ and $y \swarrow^{\text{op}} z = z \swarrow y$, for all $x \in L_1$, $y \in P$, $z \in L_3$.

Lemma 3.10: *If the lattice L_3^∂ is considered instead of L_3, the operators ${}^{\uparrow_\pi}$ and ${}^{\downarrow^N}$ satisfy Equations (1) and (2), with respect to $\swarrow^{op}$, $\&^{op}$, respectively.*

Proof: Considering L_3^∂ and $\&^{\text{op}}$ instead of L_3 and $\&$, we have that

$$f^{\downarrow^\pi}(b) = \inf_{3,\partial}\{R(a,b) \,\&_{a,b}^{\text{op}}\, f(a) \mid a \in A\}$$

for all $f \in L_1^A$. Hence, the operator ${}^{\downarrow^\pi}$, associated with $\&^{\text{op}}$, is a mapping defined as in Equation (2). Moreover, it is clear that ${}^{\downarrow^N}$, considered to be associated with $\swarrow^{\text{op}}$, is a mapping defined as in Equation (1). □

In the rest of this section, we follow the structure and spirit of the previous ones, and introduce the corresponding characterizations related to fuzzy closure, fuzzy interior, and isotone Galois connection within this framework; the proofs will be omitted.

The first proposition is related to the monotonicity of ${}^{\downarrow^\pi}$ and ${}^{\uparrow_N}$. Specifically, we consider the operators $\&_y\colon L_1 \to L_3$, $\swarrow^y\colon L_3 \to L_1$, for each $y \in P$, which are defined as $\&_y(x) = x \,\&\, y$, $\swarrow^y(z) = z \swarrow^y$, for all $x \in L_1$, $z \in L_3$.

Proposition 3.11: *The mapping ${}^{\uparrow_N}\colon L_3^B \to L_1^A$ is isotone for all formal context (A, B, R, σ) if and only if $\swarrow^y\colon L_3 \to L_1$ is isotone for all $y \in P$.*

Similarly, ${}^{\downarrow^\pi}\colon L_1^A \to L_3^B$ is isotone for all formal context (A, B, R, σ) if and only if $\&_y\colon L_1 \to L_3$ is isotone for all $y \in P$.

Remark 2: As a result of the previous proposition, we will assume hereafter in this section that the arrows $\swarrow^i$ and conjunctors $\&_i$ are isotone in their first component for all $i \in \Lambda$.

The closure and interior properties of the compositions are characterized in the following results. With respect to ${}^{\downarrow^\pi\uparrow_N}\colon L_1^A \to L_1^A$ we have

Theorem 3.12: *The equality $f \preceq_1 f^{\downarrow^\pi \uparrow_N}$ holds for all context (A, B, R, σ) and $f \in L_1^A$ if and only if the following property is satisfied:*
$(O1)$: $x \preceq_1 (x \mathbin{\&_i} y) \swarrow^i y$, *for all* $x \in L_1$, $y \in P$, $i \in \Lambda$.

The composition ${}^{\uparrow_N \downarrow^\pi}$, related to the interior property is studied below:

Theorem 3.13: *The equality $g^{\uparrow_N \downarrow^\pi} \preceq_3 g$ holds for all context (A, B, R, σ) and $g \in L_3^B$ if and only if the following property is satisfied:*
$(O2)$: $(z \swarrow^i y) \mathbin{\&_i} y \preceq_3 z$, *for all* $y \in P$, $z \in L_3$, $i \in \Lambda$.

As previously, but now with respect to Proposition 3.11 and Theorems 3.12 and 3.13, the following consequences can be obtained.

Corollary 3.14:

(1) The composition ${}^{\downarrow^\pi \uparrow_N} \colon L_1^A \to L_1^A$ is a fuzzy closure operator for all context (A, B, R, σ) if and only if Property $(O1)$ holds.
(2) The composition ${}^{\uparrow_N \downarrow^\pi} \colon L_2^B \to L_2^B$ is a fuzzy interior operator for all context (A, B, R, σ) if and only if Property $(O2)$ holds.

Corollary 3.15: *The pair $({}^{\uparrow_N}, {}^{\downarrow^\pi})$ is an isotone Galois connection for all context (A, B, R, σ) if and only if $\swarrow^y$ and $\&_y$ are isotone for all $y \in P$ and Properties $(O1)$ and $(O2)$ hold.*

As a consequence, the properties of the pair $({}^{\uparrow_N}, {}^{\downarrow^\pi})$ only depend on the mappings $\swarrow$ and $\&$.

Proposition 3.16: *The pair $({}^{\uparrow_N}, {}^{\downarrow^\pi})$ is an isotone Galois connection for all context (A, B, R, σ) if and only if the pair $(\swarrow^y, \&_y)$ is an isotone Galois connection for all $y \in P$.*

Again, in this case it is possible to show the existence of reasonable pairs of mappings $\swarrow$, $\&$ which are not components of an adjoint triple but, however, are such that the pair $(\swarrow^y, \&_y)$ is an isotone Galois connection and, hence, an object-oriented concept lattice can be considered based on this kind of mappings.

4. Conclusions

We have introduced characterizations of the different Galois connections which arise in the analysis of formal concepts, in such a way that the algebraic requirements are directly translated on the fuzzy conjunctors and implications considered in the underlying multi-adjoint context.

As a result, it turned out that the approach given in (Medina 2012, Medina *et al.* 2009) can be applied in more general situations, i.e. the conjunctors and implications in the multi-adjoint formal context need not form adjoint triples, but satisfy weaker conditions instead, as it has been shown in several examples.

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