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RESEARCH ARTICLE

On the L-fuzzy generalization of Chu correspondences

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In this paper, we focus on the framework of Chu correspondences introduced by Mori for classical formal concept analysis, and we propose a suitable extension of the framework in a more general and flexible environment based on L-fuzzy sets, and define the notions of L-Chu correspondence and of L-bond. After introducing the generalized framework, the sets of L-Chu correspondences and of L-bonds are proved to have the structure of complete lattice and, furthermore, there exists a natural anti-isomorphism between them.

1. Introduction

Formal concept analysis [11] has become an important and appealing research topic both from a theoretical perspective [4, 17, 25, 28] and from the applied one. Concept lattices provide a productive framework for a variety of problems that arise in knowledge discovery in databases [5, 14, 24], but have found applications in a lot of different fields: for instance, we can find papers ranging from ontology merging [9, 23], to the Semantic Web by using the notion of concept similarity [10], and from processing of medical records in the clinical domain [15] or structuring phenotypes/genotypes in behavior genetics [7] to the development of recommender systems [6].

Soon after the introduction of "classical" formal concept analysis, a number of different approaches for its generalization were introduced and, nowadays, there are works which extend the theory using ideas borrowed from fuzzy set theory [2, 20] or fuzzy logic reasoning [1, 8] or from rough set theory [19, 26, 29] or some integrated approaches such as fuzzy and rough [27], or rough and domain theory [18].

Taking into account that formal concept analysis relies heavily on the notion of Galois connection, which is a particular instance of adjoint functors, it is not surprising that several authors have employed the power of the tools of category theory in order to deepen our understanding of concept lattices [12, 13, 16].

In this paper we focus on the categorical approach to formal concept analysis. Our approach, broadly continues the research line which links the theory of Chu spaces with concept lattices [30] but, particularly, is based on the notion of Chu correspondences between formal contexts developed by Mori in [21, 22]. In Mori's papers, the construction of formal concepts associated to a crisp relation between objects and attributes is shown to induce a functor from the category of Chu correspondences to the category of sup-preserving maps between complete lattices. Our contribution is the development of a suitable extension of the notions and results

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to the theory of Chu correspondences in an L-fuzzy framework; after introducing the generalized framework, the sets of L-Chu correspondences and of L-bonds are proved to have the structure of complete lattice and, furthermore, there exists a natural anti-isomorphism between them.

2. Preliminary definitions

We will assume that the reader is familiar with standard notions of *classical* formal concept analysis [11], such as context and formal concept lattice. For the benefit of the reader not acquainted with the basics of the *fuzzy* extensions of the theory of formal concept analysis, we provide the preliminary notions below.

2.1 *L-fuzzy concept lattice*

To begin with, the usual set of boolean values of classical logics (containing true and false), is generalized to the algebraic notion of complete residuated lattice, which allows to provide suitable extensions in a more abstract environment.

DEFINITION 2.1 *An algebra* $\langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ *is said to be a* ***complete residuated lattice*** *if*

(1) $\langle L, \wedge, \vee, 0, 1\rangle$ *is a complete bounded lattice with the least element* 0 *and the greatest element* 1*,*
(2) $\langle L, \otimes, 1\rangle$ *is a commutative monoid,*
(3) $\otimes$ *and* $\to$ *are adjoint, i.e.* $a \otimes b \le c$ *if and only if* $a \le b \to c$*, for all* $a, b, c \in L$*, where* $\le$ *is the ordering in the lattice generated from* $\wedge$ *and* $\vee$*.*

Now, the natural extension of the notion of context is given below.

DEFINITION 2.2 *Let* L *be a complete residuated lattice, an* ***L-fuzzy context*** *is a triple* $\langle B, A, r\rangle$ *consisting of a set of objects* B*, a set of attributes* A *and an* L*-fuzzy binary relation* r*, i.e. a mapping* $r \colon B \times A \to L$*, which can be alternatively understood as an* L*-fuzzy subset of* $B \times A$

We now introduce the L-fuzzy extension provided by Bělohlávek [2, 3], where we will use the notation Y^X to refer to the set of mappings from X to Y.

DEFINITION 2.3 *Consider an* L*-fuzzy context* $\langle B, A, r\rangle$*. A pair of mappings* $\uparrow \colon L^B \to L^A$ *and* $\downarrow \colon L^A \to L^B$ *can be defined for every* $f \in L^B$ *and* $g \in L^A$ *as follows:*

$$\uparrow f(a) = \bigwedge_{o \in B} (f(o) \to r(o, a)) \qquad\qquad \downarrow g(o) = \bigwedge_{a \in A} \big(g(a) \to r(o, a)\big) \tag{1}$$

LEMMA 2.4 (See [3]) *Let* L *be a complete residuated lattice, let* $r \in L^{B\times A}$ *be an* L*-fuzzy relation between* B *and* A*. Then the pair of operators* $\uparrow$ *and* $\downarrow$ *form a Galois connection between* $\langle L^B; \subseteq\rangle$ *and* $\langle L^A; \subseteq\rangle$*, that is,* $\uparrow \colon L^B \to L^A$ *and* $\downarrow \colon L^A \to L^B$ *are antitonic and, furthermore, for all* $f \in L^B$ *and* $g \in L^A$ *we have* $f \subseteq \downarrow\uparrow f$ *and* $g \subseteq \uparrow\downarrow g$*.*

It is important to recall that, given a Galois connection, there exists a natural notion of closure which is provided below:

DEFINITION 2.5 *Consider an* L*-fuzzy context* $C = \langle B, A, r\rangle$*. An* L*-fuzzy set of objects* $f \in L^B$ *(resp. an* L*-fuzzy set of attributes* $g \in L^A$*) is said to be* ***closed in*** $\mathbf{C}$ *iff* $f = \downarrow\uparrow f$ *(resp.* $g = \uparrow\downarrow g$*).*

Lemma 2.6 (See [2]) *Under the conditions of Lemma 2.4, the following equalities hold for arbitrary $f \in L^B$ and $g \in L^A$, $\uparrow f = \uparrow\downarrow\uparrow f$ and $\downarrow g = \downarrow\uparrow\downarrow g$, that is, both $\downarrow\uparrow f$ and $\uparrow\downarrow g$ are closed in C.*

Definition 2.7 *An L-**fuzzy concept** is a pair $\langle f, g\rangle$ such that $\uparrow f = g, \downarrow g = f$. The first component f is said to be the **extent** of the concept, whereas the second component g is the **intent** of the concept.*

The set of all L-fuzzy concepts associated to a fuzzy context (B, A, r) will be denoted as L-$FCL(B, A, r)$.

An ordering between L-fuzzy concepts is defined as follows: $\langle f_1, g_1\rangle \leq \langle f_2, g_2\rangle$ if and only if $f_1 \subseteq f_2$ if and only if $g_1 \supseteq g_2$.

Theorem 2.8 *The poset $(L\text{-}FCL(B, A, r), \leq)$ is a complete lattice where*

$$\bigwedge_{j\in J} \langle f_j, g_j\rangle = \Big\langle \bigwedge_{j\in J} f_j, \uparrow \big(\bigwedge_{j\in J} f_j\big)\Big\rangle$$

$$\bigvee_{j\in J} \langle f_j, g_j\rangle = \Big\langle \downarrow \big(\bigwedge_{j\in J} g_j\big), \bigwedge_{j\in J} g_j\Big\rangle$$

2.2 *Chu correspondences*

We know recall the basic definitions and notations about crisp Chu correspondences given in [22].

Definition 2.9 *A **multifunction** from X to Y is a mapping $f : X \to 2^Y$. Note that multifunctions are also called correspondences, or set-valued or multiple-valued functions.*

*The **transposed** of a multifunction $f : X \to 2^Y$ is another multifunction ${}^tf : Y \to 2^X$ defined by ${}^tf(y) = \{x \mid y \in f(x)\}$.*

The set $Mfn(X, Y)$ of all the multifunctions from X to Y can be endowed with a poset structure by defining the ordering $f_1 \leq f_2$ as $f_1(x) \subseteq f_2(x)$ for all $x \in X$.

Given a pair of classical (crisp) contexts, a Chu correspondence defines a sort of mapping between them, as presented in the definition below:

Definition 2.10 *Let $C_i = \langle B_i, A_i, R_i\rangle$ $(i = 1, 2)$ be crisp formal contexts. A pair $f = (f_l, f_r)$ is called a **correspondence from** C_1 **to** C_2 if f_l and f_r, respectively, are multifunctions from B_1 to B_2 and from A_2 to A_1.*

*A correspondence f from C_1 to C_2 is said to be a **weak Chu correspondence** if the following equality holds for every $o_1 \in B_1$ and $a_2 \in A_2$*

$$\bigwedge_{y\in f_r(a_2)} R_1(o_1, y) = \bigwedge_{x\in f_l(o_1)} R_2(x, a_2)$$

*A weak Chu correspondence f from C_1 to C_2 is said to be **strong** or, simply, a **Chu correspondence** if $f_l(o_1) \subseteq B_2$ is closed in C_2 and $f_r(a_2) \subseteq A_1$ is closed in C_1 for every $o_1 \in B_1$ and $a_2 \in A_2$.*

3. Extending the framework

3.1 *L-fuzzy Chu correspondences*

A convenient extension of the notion of multifunction in the framework of L-fuzzy sets is provided below.

DEFINITION 3.1 *An L-**multifunction** from X to Y is a mapping $\varphi\colon X \to L^Y$.*

*The **transposed** of an L-multifunction $\varphi\colon X \to L^Y$ is an L-multifunction ${}^t\varphi\colon Y \to {}^XL$ defined by ${}^t\varphi(y)(x) = \varphi(x)(y)$.*

The set L-$Mfn(X,Y)$ of all the L-multifunctions from X to Y can be endowed with a poset structure by defining the ordering $\varphi_1 \le \varphi_2$ as $\varphi_1(x)(y) \le \varphi_2(x)(y)$ for all $x \in X$ and $y \in Y$.

In the following, we will concentrate in obtaining a suitable generalization of the previous definitions to the framework L-fuzzy sets. To begin with, let us note that a given L-fuzzy context $r\colon B \times A \to L$ can be extended to the set of L-fuzzy objects and attributes as follows. We define a new mapping $\hat{r}\colon L^B \times L^A \to L$ such that for $f \in L^B$ and $g \in L^A$ we have

$$\hat{r}(f,g) = \bigwedge_{\substack{o\in B\\ a\in A}} (f(o) \otimes g(a) \to r(o,a)).$$

LEMMA 3.2 *The previous definition is a suitable generalization of Bělohlávek's Galois connection introduced in Defn 2.3.*

Proof Given a singleton $\{x\} \subseteq B$, consider its characteristic function $\chi_x \in L^B$ defined by $\chi_x(x) = 1$ and $\chi_x(o) = 0$ for all $o \in B, o \ne x$. Then

$$\begin{aligned}
\hat{r}(\chi_x, g) &= \bigwedge_{\substack{o\in B\\ a\in A}} (\chi_x(o) \otimes g(a) \to r(o,a))\\
&= \bigwedge_{\substack{o\in B\\ o\ne x\\ a\in A}} (\chi_x(o) \otimes g(a) \to r(o,a)) \wedge \bigwedge_{a\in A} (\chi_x(x) \otimes g(a) \to r(o,a))\\
&= \bigwedge_{\substack{o\in B\\ o\ne x\\ a\in A}} (0 \otimes g(a) \to r(o,a)) \wedge \bigwedge_{a\in A} (1 \otimes g(a) \to r(x,a))\\
&= \bigwedge_{a\in A} (g(a) \to r(x,a))
\end{aligned}$$

Therefore, it coincides with Bělohlávek's definition, in which the element x has been substituted by the characteristic function χ_x.

A similar result can be obtained by fixing a singleton in the set of attributes. ■

DEFINITION 3.3 *Consider two L-fuzzy contexts $C_i = \langle B_i, A_i, r_i\rangle, (i = 1,2)$, then the pair $\varphi = (\varphi_l, \varphi_r)$ is called a **correspondence** from C_1 to C_2 if φ_l and φ_r are L-multifunctions, respectively, from B_1 to B_2 and from A_2 to A_1 (that is, $\varphi_l : B_1 \to L^{B_2}$ and $\varphi_r : A_2 \to L^{A_1}$).*

*The L-correspondence φ is said to be a **weak L-Chu correspondence** if the equality $\hat{r}_1(\chi_{o_1}, \varphi_r(a_2)) = \hat{r}_2(\varphi_l(o_1), \chi_{a_2})$ holds for all $o_1 \in B_1$ and $a_2 \in A_2$. By unfolding the definition of $\hat{r}_i$ this means that*

$$\bigwedge_{a_1\in A_1} (\varphi_r(a_2)(a_1) \to r_1(o_1,a_1)) = \bigwedge_{o_2\in B_2} (\varphi_l(o_1)(o_2) \to r_2(o_2,a_2)) \tag{2}$$

*A weak Chu correspondence φ is an **L-Chu correspondence** if $\varphi_l(o_1)$ is closed in C_2 and $\varphi_r(a_2)$ is closed in C_1 for all $o_1 \in B_1$ and $a_2 \in A_2$. We will denote the set of all Chu correspondences from C_1 to C_2 by L-$ChuCors(C_1, C_2)$.*

Now, we will show that this definition allows us to provide a suitable generalization of Mori's definition of weak Chu correspondence and Chu correspondence as follows.

LEMMA 3.4 *Every weak 2-Chu correspondence is a (crisp) weak Chu correspondence. Furthermore, 2-$ChuCors(C_1, C_2) = ChuCors(C_1, C_2)$.*

Proof Recall that in the classical case, hence we are working with crisp relations. Then

$$\begin{aligned}
\bigwedge_{a_1 \in \varphi_r(a_2)} r_1(o_1, a_1) &= \bigwedge_{a_1 \in \varphi_r(a_2)} (1 \to r_1(o_1, a_1)) \\
&= \bigwedge_{a_1 \in \varphi_r(a_2)} (\varphi_r(a_2)(a_1) \to r_1(o_1, a_1)) \\
&= \bigwedge_{a_1 \in A_1} (\varphi_r(a_2)(a_1) \to r_1(o_1, a_1)) \\
&= \widehat{r}_1(\chi_{o_1}, \varphi_r(a_2)) \qquad \text{(by property (2) of weak } L\text{-Chu)} \\
&= \widehat{r}_2(\varphi_l(o_1), \chi_{a_2}) \qquad (\ldots \text{ and by a similar chain of equalities)} \\
&= \bigwedge_{o_2 \in \varphi_l(o_1)} r_2(o_2, a_2)
\end{aligned}$$

Thus, any weak 2-Chu correspondence is actually a crisp weak Chu correspondence.

The property of being strong is preserved straightforwardly because of the conservativeness of the extension of the arrows ↑ and ↓. ■

3.2 *On bonds and L-bonds*

The notion of bond was introduced in [11] as a means of characterizing complete sublattices of a direct product for which the projection mappings are surjective. However, our aim in this section is more related to the fact that bonds are mappings between contexts and, following [22], there might be some relation with the notion of Chu correspondence between contexts.

Firstly, we recall the classical definition of bond and, then, we extend it to the L-fuzzy framework.

DEFINITION 3.5 *A **bond** from a context $C_1 = \langle B_1, A_1, R_1\rangle$ to a context $C_2 = \langle B_2, A_2, R_2\rangle$ is a relation $R_b \subseteq B_1 \times A_2$ for which the following holds:*

- *$\{a_2 \in A_2 : (o_1, a_2) \in R_b\}$ is an intent of C_2 for every $o_1 \in B_1$*
- *$\{o_1 \in B_1 : (o_1, a_2) \in R_b\}$ is an extent of C_1 for every $a_2 \in A_2$.*

Now, we introduce our candidate for the L-fuzzy extension of the notion of bond.

DEFINITION 3.6 *An L-**bond** between two formal contexts $C_1 = \langle B_1, A_1, r_1\rangle$ and $C_2 = \langle B_2, A_2, r_2\rangle$ is a multifunction $b : B_1 \to L^{A_2}$ satisfying the condition that for all $o_1 \in B_1$ and $a_2 \in A_2$ both $b(o_1)$ and ${}^t b(a_2)$ are closed L-fuzzy sets of, respectively, attributes in C_2 and objects in C_1. The set of all bonds from C_1 to C_2 is denoted as L-$Bonds(C_1, C_2)$.*

4. Relating *L*-Chu correspondences and *L*-bonds

Every multifunction $b : B_1 \to L^{A_2}$ is a relation, thus an L-bond can be seen as a relation between B_1 and A_2. This certainly suggests a possible relationship between L-Chu correspondences and L-bonds.

Definition 4.1 *Let $C_1 = \langle B_1, A_1, r_1\rangle$ and $C_2 = \langle B_2, A_2, r_2\rangle$ be L-fuzzy contexts:*

- *Let $b\colon C_1 \to C_2$ be an L-bond. We can define a correspondence $\varphi_b\colon C_1 \to C_2$ by*

$$\varphi_{bl}(o_1) = \downarrow_2 (b(o_1)) \in L^{B_2} \text{ for } o_1 \in B_1$$
$$\varphi_{br}(a_2) = \uparrow_1 ({}^t b(a_2)) \in L^{A_1} \text{ for } a_2 \in A_2$$

- *Conversely, consider an L-Chu correspondence φ from C_1 to C_2, and define a multifunction $b_\varphi\colon B_1 \to L^{A_2}$ by*

$$b_\varphi(o_1) = \uparrow_2 (\varphi_l(o_1))$$

Remark 1 Note that $\big(\downarrow_1\varphi_r(a_2)\big)(o_1) = \big(\uparrow_2\varphi_l(o_1)\big)(a_2)$ holds since φ is weak Chu. As a result, we can write the transposed of the bond as ${}^t b_\varphi(a_2) = \downarrow_1(\varphi_r(a_2))$.

The following proposition states that every L-bond defines an L-Chu correspondence, and vice versa.

Proposition 4.2 *With the definitions given above*

(1) φ_b is an L-Chu correspondence from C_1 to C_2.
(2) b_φ is an L-bond from C_1 to C_2.

Proof Both proofs follow as a result of more or less straightforward chains of computations. We will only include one of them.

(1) Let $o_1 \in B_1$ and $a_2 \in A_2$. Then

$$\begin{aligned}
\widehat{r_2}(\varphi_{bl}(o_1), \chi_{a_2}) &= \bigwedge_{o_2 \in B_2} (\varphi_{bl}(o_1)(o_2) \to r_2(o_2, a_2)) \\
&= \bigwedge_{o_2 \in B_2} (\downarrow_2 (b(o_1))(o_2) \to r_2(o_2, a_2)) \\
&= \uparrow_2 (\downarrow_2 (b(o_1)))(a_2) \\
&= b(o_1)(a_2) = {}^t b(a_2)(o_1) \\
&= \downarrow_1 (\uparrow_1 ({}^t b(a_2)))(o_1) \\
&= \bigwedge_{a_1 \in A_1} (\uparrow_1 ({}^t b(a_2))(a_1) \to r_1(o_1, a_1)) \\
&= \bigwedge_{a_1 \in A_1} (\varphi_{br}(a_2)(a_1) \to r_1(o_1, a_1)) = \widehat{r_1}(\chi_{o_1}, \varphi_{br}(a_2))
\end{aligned}$$

■

The previous proposition suggests a close relationship between the notions of L-Chu correspondence and L-bond between two formal contexts. We continue with the introduction of several technical lemmas which will be used later.

The first one states that the two mappings between L-bonds and L-Chu correspondences are closely related.

LEMMA 4.3 *Let $C_1 = \langle B_1, A_1, r_1 \rangle$ and $C_1 = \langle B_2, A_2, r_2 \rangle$ are two L-fuzzy contexts. For all L-Chu correspondences $\varphi \in$ L-ChuCors(C_1, C_2) and for all $b \in$ L-Bonds(C_1, C_2) holds*

(1) $b_{\varphi_b} = b$
(2) $\varphi_{b_\varphi} = \varphi$.

Proof Let $o_1 \in B_1$ be an arbitrary object of C_1 and $a_2 \in A_2$ an arbitrary attribute of C_2

(1) $b_{\varphi_b}(o_1) = \uparrow_2(\varphi_{bl}(o_1)) = \uparrow_2(\downarrow_2(b(o_1))) = b(o_1)$
(2) a) $\varphi_{b_\varphi l}(o_1) = \downarrow_2(b_\varphi(o_1)) = \downarrow_2(\uparrow_2(\varphi_l(o_1))) = \varphi_l(o_1)$
b) $\varphi_{b_\varphi r}(a_2) = \uparrow_1({}^t b_\varphi(a_2)) = \uparrow_1(\downarrow_1(\varphi_r(a_2))) = \varphi_r(a_2)$

■

LEMMA 4.4 *Let $\langle B, A, r \rangle$ be an L-fuzzy context and $(\uparrow, \downarrow)$ the mappings defined in* (1). *Let $f_i \in L^B$ and $g_i \in L^A$ for all $i \in I$. Then*

$$\downarrow (\bigvee_{i \in I} g_i) = \bigwedge_{i \in I} \downarrow (g_i) \quad \textit{and} \quad \uparrow (\bigvee_{i \in I} f_i) = \bigwedge_{i \in I} \uparrow (f_i)$$

Proof Let $o \in B$ be an arbitrary object.

$$\begin{aligned} \downarrow (\bigvee_{i \in I} (g_i))(o) &= \bigwedge_{a \in A} (\bigvee_{i \in I} g_i(a) \to r(o, a)) \\ &= \bigwedge_{a \in A} \bigwedge_{i \in I} (g_i(a) \to r(o, a)) \\ &= \bigwedge_{i \in I} (\bigwedge_{a \in A} (g_i(a) \to r(o, a))) = \bigwedge_{i \in I} (\downarrow (g_i)(o)) \end{aligned}$$

The second equality is proved similarly. ■

LEMMA 4.5 *Let C_1, C_2 be L-fuzzy formal contexts, and let $b_i \in$ L-Bonds(C_1, C_2) for all $i \in I$, and consider the following operations*

- $(\bigwedge_{i \in I} b_i)(o) = \bigwedge_{i \in I} (b_i(o))$ *for all $o \in B_1$.*
- $(\bigvee_{i \in I} b_i)(o) = \uparrow_2 \downarrow_2 (\bigvee_{i \in I} (b_i(o))) = \uparrow_2 (\bigwedge_{i \in I} \downarrow_2 (b_i(o)))$ *for all $o \in B_1$.*

The previous operations provide L-Bonds(C_1, C_2) with a complete lattice structure.

Proof

(1) Let $o_1 \in B_1$ be an arbitrary object and, firstly, let us prove that $\bigwedge_{i \in I} b_i(o_1)$ is closed.

Note that $\bigwedge_{i \in I} (b_i(o_1)) \subseteq b_i(o_1)$ for all $i \in I$ and that, by monotonicity of the arrows and closedness of $b_i(o_1)$, we have

$$\uparrow_2 \downarrow_2 (\bigwedge_{i \in I} (b_i(o_1))) \subseteq \uparrow_2 \downarrow_2 (b_i(o_1)) = b_i(o_1) \text{ for all } i \in I$$

The closedness of $\bigwedge_{i\in I} b_i(o_1)$ is proved by showing the two inclusions:

$$\uparrow_2\downarrow_2 (\bigwedge_{i\in I}(b_i(o_1))) \subseteq \bigwedge_{i\in I} b_i(o_1) \text{ follows from properties of infimum}$$

$$\uparrow_2\downarrow_2 (\bigwedge_{i\in I}(b_i(o_1))) \supseteq \bigwedge_{i\in I} b_i(o_1) \text{ follows from properties of closure operator.}$$

The closedness of the transposed follows from the fact that ${}^t(\bigwedge_{i\in I} b_i) = \bigwedge_{i\in I} {}^tb_i$. Now, the same procedure can be applied in order to show that $\downarrow_1\uparrow_1 (\bigwedge_{i\in I}({}^tb_i(a_2))) = \bigwedge_{i\in I} {}^tb_i(a_2)$ for arbitrary $a_2 \in A_2$.

(2) Similar. The main point is that, as $\bigvee_{i\in I}(b_i(o_1))$ needs not be closed in L^{A_2}, its closure has to be considered.

■

Lemma 4.6 *Let C_1, C_2 be L-fuzzy formal contexts. L-ChuCors(C_1, C_2) is a complete lattice in which, for $\varphi_{li} \in$ L-ChuCors(C_1, C_2) for all $i \in I$, the infimum and the supremum are defined as follows:*

(1) $(\bigwedge_{i\in I} \varphi_{li})(o) = \bigwedge_{i\in I}(\varphi_{li}(o))$ *for all* $o \in B_1$.

(2) $(\bigvee_{i\in I} \varphi_{li})(o) = \uparrow_2\downarrow_2 (\bigvee_{i\in I}(\varphi_{li}(o)))$ *for all* $o \in B_1$.

Proof We will only prove that the constructions given in the statement really are L-Chu correspondences.

(1) The property of weak Chu correspondence is proved below:

$$\begin{aligned}
\widehat{r_1}(\chi_{o_1}, \bigwedge_{i\in I} \varphi_{r_i}(a_2)) &= \widehat{r_1}(\chi_{o_1}, \bigwedge_{i\in I} \uparrow_1 ({}^tb_i(a_2))) = \widehat{r_1}(\chi_{o_1}, \uparrow_1 (\bigvee_{i\in I} {}^tb_i(a_2))) \\
&= \bigwedge_{a_1\in A_1} (\uparrow_1 (\bigvee_{i\in I} {}^tb_i(a_2))(a_1) \to r_1(o_1, a_1)) \\
&= \downarrow_1\uparrow_1 (\bigvee_{i\in I} {}^tb_i(a_2))(o_1) = \bigvee_{i\in I} {}^tb_i(a_2)(o_1) \\
&= \bigvee_{i\in I} b_i(o_1)(a_2) = \uparrow_2\downarrow_2 (\bigvee_{i\in I} b_i(o_1)(a_2)) \\
&= \uparrow_2 \bigwedge_{i\in I}(\downarrow_2 (b_i(o_1)(a_2))) \\
&= \bigwedge_{o_2\in B_2} (\bigwedge_{i\in I} \downarrow_2 (b_i(o_1))(o_2) \to r_2(o_2, a_2)) \\
&= \widehat{r_2}(\bigwedge_{i\in I} \downarrow_2 (b_i(o_1)), \chi_{a_2}) = \widehat{r_2}(\bigwedge_{i\in I} \varphi_l(o_1), \chi_{a_2})
\end{aligned}$$

Finally, as the intersection of closed sets is closed, the we have defined indeed a strong L-Chu correspondence.

(2) In this second case, the strong property arises directly from the definition,

and we only have to focus on the weak property of L-Chu correspondence:

$$
\begin{aligned}
\widehat{r_1}(\chi_{o_1}, \uparrow_1\downarrow_1 (\bigvee_{i\in I} \varphi_{r_i}(a_2))) &= \bigwedge_{a_1\in A_1} \big(\uparrow_1\downarrow_1 (\bigvee_{i\in I} \varphi_{r_i}(a_2))(a_1) \to r_1(o_1, a_1)\big) \\
&= \downarrow_1\uparrow_1\downarrow_1 (\bigvee_{i\in I} \varphi_{r_i}(a_2))(o_1) \\
&= \downarrow_1\uparrow_1 (\bigwedge_{i\in I} \downarrow_1 (\varphi_{r_i}(a_2)))(o_1) \\
&= \downarrow_1\uparrow_1 (\bigwedge_{i\in I} {}^t b_{\varphi_{r_i}}(a_2))(o_1) \\
&= \bigwedge_{i\in I} {}^t b_{\varphi_{r_i}}(a_2)(o_1) = \bigwedge_{i\in I} b_{\varphi_{l_i}}(o_1)(a_2) \\
&= \uparrow_2\downarrow_2 (\bigwedge_{i\in I} b_{\varphi_{l_i}}(o_1))(a_2) \\
&= \uparrow_2\downarrow_2 (\bigwedge_{i\in I} \uparrow_2 (\varphi_{l_i}(o_1)))(a_2) \\
&= \uparrow_2\downarrow_2\uparrow_2 (\bigwedge_{i\in I} \varphi_{l_i}(o_1))(a_2) \\
&= \bigwedge_{o_2\in B_2} \big(\downarrow_2\uparrow_2 (\bigvee_{i\in I} \varphi_{l_i}(o_1))(o_2) \to r_2(o_2, a_2)\big) \\
&= \widehat{r_2}(\downarrow_2\uparrow_2 (\bigvee_{i\in I} \varphi_{l_i}(o_1)), \chi_{a_2})
\end{aligned}
$$

■

The mapping which assigns the bond b_φ to each Chu correspondence φ (see Defn 4.1) is a strong link between L-Chu correspondences and L-bonds. We will prove below that it is actually an order-reversing lattice isomorphism between $L\text{-}ChuCors(C_1, C_2)$ and $L\text{-}Bonds(C_1, C_2)$.

THEOREM 4.7 *The lattice $L\text{-}ChuCors(C_1, C_2)$ and the opposite lattice of L-bonds $L\text{-}Bonds(C_1, C_2)^*$ are isomorphic.*

Proof As stated above, the isomorphism is defined by the mapping which assigns the bond b_φ to each Chu correspondence φ. The details of the prove that it is a lattice isomorphism are the following:

(1) The mapping is a bijection as a consequence of Lemma 4.3.
(2) Let us consider two arbitrary Chu correspondences $(\varphi_{l1}, \varphi_{r1})$ and $(\varphi_{l2}, \varphi_{r2})$, such that $\varphi_{l1}(o_1) \leq \varphi_{l2}(o_1)$ and $\varphi_{r1}(a_2) \leq \varphi_{r2}(a_2)$ for all $o_1 \in B_1$ and $a_2 \in A_2$.

   Because of antitonicity of $\uparrow_2$ and $\downarrow_1$, the ordering in $L\text{-}Bond(C_1, C_2)$ is reversed, and we obtain

$$
\begin{aligned}
b_{\varphi_{l1}}(o_1) = \uparrow_2 (\varphi_{l1}(o_1)) \geq \uparrow_2 (\varphi_{l2}(o_1)) = b_{\varphi_{l2}}(o_1) \text{ and} \\
{}^t b_{\varphi_{r1}}(a_2) = \downarrow_1 (\varphi_{r1}(a_2)) \geq \downarrow_1 (\varphi_{r2}(a_2)) = {}^t b_{\varphi_{r2}}(a_2).
\end{aligned}
$$

(3) Finally, we show how the mapping behaves with respect to suprema and infima. Firstly, we see that the image of a supremum is the infimum of the

images.

$$b_{\bigvee_{i\in I}\varphi_i}(o_1) = \uparrow_2 ((\bigvee_{i\in I}\varphi_{li})(o_1)) = \uparrow_2 (\downarrow_2\uparrow_2 (\bigvee_{i\in I}(\varphi_{li}(o_1))))$$
$$= \uparrow_2\downarrow_2 (\bigwedge_{i\in I}(\uparrow_2 (\varphi_{li}(o_1)))) = \uparrow_2\downarrow_2 (\bigwedge_{i\in I}(b_{\varphi_{li}}(o_1))) = \bigwedge_{i\in I} b_{\varphi_i}(o_1)$$

Secondly, we see that the image of an infimum is the supremum of the images.

$$b_{\bigwedge_{i\in I}\varphi_i}(o_1) = \uparrow_2 ((\bigwedge_{i\in I}\varphi_{li})(o_1)) = \uparrow_2 (\bigwedge_{i\in I}(\varphi_{li}(o_1)))$$
$$= \uparrow_2 (\bigwedge_{i\in I}\downarrow_2\uparrow_2 (\varphi_{li}(o_1))) = \uparrow_2\downarrow_2 (\bigvee_{i\in I}\uparrow_2 (\varphi_{li}(o_1)))$$
$$= \uparrow_2\downarrow_2 (\bigvee_{i\in I}(b_{\varphi_i}(o_1))) = (\bigvee_{i\in I} b_{\varphi_i})(o_1)$$

■

5. Conclusion

We have introduced a generalized approach of Chu correspondences in the context of L-fuzzy sets. The notions of L-Chu correspondence and L-bond have been defined and the sets of L-Chu correspondences and L-bonds have been shown to have the structure of complete lattice. Finally, given two contexts C_1 and C_2, we have proved that there exists a lattice anti-isomorphism between $L\text{-}ChuCors(C_1, C_2)$ and $L\text{-}Bonds(C_1, C_2)$.

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