Concept-forming operators on multilattices *

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Abstract. Adjoint pairs or adjoint triples defined on lattices have proven to be a useful tool when working in fuzzy formal concept analysis. This paper shows that adjoint pairs and triples can play as well an important role within the framework of multilattices, especially in order to form the Galois connections needed to build concept multilattices.

Keywords: Fuzzy formal concept analysis; Galois connection; Multilattices

1 Introduction

The notion of adjoint pairs (or adjoint triples) has been fruitfully used in areas such as extended logic programming or fuzzy formal concept analysis, as an important tool to deal with uncertainty, imprecise data or incomplete information which provides different fuzzifications of the classical framework of these theories, by considering arbitrary complete residuated lattices as underlying set of truth values.

In the literature we can find many approaches which fuzzify the classical concept lattices given by Ganter and Wille [11] allowing some uncertainty in data as we can see in the papers of Burusco and Fuentes-González [4] where fuzzy concept lattices were first presented, and later further developed by Pollandt [20] and Bělohlávek [1] or the work of Georgescu and Popescu [12] which allows non-commutative operators. Bělohlávek [2] provided a further method to extend the fuzzy concept lattice by using L-equalities. This approach was later extended in an asymmetric way, although only for the case of classical equality ($L = \{0, 1\}$) by Krajči, who introduced the so-called generalized concept lattices in [15,14].

We can also cite another approach proposed by Medina et al in [19,18] which introduces the multi-adjoint concept lattices, joining the multi-adjoint philosophy with concept lattices. To do this the authors needed to generalize the adjoint pairs into what they called adjoint triples [6].

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All these approaches have in common that the underlying set of values are lattices; recently, in [21] the construction of concept lattice was given on the more general structure called multilattice, where the restrictions imposed on a (complete) lattice, namely, the "existence of least upper bounds and greatest lower bounds" is relaxed to "existence of *minimal* upper bounds and *maximal* lower bounds" is relaxed to the existence if minimal (maximal) elements of the upper(lower) bounds of the subsets.

This is not merely another mathematical abstraction, but a way to reflect the fact that, in real life, there are many things which cannot be compared, in the sense that ones are better than others. And this also applies to the "best" upper bounds of a set. This idea is what led us to consider multilattices as underlying set of truth values in our work in formal concept analysis.

The theory of multilattices is not a new one since there was a first definition used in [3,13]. This definition was improved later by Cordero et al in [5,16], with the original aim of providing some advances to the theory of mechanized deduction in temporal logics.

Multilattices, in the sense of the paragraph above, also arise in a natural manner in the research area concerning fuzzy extensions of logic programming [17]. For instance, one of the hypotheses of the main termination result for sorted multi-adjoint logic programs [7] has been weakened only when the underlying set of truth-values is a multilattice [8].

Back to formal concept analysis, the authors introduced in [21] the fuzzy concept multilattice, but the question on how to form the Galois connections to build them was still open.

In this paper, we show that the usual concept forming operators when working on an adjoint triple, directly generate a Galois connection, regardless the underlying framework being that of multilattices. Technically, the point is a suitable generalization of the notion of left-continuity used by Krajči in [15,14] to the framework multilattices. This generalization together with some boundary conditions is proved to be equivalent, as in the lattice-based case, to the existence of adjoint triples and ensures the existence of the Galois connections.

The plan of this paper is the following: in Section 2 we present the main definitions and results to understand the paper. Section 3 presents the formal concept multilattice together with their first properties, while in Section 4 we give the properties needed for the forming concept operators to form Galois connections; the paper ends with a worked example, some conclusions and prospects for future work.

2 Preliminaries

In order to make this paper self-contained, this section introduces several well known notions of lattice theory as a starting point to consider later their extensions defined on multilattices; we will recall some parts of the Galois connection theory, as well.

Definition 1. *A* complete lattice *is a poset,* $(L, \preceq)$*, where every subset of* L *has supremum and infimum.*

The definition of semilattice arises when, instead of the existence of both supremum and infimum for every subset, we only ask for the existence of one of them (either supremum or infimum). A multilattice is an structure that generalizes the notions of lattice and semilattice. Before introducing its definition, we will recall some notions which we will use to in the definition of multilattices.

Definition 2. *Let* $(P, \leq)$ *be a poset and* $K \subseteq P$*, we say that:*

- K *is an* antichain *if its elements are pairwise uncomparable, i.e., for every* $x, y \in K$ *we have that* $x \nleq y$ *and* $y \nleq x$*.*
- K *is a* chain *if for pair* $x, y \in K$ *we have either* $x \leq y$ *or* $y \leq x$*.*
- $(P, \leq)$ *is called* coherent *if every chain has supremum and infimum.*

Definition 3. *A* complete multilattice *is a coherent poset without infinite antichains,* $(M, \leq)$*, where for every subset, the set of its upper (lower) bounds has minimal (maximal) elements.*

Each minimal (maximal) element of the upper (lower) bounds of a subset is called *multisupremum (multinfimum)*. The set of all multisuprema (multinfima) will be denoted by multisup (multinf).

Example 1. Figure 1 shows the Hasse diagram of the smaller multilattice which is not a lattice. This multilattice is denoted as $M6$.

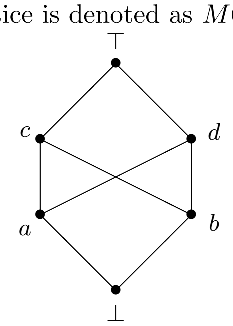

Fig. 1. Multilattice $(M6, \leq)$

We can check that, given the subset $\{a, b\}$, there is just one multiinfimum (which coincides with the infimum), since multinf$\{a, b\} = \{\bot\}$, and the multisuprema are multisup$\{a, b\} = \{c, d\}$, hence, there is not a supremum but two multisuprema $\{c, d\}$. Moreover, if we consider the subset $\{c, d\}$ we have that multinf$\{c, d\} = \{a, b\}$, then there is not an infimum but two multiinfima $\{a, b\}$ and multisup$\{a, b\} = \{\top\}$.

The notion of Galois connection [9,10], which we recall here, will play a key role hereafter.

Definition 4. *Let* $^{\downarrow}: P \to Q$ *and* $^{\uparrow}: Q \to P$ *be two maps between the posets* $(P, \leq)$ *and* $(Q, \leq)$*. The pair* $(^{\uparrow}, ^{\downarrow})$ *is called a* Galois connection *if:*

- $p_1 \leq p_2$ *implies* ${p_2}^{\downarrow} \leq {p_1}^{\downarrow}$ *for every* $p_1, p_2 \in P$*;*
- $q_1 \leq q_2$ *implies* ${q_2}^{\uparrow} \leq {q_1}^{\uparrow}$ *for every* $q_1, q_2 \in Q$*;*
- $p \leq p^{\uparrow\downarrow}$ *and* $q \leq q^{\downarrow\uparrow}$ *for all* $p \in P$ *and* $q \in Q$*.*

An interesting property of a Galois connection $(^{\uparrow}, ^{\downarrow})$ is that $^{\downarrow} = {}^{\downarrow\uparrow\downarrow}$ and $^{\uparrow} = {}^{\uparrow\downarrow\uparrow}$ where the chain of arrows means their composition.

Once we have a Galois connection we can focus on the pairs of elements (p, q) which are the image of each other by the application of the corresponding arrow. These pairs are called concepts, as stated in the following definition.

Definition 5. *A pair* (p, q) *is called a* concept *if* $p^{\downarrow} = q$ *and* $q^{\uparrow} = p$*.*

In the case that P and Q are lattices, the following result holds as well.

Theorem 1 ([9]). *Let* $(L_1, \preceq_1)$ *and* $(L_2, \preceq_2)$ *be two complete lattices and* $(^{\uparrow}, ^{\downarrow})$ *a Galois connection between them, then we have that the set* $\mathcal{C} = \{(x, y) \mid x \in L_1, y \in L_2$ *with* $x^{\downarrow} = y, y^{\uparrow} = x\}$ *is a complete lattice with the following ordering* $(x_1, y_1) \preceq (x_2, y_2)$ *if and only if* $x_1 \preceq_1 x_2$ *(or equivalently* $y_2 \preceq_2 y_1$*), where the supremum and the infimum are given below:*

$$\bigwedge_{i \in I} (x_i, y_i) = \left(\bigwedge_{i \in I} x_i, (\bigvee_{i \in I} y_i)^{\uparrow\downarrow} \right)$$

$$\bigvee_{i \in I} (x_i, y_i) = \left((\bigvee_{i \in I} x_i)^{\downarrow\uparrow}, \bigwedge_{i \in I} y_i \right)$$

The main aim in the next section will be to generalize the theorem above by considering multilattices.

3 Fuzzy formal concept multilattices

The fuzzy formal concept multilattices were introduced in [21], in this section we recall the main definitions and results. From now on, two sets A and B, and two multilattices $(M_1, \leq_1)$ and $(M_2, \leq_2)$ will be fixed. Moreover, we will denote by M_1^A and M_2^B the sets of all mappings from A to M_1 and from B to M_2 respectively.

The following proposition is a technical result which shows some extra differences with respect to the standard theory of lattices.

Proposition 1 ([21]). *Let* $(M_1, \leq_1)$ *and* $(M_2, \leq_2)$ *be two complete multilattices,* A *and* B *two sets and* $(^{\uparrow}, ^{\downarrow})$ *a Galois connection between* M_1^A *and* M_2^B*. If* $\{(g_i, f_i)\}_{i \in I}$ *is a set of concepts we have that*

$$\textit{multinf}\{{f_i}^{\downarrow} \mid i \in I\} \subseteq (\textit{multisup}\{f_i \mid i \in I\})^{\downarrow} \qquad (1)$$

$$\mathit{multinf}\{{g_i}^{\uparrow} \mid i \in I\} \subseteq (\mathit{multisup}\{g_i \mid i \in I\})^{\uparrow} \tag{2}$$

where $(\mathit{multisup}\{f_i \mid i \in I\})^{\downarrow} = \{f^{\downarrow}_{mult} \mid f_{mult} \in \mathit{multisup}\{f_i \mid i \in I\}\}$ *and* $(\mathit{multisup}\{g_i \mid i \in I\})^{\uparrow}$ *is given similarly.*

We cannot get always the equality in this theorem as we can see If we consider the multilattice of Figure 1 and the following Galois connection, $^{\uparrow} = {}^{\downarrow} \colon M6 \to M6$ defined by:

$$\bot^{\uparrow} = \top \ ; \ a^{\uparrow} = b^{\uparrow} = c^{\uparrow} = c \ ; \ d^{\uparrow} = \bot \ ; \ \top^{\uparrow} = \bot$$

It is routine to prove that the pair $(^{\uparrow}, ^{\downarrow})$ is a Galois connection. On one hand, we obtain that

$$\text{multinf}\{a^{\uparrow}, b^{\uparrow}\} = \text{multinf}\{c\} = c$$

however, on the other hand:

$$(\text{multisup}\{a, b\})^{\uparrow} = (\{c, d\})^{\uparrow} = \{c^{\uparrow}, d^{\uparrow}\} = \{c, \bot\}$$

which proves that we cannot always get the equality.

As a consequence of the previous proposition, one obtains that, given the set of all concepts $\mathcal{C} = \{(g, f) \mid f \in M_1^A, g \in M_2^B, g^{\uparrow} = f, f^{\downarrow} = g\}$, and the ordering defined as $(g_1, f_1) \leq (g_2, f_2)$ if and only if $g_1 \leq_1 g_2$ (if and only if $f_2 \leq_2 f_1$), then this set is a complete multilattice which is a result similar to Theorem 1, but now with respect to multilattices.

Theorem 2 ([21]). *If* $(M_1, \leq_1)$ *and* $(M_2, \leq_2)$ *be two complete multilattices,* A *and* B *two sets and* $(^{\uparrow}, ^{\downarrow})$ *a Galois connection between* M_1^A *and* M_2^B*, then we have that* $(\mathcal{C}, \leq)$ *is a complete multilattice where for every set of concepts* $\{(g_i, f_i)\}_{i \in I}$*:*

$$\mathit{multinf}\{(g_i, f_i)\} = (\mathit{multinf}\{g_i\}, (\mathit{multinf}\{g_i\})^{\uparrow}) \tag{3}$$
$$\mathit{multisup}\{(g_i, f_i)\} = ((\mathit{multinf}\{f_i\})^{\downarrow}, \mathit{multinf}\{f_i\}) \tag{4}$$

Hence, with multilattices we obtain a concept multilattice, although if one of the multilattices is, indeed, a lattice, then we obtain a concept lattice.

Proposition 2 ([21]). *Considering the framework of the previous theorem, if* $(M_1, \leq_1)$ *or* $(M_2, \leq_2)$ *is a lattice, then* $(\mathcal{C}, \leq)$ *is a lattice.*

Therefore, given a Galois connection on multilattices, the set of concepts associated with this connection forms a complete multilattice. The next step is to use this result in order to obtain information from a specific relational database, considering the flexibility of multilattices, in a similar way as in formal concept analysis. This will be the aim of the following section.

4 Concept-forming operators on multilattices

This section studies some conditions which guarantee that the concept-forming operators on multilattices, defined using the same syntactical form than the extension and intension operators of fuzzy concept lattices, form a Galois connection.

The property that we need to require is a generalization of the left-continuity given by Krajči in [15].

Definition 6. *Let* $(M_1, \preceq_1)$*,* $(M_2, \preceq_2)$ *be two multilattices and* $(P_3, \leq)$ *a poset, and* $\&\colon M_1 \times M_2 \to P_3$ *a mapping between them, we say that* $\&$ *is:*

1. left-continuous in the first argument *if for every non-empty subset* $K_1 \subseteq M_1$ *and elements* $m_2 \in M_2$ *and* $p \in P$ *such that* $k \,\&\, m_2 \leq p$ *for every* $k \in K_1$*, then for* ***every*** $m_1 \in \mathit{multisup}\{K_1\}$*, we have that* $m_1 \,\&\, m_2 \leq p$*.*
2. left-continuous in the second argument *if for every non-empty subset* $K_2 \subseteq M_2$ *and elements* $m_1 \in M_1$ *and* $p \in P$ *such that* $m_1 \,\&\, k \leq p$ *for every* $k \in K_2$*, then for* ***every*** $m_2 \in \mathit{multisup}\{K_2\}$ *we have that* $m_1 \,\&\, m_2 \leq p$*.*
3. left-continuous *if it is left-continuous in both arguments.*

The previous property can be weakened by not assuming that each multisuprema must satisfy the inequality, but at least some of them. Formally, $\&$ *is*

1'. soft left-continuous in the first argument *if for every non-empty subset* $K_1 \subseteq M_1$ *and elements* $m_2 \in M_2$ *and* $p \in P$ *such that* $k \,\&\, m_2 \leq p$ *for every* $k \in K_1$*, then* ***there exists*** $m_1 \in \mathit{multisup}\{K_1\}$ *satisfying* $m_1 \,\&\, m_2 \leq p$*.*
2'. soft left-continuous in the second argument *if for every non-empty subset* $K_2 \subseteq M_2$ *and elements* $m_1 \in M_1$ *and* $p \in P$ *such that* $m_1 \,\&\, k \leq p$ *for every* $k \in K_2$*, then* ***there exists*** $m_2 \in \mathit{multisup}\{K_2\}$ *satisfying* $m_1 \,\&\, m_2 \leq p$*.*
3'. soft left-continuous *if it is soft left-continuous in both arguments.*

Alternatively, given elements $m_2 \in M_2$ and $p \in P$, if we denote by X_1 the subset of M_1 of elements m_1 satisfying $m_1 \,\&\, m_2 \leq p$, then

- $\&$ is **left-continuous** in the first argument if multisup$\{K_1\} \subseteq X_1$ for all non-empty subset $K_1 \subseteq X_1$.
- $\&$ is **soft left-continuous** in the first argument if multisup$\{K_1\} \cap X_1 \neq \varnothing$ for all non-empty subset $K_1 \subseteq X_1$.

Similarly for (soft) left-continuity in the second argument.

It is obvious that left-continuity implies soft left-continuity; moreover, both definitions collapse in the left-continuity used by Krajči when working with lattices.

Example 2. Given the multilattice given in Fig. 2 , we can consider the conjunctor $\&\colon M6^* \times M6^* \to M6^*$, defined as

$$x \,\&\, y = \begin{cases} x & \text{if } y = \top \\ y & \text{if } x = \top \\ \bot & \text{if } x \in \{\bot, b\} \text{ or } y \in \{\bot, b\} \\ a & \text{otherwise} \end{cases}$$

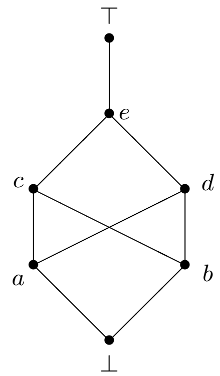

Fig. 2. Multilattice $(M6^*, \preceq)$

for all $x, y \in M6^*$.

This conjunctor is commutative. Moreover, $\&$ is soft left-continuous although it is not left-continuous.

For instance, if we consider $K = \{a, b\}$, $y = \top$ and $z = c$, we obtain that

$$a \,\&\, \top = a \preceq c, \qquad b \,\&\, \top = b \preceq c$$

but $d \in \text{multisup}\{K\}$ and $d \,\&\, \top = d \not\preceq c$. Therefore, $\&$ is not left-continuous.

However, $\&$ is soft left-continuous. It is easy to check that, given $y, z \in M6^*$, for any non-empty subset $K \subseteq M6^*$, such that $k \,\&\, y \preceq z$, for every $k \in K$, then there exists $k^* \in \text{multisup}\{K\}$ satisfying that $k^* \,\&\, y \preceq z$.

Since $\&$ is commutative, we also obtain that it is left-continuous in the second argument.

Now, following [19], left-continuity will be related to the notion of adjoint triples [6], which are formed by three mappings: a non-commutative conjunctor and two residuated implications that satisfy the well-known adjoint property.

Definition 7. *Let* $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ *be posets and* $\& \colon P_1 \times P_2 \to P_3$, $\swarrow \colon P_3 \times P_2 \to P_1$, $\nwarrow \colon P_3 \times P_1 \to P_2$ *be mappings, then* $(\&, \swarrow, \nwarrow)$ *is an* adjoint triple *with respect to* P_1, P_2, P_3 *if:*[4]

$$x \leq_1 z \swarrow y \quad \textit{iff} \quad x \,\&\, y \leq_3 z \quad \textit{iff} \quad y \leq_2 z \nwarrow x$$

where $x \in P_1$, $y \in P_2$ *and* $z \in P_3$.

These operators are an straightforward generalization of a t-norm and its residuated implication. In particular, the Gödel, product and Łukasiewicz t-norms together with their residuated implications form adjoint triples.

When working in the framework of lattices, in [19] it was proven that if the left-continuous operator satisfies that it is increasing in both arguments and the

[4] Note that the antecedent will be evaluated on the right side, while the consequent will be evaluated on the left side, as in logic programming framework.

following boundary conditions $\bot_1 \& y = \bot_3$ and $x \& \bot_2 = \bot_3$ hold, then it is an adjoint triple, and vice versa, the conjunctor in every adjoint triple is always left-continuous and satisfies the previous properties.

In the following results we aim at proving a similar relation between the soft left-continuity and the adjoint triples defined on multilattices.

Proposition 3. *Let $(M_1, \preceq_1)$, $(M_2, \preceq_2)$ and $(M_3, \preceq_3)$ be three multilattices and $\& \colon M_1 \times M_2 \to M_3$ a soft left-continuous operator satisfying $\bot_1 \& y = \bot_3$ and then for every $y \in M_2$ and $z \in M_3$ the set $X = \{x \in M_1 \mid x \& y \preceq_3 z\}$ has a maximum element.*

Similarly, assuming the boundary condition $x \& \bot_2 = \bot_3$, we have that for every $x \in M_1$ and $z \in M_3$ the set $Y = \{y \in M_2 \mid x \& y \preceq_3 z\}$ has a maximum.

Proof. X is a non-empty set since it is easy to see that at least $\bot_1 \in M_1$. By construction of X we have that for every $x \in X$ we have that $x \& y \preceq_3 z$, by definition of soft left-continuity we have that there is $x_1 \in \text{multisup}\{X\}$ such that $x_1 \& y \leq z$, so $x_1 \in X$. Therefore, $\text{multisup}\{X\} = \{x_1\}$ and x_1 is the maximum of X.

The proof for Y is similar. □

The following result provides the equivalence between adjoint triples and soft left-continuous conjunctors.

Theorem 3. *Let $(M_1, \preceq_1)$, $(M_2, \preceq_2)$ and $(M_3, \preceq_3)$ be three multilattices and $\& \colon M_1 \times M_2 \to M_3$ an operator which is increasing in both arguments, then the following conditions are equivalent:*

1. *$\&$ is soft left-continuous and $\bot_1 \& y = \bot_3$ and $x \& \bot_2 = \bot_3$.*
2. *There exist $\swarrow \colon M_3 \times M_2 \to M_1$, $\nwarrow \colon M_3 \times M_1 \to M_2$ such that $(\&, \swarrow, \nwarrow)$ is an adjoint triple.*

Proof. (1) implies (2). We have to define a suitable pair of implications, and check the adjoint properties.

For every $y \in M_2$ and $z \in M_3$, we have that the set $X_{y,z} = \{x \in M_1 \mid x \& y \preceq_3 z\}$ has a maximum by Proposition 3. So we can define.

$$z \swarrow y = \max\{X_{y,z}\} = \max\{x \in M_1 \mid x \& y \preceq z\}$$

Assume $x \in M_1$, $y \in M_2$ and $z \in M_3$ are elements satisfying $x \& y \preceq_3 z$, then x belongs to the set $X_{y,z}$, so $x \preceq_1 \max\{X_{y,z}\} = z \swarrow y$.

Conversely, let $x \in M_1$, $y \in M_2$ and $z \in M_3$ elements satisfying $x \preceq_1 z \swarrow y$ and consider the set $X_{y,z}$. This set has maximum, by Proposition 3, which is $z \swarrow y$ so we have that $(z \swarrow y) \& y \preceq_3 z$, but $x \preceq_1 z \swarrow y$ and $\&$ is increasing in the first argument so

$$x \& y \preceq_3 (z \swarrow y) \& y \preceq_3 z$$

If we define, for every $x \in M_1$ and $z \in M_3$, $z \nwarrow x = \max\{y \in M_2 \mid x \& y \preceq z\}$ we can prove, in a similar way, that $\nwarrow$ satisfies all the conditions required.

So we obtain that $(\&, \swarrow, \nwarrow)$ is an adjoint triple.

(2) implies (1).

Let us consider $y \in M_2$ and $z \in M_3$ and the non-empty subset $X_{y,z} = \{x \in M_1 \mid x \,\&\, y \preceq_3 z\}$. Given $x \in X_{y,z}$, then $x \,\&\, y \preceq_3 z$ and, applying the adjoint property, $x \preceq_1 z \swarrow y$ holds, for every $x \in X_{y,z}$. Therefore, $z \swarrow y$ belongs to the upper bounds of $X_{y,z}$.

Hence, as M_1 is a multilattice, there is $x_1 \in \text{multisup}\{X_{y,z}\}$, such that $x_1 \preceq_1 z \swarrow y$ and, applying the adjoint property again, we have that there exists $x_1 \in \text{multisup}\{X_{y,z}\}$ such that $x_1 \,\&\, y \preceq_3 z$.

Finally, we prove the boundary conditions. Since $\bot_1 \preceq_1 \bot_3 \swarrow y$, for all $y \in M_2$, then, applying the adjoint property, we obtain $\bot_1 \,\&\, y \preceq_3 \bot_3$, which leads us to $\bot_1 \,\&\, y = \bot_3$. The other equality follows similarly.

Thus, & is soft left-continuous in the first argument. The proof for soft left-continuous in the second argument can be proven analogously.

Therefore we have that & is soft left-continuous. □

Now that we have proven this equivalence we will go on in our purpose of defining the forming concept operators. Hence, we consider an adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to three multilattices $(M_1, \preceq_1)$, $(M_2, \preceq_2)$, $(M_3, \preceq_3)$, and a *context*, that is, a tuple (A, B, R), where A and B are non-empty sets (usually interpreted as attributes and objects, respectively) and R is a M_3-fuzzy relation $R\colon A \times B \to M_3$.

The following result shows that the usual syntactic structure of the forming concept operators works on multilattices.

Theorem 4. *Given three multilattices $(M_1, \preceq_1)$, $(M_2, \preceq_2)$ and $(M_3, \preceq_3)$, an adjoint triple between them $(\&, \swarrow, \nwarrow)$, and $g \in M_2^B$ and $f \in M_1^A$, we have that there exist*

$$\inf\{R(a,b) \swarrow g(b) \mid b \in B\} \quad \textit{and} \quad \inf\{R(a,b) \nwarrow f(a) \mid a \in A\}$$

Proof. As we are working on multilattices we have guaranteed the existence of the sets of multinfima of these subsets, the idea is to prove that those sets are actually singletons, leading to the existence of infimum.

Given x_1 and $x_2 \in \text{multinf}\{R(a,b) \swarrow g(b) \mid b \in B\}$, we have that $x_1 \preceq_1 R(a,b) \swarrow g(b)$ and $x_2 \preceq_1 R(a,b) \swarrow g(b)$, for every $b \in B$. Since $(\&, \swarrow, \nwarrow)$ is an adjoint triple, we obtain that $x_1 \,\&\, g(b) \preceq_3 R(a,b)$ and $x_2 \,\&\, g(b) \preceq_3 R(a,b)$, for every $b \in B$.

By Theorem 3, we can use that & is soft left-continuous, then there exists $x \in \text{multisup}\{x_1, x_2\}$, such that $x \,\&\, g(b) \preceq_3 R(a,b)$. Hence, as $(\&, \swarrow, \nwarrow)$ is an adjoint triple, we have that $x \preceq_1 R(a,b) \swarrow g(b)$, for every $b \in B$. Hence, x is a lower bound of the set $\{R(a,b) \swarrow g(b) \mid b \in B\}$; as x_1 and x_2 are maximal lower bounds, we obtain that $x = x_1 = x_2$. Thus, all multinfima collapse in one and, so, there is an infimum.

The proof for the other set is similar. □

As a consequence of the previous result, we can define the following mappings ${}^{\uparrow}\colon M_2^B \to M_1^A$ and ${}^{\downarrow}\colon M_1^A \to M_2^B$ as follows:

$$g^{\uparrow}(a) = \inf\{R(a,b) \swarrow g(b) \mid b \in B\} \tag{5}$$
$$f^{\downarrow}(b) = \inf\{R(a,b) \nwarrow f(a) \mid a \in A\} \tag{6}$$

Applying the previous theorem on the just defined mappings we get the following:

Corollary 1. *Given three multilattices $(M_1, \preceq_1)$, $(M_2, \preceq_2)$ and $(M_3, \preceq_3)$, an adjoint triple between them $(\&, \swarrow, \nwarrow)$, and $g \in M_2^B$ and $f \in M_1^A$, we have that there exist the infimum of the sets*

$$\{R(a,b) \nwarrow g^{\uparrow}(a) \mid b \in B\}$$
$$\{R(a,b) \swarrow f^{\downarrow}(b) \mid a \in A\}$$

Now, we know that the usual pair of derivation operators is well-defined even in the framework of multilattices, and can state and prove the main result in this work:

Theorem 5. *The pair $({}^{\uparrow}, {}^{\downarrow})$ is a Galois connection between M_1^A and M_2^B.*

Proof. The mapping ${}^{\uparrow}$ is decreasing: consider $g_1 \preceq_2 g_2$ since $\nwarrow$ is decreasing in the second argument we would have, for all $b \in B$,

$$R(a,b) \swarrow g_2(b) \preceq_1 R(a,b) \swarrow g_1(b)$$

By Theorem 4, both subsets $\{R(a,b) \swarrow g_i(b) \mid b \in B\}$ have infimum (the definition of $g^{\uparrow}$). Therefore, we obtain $g_2^{\uparrow} \preceq_1 g_1^{\uparrow}$.

Similarly, we obtain that ${}^{\downarrow}$ is decreasing.

Now we will prove that $g \preceq_2 g^{\uparrow\downarrow}$ for every $g \in M_2^B$. Given $a \in A$ and $b \in B$, by definition of $g^{\uparrow}(a)$, we have that, $g^{\uparrow}(a) \preceq_1 R(a,b) \swarrow g(b)$. Now, by the adjoint property, we obtain that this is equivalent to $g(b) \preceq_2 R(a,b) \nwarrow g^{\uparrow}(a)$, for every $a \in A$. Therefore, by Corollary 1, the inequality is obtained:

$$g(b) \preceq_2 \inf\{R(a,b) \nwarrow g^{\uparrow}(a)\} = g^{\uparrow\downarrow}(b)$$

The proof for $f \leq_1 f^{\downarrow\uparrow}$, for every $f \in M_1^A$, is similar to the previous one. □

5 A worked example

Imagine that we are going to travel to a city and we have to decide which hotel is the best for us. In this example, in order not to complicate the calculation we will take into account seven different hotels, as objects, and two attributes, which will be *price* and *location*. Hence, we have as set of objects $B = \{H1, H2, H3, H4, H5, H6, H7\}$ and as set of attributes $A = \{\text{price}, \text{location}\}$, both evaluated in $M6^*$ and the $M6^*$-fuzzy relation, $R\colon A \times B \to M6^*$, between them, defined in Table 1:

Table 1. Relation R

R	price	location
H1	d	$\bot$
H2	c	a
H3	$\top$	b
H4	a	d
H5	b	e
H6	a	b
H7	d	c

Evaluating the hotels in a multilattice comes from the idea that the hotels are ordered thinking of the number of stars they have. We can state, for example that any four-star hotel is better than any three-star hotel, but if both hotels are four-star ones we cannot distinguish between them at the beginning.

In the case of the location, we can say some locations are better than others but we cannot compare locations that are, for example, one kilometer far from downtown but in different directions.

Concerning prices, something similar occurs in that we cannot distinguish between prices which are alike.

If we look at the relation in Table 1, the values $R(H5, \text{price}) = b$ and $R(H6, \text{price}) = a$ mean that the hotels $H5$ and $H6$ have similar prices but we cannot decide which is best taking into account just this attribute. This is an example of the underlying usefulness of using multilattices for their evaluation.

We are trying to choose a hotel to stay in according to our preferences in prizes and location. For instance, if our preferences are: $g_0(\text{price}) = a$ and $g_0(\text{location}) = d$, then we can consider the Galois connection $(^\uparrow, ^\downarrow)$ and the multilattice concept $(\mathcal{C}, \preceq)$ associated with the multilattice, the adjoint triple and the context, and compute the concept that better interprets our preferences.

Let us consider, on the multilattice $M6^*$, the conjunctor $\&$, given in Example 2. Since $\&$ is soft left-continuous and $\bot \& y = \bot$ and $x \& \bot = \bot$, for all $x, y \in M6^*$, then, by Theorem 3, there exist $\swarrow\colon M6^* \times M6^* \to M6^*$, $\nwarrow\colon M6^* \times M6^* \to M6^*$ such that $(\&, \swarrow, \nwarrow)$ is an adjoint triple. Moreover, as $\&$ is commutative, $\swarrow = \nwarrow$. It is routine calculation that this mapping is exactly:

$$z \swarrow y = z \nwarrow y = \begin{cases} \top & \text{if } y \leq z \\ z & \text{if } y = \top \\ b & \text{if } y \notin \{\bot, b, \top\} \text{ and } z \in \{\bot, b\} \\ e & \text{otherwise} \end{cases}$$

Furthermore, given a context, by Theorem 5, the maps $^\uparrow\colon (M6^*)^B \to (M6^*)^A$, $^\downarrow\colon (M6^*)^A \to (M6^*)^B$, defined as in Equations (5) and (6) form a Galois connection, and so, by Theorem 2, the pair $(\mathcal{C}, \preceq)$ is a complete multilattice, where $\mathcal{C} = \{(g, f) \mid f \in (M6^*)^A, g \in (M6^*)^B, g^\uparrow = f, f^\downarrow = g\}$ is the set of concepts.

Now, given the adjoint triple above $(\&, \swarrow, \nwarrow)$, we will use the context in Table 1 to get practical information when we are trying to choose a hotel for our holidays.

Applying the definitions of the concept forming operators, we obtain for $H1$:

$$g_0^{\uparrow}(H1) = \inf\{d \swarrow a, \bot \swarrow d\} = \inf\{\top, b\} = b$$

And for the others:

$$g_0^{\uparrow}(H2) = e\ ,\ g_0^{\uparrow}(H3) = b\ ,\ g_0^{\uparrow}(H4) = \top\ ,\ g_0^{\uparrow}(H5) = b\ ,\ g_0^{\uparrow}(H6) = b\ ,\ g_0^{\uparrow}(H7) = e$$

On the other hand, we have that

$$\begin{aligned} g_0^{\uparrow\downarrow}(\text{price}) &= \inf\{d \nwarrow b, c \nwarrow e, \top \nwarrow b\ , a \nwarrow \top, b \nwarrow b, a \nwarrow b, a \nwarrow e\} \\ &= \inf\{\top, \top, a, \top, e, e\} = a \end{aligned}$$

In a similar way we obtain that $g_0^{\uparrow\downarrow}(\text{location}) = d$. Thus, according to our preference established by g_0, we have that our best choice is $H4$, although $H2$ and $H7$ are really good ones too.

6 Conclusions and future work

We have proved that the structure of adjoint triple allows for constructing Galois connections on multilattices following strictly the same syntactic definitions as those in the framework of lattices. The key observation has been the notion of *soft* left-continuity which, under certain conditions, has been shown to be equivalent to that of adjoint triple.

As future work, on the one hand, it is worth to consider other possible constructions of Galois connections stemming from the use of multilattices, i.e. when not working on a structure given by an adjoint triple, or without soft left-continuity; on the other hand, we will focus on obtaining a representation theorem for fuzzy concept multilattices.

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