Functorial connection between L-ChuCors and a category of supremum preserving mappings

Ondrej Krídlo[1], Stanislav Krajči[1], and Manuel Ojeda-Aciego[2]

[1] University of Pavol Jozef Šafárik, Košice, Slovakia*
[2] Dept. Matemática Aplicada, Univ. Málaga, Spain**

Abstract. A category L-ChuCors of L-Chu correspondences between formal L-fuzzy contexts provides a categorical view on Formal Context Analysis. In this paper some interesting and useful properties are shown. The main target of this paper is to introduce a functor between L-ChuCors and a category of supremum preserving mappings between completely L-ordered sets.

1 Introduction

Formal concept analysis (FCA) introduced by Ganter and Wille [7] has become an extremely useful theoretical and practical tool for formally describing structural and hierarchical properties of data with "object-attribute" character. Bělohlávek in [2, 3] provided an L-fuzzy extension of the main notions of FCA, such as context and concept, by extending its underlying interpretation on classical logic to the more general framework of L-fuzzy logic [9].

We aim at formally describing some structural properties of intercontextual relationships [8, 14] of L-fuzzy formal contexts by using category theory [1]. Our approach, broadly continues the research line which links the theory of Chu spaces with concept lattices [17] but, particularly, is based on the notion of Chu correspondences between formal contexts developed by Mori in [15,16]. In Mori's papers, the construction of formal concepts associated to a crisp relation between objects and attributes is shown to induce a functor from the category of Chu correspondences to the category of sup-preserving maps between complete lattices. The category L-ChuCors is formed by considering the class of L-fuzzy formal contexts as objects and the L-fuzzy Chu correspondences as arrows between objects. The main result here is to introduce a functor between L-ChuCors and a category of supremum preserving mappings between completely L-ordered sets.

In order to obtain a mostly self-contained document, the next section introduces the basic definitions concerning the L-ordered sets, the L-fuzzy extension of formal concept analysis, as well as those concerning L-Chu correspondences and L-bonds, the main results on these topics are stated too. The core of the paper starts at Section 3 with the introduction of the internal Hom functor

* Partially supported by grant VEGA 1/0131/09 and APVV-0035-10.
** Partially supported by Spanish Ministry of Science project TIN09-14562-C05-01 and Junta de Andalucía project FQM-5233.

$C_1 \multimap C_2$ between L-fuzzy contexts C_1 and C_2, then a Galois functor is defined between the categories L-ChuCors and Slat; finally, the results of the two previous sections are merged in order to generate a new functor between L-ChuCors and L-Slat.

2 Preliminaries

In this section we introduce the preliminary definitions concerning L-fuzzy formal concept analysis, mainly following Bělohlávek's approach [2], and the results about L-Chu correspondences on which the present work is built [12, 13].

Definition 1. *An algebra* $\langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ *is said to be a* ***complete residuated lattice*** *if*

1. $\langle L, \wedge, \vee, 0, 1\rangle$ *is a complete bounded lattice with the least element* 0 *and the greatest element* 1*,*
2. $\langle L, \otimes, 1\rangle$ *is a commutative monoid,*
3. $\otimes$ *and* $\to$ *are adjoint, i.e.* $a \otimes b \le c$ *if and only if* $a \le b \to c$*, for all* $a, b, c \in L$*, where* $\le$ *is the ordering in the lattice.*

Now, the natural extension of the notion of context is given below.

Definition 2. *Let L be a complete residuated lattice, an* ***L-fuzzy context*** *is a triple $\langle B, A, r\rangle$ consisting of a set of objects B, a set of attributes A and an L-fuzzy binary relation r, i.e. a mapping $r\colon B \times A \to L$, which can be alternatively understood as an L-fuzzy subset of $B \times A$*

We now introduce the L-fuzzy extension in [2], where we will use the notation Y^X to refer to the set of mappings from X to Y.

Definition 3. *Consider an L-fuzzy context $\langle B, A, r\rangle$. A pair of mappings $\uparrow\colon L^B \to L^A$ and $\downarrow\colon L^A \to L^B$ can be defined for every $f \in L^B$ and $g \in L^A$ as follows:*

$$\uparrow f(a) = \bigwedge_{o \in B} (f(o) \to r(o, a)) \qquad \downarrow g(o) = \bigwedge_{a \in A} \big(g(a) \to r(o, a)\big) \tag{1}$$

Lemma 1. *Let L be a complete residuated lattice, let $r \in L^{B \times A}$ be an L-fuzzy relation between B and A. Then the pair of operators $\uparrow$ and $\downarrow$ form a Galois connection between $\langle L^B; \subseteq\rangle$ and $\langle L^A; \subseteq\rangle$, that is, $\uparrow\colon L^B \to L^A$ and $\downarrow\colon L^A \to L^B$ are antitonic and, furthermore, for all $f \in L^B$ and $g \in L^A$ we have $f \subseteq {\downarrow\uparrow} f$ and $g \subseteq {\uparrow\downarrow} g$.*

Definition 4. *Consider an L-fuzzy context $C = \langle B, A, r\rangle$. An L-fuzzy set of objects $f \in L^B$ (resp. an L-fuzzy set of attributes $g \in L^A$) is said to be* ***closed in*** **C** *iff $f = {\downarrow\uparrow} f$ (resp. $g = {\uparrow\downarrow} g$).*

Lemma 2. *Under the conditions of Lemma 1, the following equalities hold for arbitrary $f \in L^B$ and $g \in L^A$, $\uparrow f = {\uparrow\downarrow\uparrow} f$ and $\downarrow g = {\downarrow\uparrow\downarrow} g$, that is, both ${\downarrow\uparrow} f$ and ${\uparrow\downarrow} g$ are closed in C.*

Definition 5. *An L-**fuzzy concept** is a pair $\langle f, g\rangle$ such that $\uparrow f = g, \downarrow g = f$. The first component f is said to be the **extent** of the concept, whereas the second component g is the **intent** of the concept.*

The set of all L-fuzzy concepts associated to a fuzzy context (B, A, r) will be denoted as L-$FCL(B, A, r)$.

An ordering between L-fuzzy concepts is defined as follows: $\langle f_1, g_1\rangle \le \langle f_2, g_2\rangle$ if and only if $f_1 \subseteq f_2$ if and only if $g_1 \supseteq g_2$.

Theorem 1. *The poset $(L\text{-}FCL(B, A, r), \le)$ is a complete lattice where*

$$\bigwedge_{j\in J} \langle f_j, g_j\rangle = \Big\langle \bigwedge_{j\in J} f_j, \uparrow (\bigwedge_{j\in J} f_j)\Big\rangle$$
$$\bigvee_{j\in J} \langle f_j, g_j\rangle = \Big\langle \downarrow (\bigwedge_{j\in J} g_j), \bigwedge_{j\in J} g_j\Big\rangle$$

2.1 *L*-Chu correspondences and *L*-Bonds

We now recall the basic definitions and results about L-fuzzy Chu correspondences given in [12].

Definition 6. *An L-**multifunction** from X to Y is a mapping $\varphi\colon X \to L^Y$.*

*The **transposed** of an L-multifunction $\varphi\colon X \to L^Y$ is an L-multifunction ${}^t\varphi\colon Y \to L^X$ defined by ${}^t\varphi(y)(x) = \varphi(x)(y)$.*

The set L-Mfn(X, Y) of all the L-multifunctions from X to Y can be endowed with a poset structure by defining the ordering $\varphi_1 \le \varphi_2$ as $\varphi_1(x)(y) \le \varphi_2(x)(y)$ for all $x \in X$ and $y \in Y$.

Definition 7. *Consider two L-fuzzy contexts $C_i = \langle B_i, A_i, r_i\rangle, (i = 1, 2)$, then the pair $\varphi = (\varphi_L, \varphi_R)$ is called a **correspondence** from C_1 to C_2 if φ_L and φ_R are L-multifunctions, respectively, from B_1 to B_2 and from A_2 to A_1 (that is, $\varphi_L : B_1 \to L^{B_2}$ and $\varphi_R : A_2 \to L^{A_1}$).*

*The L-correspondence φ is said to be a **weak** L**-Chu correspondence** if the equality $\hat{r}_1(\chi_{o_1}, \varphi_R(a_2)) = \hat{r}_2(\varphi_L(o_1), \chi_{a_2})$ holds for all $o_1 \in B_1$ and $a_2 \in A_2$. By unfolding the definition of $\hat{r}_i$ this means that*

$$\bigwedge_{a_1\in A_1} (\varphi_R(a_2)(a_1) \to r_1(o_1, a_1)) = \bigwedge_{o_2\in B_2} (\varphi_L(o_1)(o_2) \to r_2(o_2, a_2)) \qquad (2)$$

*A weak Chu correspondence φ is an L**-Chu correspondence** if $\varphi_L(o_1)$ is closed in C_2 and $\varphi_R(a_2)$ is closed in C_1 for all $o_1 \in B_1$ and $a_2 \in A_2$. We will denote the set of all Chu correspondences from C_1 to C_2 by L-$ChuCors(C_1, C_2)$.*

Definition 8. *An L-**bond** between two formal contexts $C_1 = \langle B_1, A_1, r_1\rangle$ and $C_2 = \langle B_2, A_2, r_2\rangle$ is a multifunction $\beta : B_1 \to L^{A_2}$ satisfying the condition that for all $o_1 \in B_1$ and $a_2 \in A_2$ both $\beta(o_1)$ and ${}^t\beta(a_2)$ are closed L-fuzzy sets of, respectively, attributes in C_2 and objects in C_1. The set of all bonds from C_1 to C_2 is denoted as L-Bonds(C_1, C_2).*

Definition 9. *Let $C_1 = \langle B_1, A_1, r_1\rangle$ and $C_2 = \langle B_2, A_2, r_2\rangle$ be L-fuzzy contexts:*

- *Let $\beta\colon C_1 \to C_2$ be an L-bond. We define a correspondence $\varphi_\beta\colon C_1 \to C_2$ by*

$$\varphi_{\beta L}(o_1) = \downarrow_2 (\beta(o_1)) \in L^{B_2} \text{ for } o_1 \in B_1$$
$$\varphi_{\beta R}(a_2) = \uparrow_1 ({}^t\beta(a_2)) \in L^{A_1} \text{ for } a_2 \in A_2$$

- *Conversely, consider an L-Chu correspondence φ from C_1 to C_2, and define a multifunction $\beta_\varphi\colon B_1 \to L^{A_2}$ by*

$$b_\varphi(o_1) = \uparrow_2 (\varphi_L(o_1))$$

Lemma 3. *With the definitions given above*

1. *φ_β is an L-Chu correspondence from C_1 to C_2.*
2. *β_φ is an L-bond from C_1 to C_2.*

Lemma 4. *Let C_1, C_2 be L-fuzzy formal contexts. L-Bonds(C_1, C_2) is a complete lattice. Let $b_i \in$ L-Bonds(C_1, C_2) for all $i \in I$, then*

1. $(\bigwedge_{i\in I} b_i)(o) = \bigwedge_{i\in I}(b_i(o))$
2. $(\bigvee_{i\in I} b_i)(o) = \uparrow_2\downarrow_2 (\bigvee_{i\in I}(b_i(o))) = \uparrow_2 (\bigwedge_{i\in I} \downarrow_2 (b_i(o)))$

for all $o \in B_1$.

Lemma 5. *Let C_1, C_2 be L-fuzzy formal contexts. L-ChuCorrs(C_1, C_2) is a complete lattice. Let $\varphi_{Li} \in$ L-ChuCorrs(C_1, C_2) for all $i \in I$, then*

1. $(\bigwedge_{i\in I} \varphi_{Li})(o) = \bigwedge_{i\in I}(\varphi_{Li}(o))$
2. $(\bigvee_{i\in I} \varphi_{Li})(o) = \uparrow_2\downarrow_2 (\bigvee_{i\in I}(\varphi_{Li}(o)))$

for all $o \in B_1$.

Theorem 2. *The lattice L-ChuCors(C_1, C_2) and the opposite lattice of L-bonds L-Bonds$(C_1, C_2)^*$ are isomorphic, and the mapping which assigns to each Chu correspondence φ the bond b_φ provides such isomorphism.*

Finally, let us recall the following relationship between the right and left sides of L-fuzzy Chu correspondences has been presented in [13].

Definition 10. *Consider a mapping $\varpi : X \to L^Y$. Lets define new mappings $\varpi_* : L^X \to L^Y$ and $\varpi^* : L^Y \to L^X$ for all $f \in L^X$ and $g \in L^Y$ put*

1. $\varpi_*(f)(y) = \bigvee_{x\in X}(f(x) \otimes \varpi(x)(y))$
2. $\varpi^*(g)(x) = \bigwedge_{y\in Y} \varpi(x)(y) \to g(y)$

Lemma 6. *Let $C_i = \langle B_i, A_i, r_i\rangle$ for $i = 1, 2$ be L-fuzzy contexts. Let $\varphi = (\varphi_L, \varphi_R) \in$ L-ChuCors(C_1, C_2). Then for all $f \in L^{B_1}$ and $g \in L^{A_2}$ holds*

$$\uparrow_2 (\varphi_{L*}(f)) = \varphi_R^*(\uparrow_1 (f)) \text{ and } \downarrow_1 (\varphi_{R*}(g)) = \varphi_L^*(\downarrow_2 (g))$$

Lemma 7. *Let $C_i = \langle B_i, A_i, r_i\rangle$ for $i = 1, 2$ be L-fuzzy contexts. If $\varphi = (\varphi_L, \varphi_R) \in$ $L\text{-ChuCors}(C_1, C_2)$, then for all $o_1 \in B_1$ and $a_2 \in A_2$ holds*

$$\varphi_L(o_1) = \downarrow_2 (\varphi_R^*(\uparrow_1 (\chi_{o_1}))) \text{ and } \varphi_R(a_2) = \uparrow_1 (\varphi_L^*(\downarrow_2 (\chi_{a_2})))$$

Lemma 8. *Let $C_i = \langle B_i, A_i, r_i\rangle$ be an L-context and $\varphi \in L\text{-ChuCors}(C_1, C_2)$. Then*

1. $\uparrow_2 (\varphi_{L*}(\downarrow_1\uparrow_1 (\bigvee_{i\in I} f_i)))(a_2) = \uparrow_2 (\bigvee_{i\in I} \varphi_{L*}(f_i))(a_2)$
2. $\downarrow_1 (\varphi_{R*}(\uparrow_2\downarrow_2 (\bigvee_{i\in I} g_i)))(o_1) = \downarrow_1 (\bigvee_{i\in I} \varphi_{R*}(g_i))(o_1)$

2.2 The category *L*-ChuCors

Now a category of L-Chu correspondences between L-fuzzy formal contexts will be showed.

- **objects** L-fuzzy formal contexts
- **arrows** L-Chu correspondences
- **identity arrow** $\iota : C \to C$ of L-context $C = \langle B, A, r\rangle$
  - $\iota_l(o) = \downarrow\uparrow (\chi_o)$, for all $o \in B$
  - $\iota_r(a) = \uparrow\downarrow (\chi_a)$, for all $a \in A$
- **composition** $\varphi_2 \circ \varphi_1 : C_1 \to C_3$ **of arrows** $\varphi_1 : C_1 \to C_2$, $\varphi_2 : C_2 \to C_3$ ($C_i = \langle B_i, A_i, r_i\rangle$, $i \in \{1, 2\}$)
  - $(\varphi_2 \circ \varphi_1)_L : B_1 \to L^{B_3}$ and $(\varphi_2 \circ \varphi_1)_R : A_3 \to L^{A_1}$
  - $(\varphi_2 \circ \varphi_1)_L(o_1) = \downarrow_3\uparrow_3 (\varphi_{2L*}(\varphi_{1L}(o_1)))$
    where
    $$\varphi_{2L*}(\varphi_{1L}(o_1))(o_3) = \bigvee_{o_2\in B_2} \varphi_{1L}(o_1)(o_2) \otimes \varphi_{2L}(o_2)(o_3)$$
  - $(\varphi_2 \circ \varphi_1)_R(a_3) = \uparrow_1\downarrow_1 (\varphi_{1R*}(\varphi_{2R}(a_3)))$
    where
    $$\varphi_{1R*}(\varphi_{2R}(a_3))(a_1) = \bigvee_{a_2\in A_2} \varphi_{2R}(a_3)(a_2) \otimes \varphi_{1R}(a_2)(a_1)$$
- **associativity of composition** $\varphi_3 \circ (\varphi_2 \circ \varphi_1) = (\varphi_3 \circ \varphi_2) \circ \varphi_1$ for all $\varphi_1, \varphi_2, \varphi_3 \in L\text{-ChuCors}$ such that $\varphi_i : C_i \to C_{i+1}$ for $i \in \{1, 2, 3\}$, where C_1, C_2, C_3, C_4 are L-fuzzy contexts.

2.3 *L*-ordered sets of *L*-concepts and *L*-Chu correspondences

The definitions and results concerning L-ordered sets of L-concepts is taken from [4, 5]. Note that following the usual convention, ordered-like relations are written in infix form, that is, $R(x, y)$ will be written as xRy.

Definition 11. *A binary L-relation $\approx$ on X is called an L-**equality** if it satisfies*

1. $(x \approx x) = 1$, *(reflexivity),*
2. $(x \approx y) = (y \approx x)$, *(symmetry),*
3. $(x \approx y) \otimes (y \approx z) \leq (x \approx z)$, *(transitivity),*
4. $(x \approx y) = 1$ *implies* $x = y$

Definition 12. *An L-**ordering** (or fuzzy ordering) on a set X endowed with an L-equality relation* $\approx$ *is a binary L-relation* $\preceq$ *which is compatible w.r.t.* $\approx$ *and satisfies*

1. $x \preceq x = 1$, *(reflexivity),*
2. $(x \preceq y) \wedge (y \preceq x) \leq x \approx y$, *(antisymmetry),*
3. $(x \preceq y) \otimes (y \preceq z) \leq x \preceq z$, *(transitivity).*

If $\preceq$ *is an L-order on a set X with an L-equality* $\approx$*, we call the pair* $\langle\langle X, \approx\rangle, \preceq\rangle$ *an L-ordered set.*

Definition 13. *An L-set* $f \in L^X$ *is said to be an L-**singleton** in* $\langle X, \approx\rangle$ *if it is compatible w.r.t.* $\approx$ *(i.e.* $f(x) \otimes (x \approx y) \leq f(y)$*, for all* $x, y \in X$*) and the following holds:*

1. there exists $x \in X$ *with* $f(x) = 1$
2. $f(x) \otimes f(y) \leq (x \approx y)$*, for all* $x, y \in X$.

Definition 14. *For an L-ordered set* $\langle\langle X, \approx\rangle \preceq\rangle$ *and* $f \in L^X$ *we define the L-sets* $\inf(f)$ *and* $\sup(f)$ *in X by*

– $\inf(f)(x) = (\mathcal{L}(f))(x) \wedge (\mathcal{UL}(f))(x)$
– $\sup(f)(x) = (\mathcal{U}(f))(x) \wedge (\mathcal{LU}(f))(x)$

where

– $\mathcal{L}(f)(x) = \bigwedge_{y \in X}(f(y) \to (x \preceq y))$
– $\mathcal{U}(f)(x) = \bigwedge_{y \in X}(f(y) \to (y \preceq x))$

$\inf(f)$ *and* $\sup(f)$ *are called* infimum *or* supremum*, respectively.*

Definition 15. *An L-ordered set* $\langle\langle X, \approx\rangle \preceq\rangle$ *is said to be **completely L-ordered** if for any* $f \in L^X$ *both* $\sup(f)$ *and* $\inf(f)$ *are* $\approx$*-singletons.*

Lemma 9. *For an L-ordered set* $\langle\langle X, \approx\rangle, \preceq\rangle$ *and* $f \in L^X$ *we have that* $\inf(f)$ *is an* $\approx$*-singleton if and only if there is some* $x \in X$ *such that* $(\inf(f))(x) = 1$. *The same is true for suprema.*

Now, given a formal context C, we will consider a completely L-ordered set based on the on the set of formal concepts $L\text{-}FCL(C)$.

Definition 16. *Let us define an L-equality* $\approx_1$ *and L-ordering* $\preceq_1$ *on the set of formal concepts* $L\text{-}FCL(C)$ *of context C:*

- $\langle f_1, g_1\rangle \preceq_1 \langle f_2, g_2\rangle = \bigwedge_{o \in B}(f_1(o) \to f_2(o))$
- $\langle f_1, g_1\rangle \approx_1 \langle f_2, g_2\rangle = \bigwedge_{o \in B}(f_1(o) \leftrightarrow f_2(o))$

Definition 17. *Let $C = \langle B, A, r\rangle$ be an L-fuzzy formal context and γ be an L-set from $L^{L\text{-}FCL(C)}$. We define L-sets of objects and attributes $\bigcup_B \gamma$ and $\bigcup_A \gamma$, respectively, as follows:*

- $(\bigcup_B \gamma)(o) = \bigvee_{\langle f,g\rangle \in L\text{-}FCL(C)} (\gamma(\langle f,g\rangle) \otimes f(o))$, *for* $o \in B$
- $(\bigcup_A \gamma)(a) = \bigvee_{\langle f,g\rangle \in L\text{-}FCL(C)} (\gamma(\langle f,g\rangle) \otimes g(a))$, *for* $a \in A$

Theorem 3 ([4, 5]). *Let $C = \langle B, A, r\rangle$ be an L-context. $\langle\langle L\text{-}FCL(C), \approx\rangle, \preceq\rangle$ is a completely L-ordered set in which infima and suprema can be described as follows: for an L-set $\gamma \in L^{L-FCL(C)}$ we have:*

$$^1\inf(\gamma) = \{\langle\downarrow(\bigcup_A \gamma), \uparrow\downarrow(\bigcup_A \gamma)\rangle\}$$

$$^1\sup(\gamma) = \{\langle\downarrow\uparrow(\bigcup_B \gamma), \uparrow(\bigcup_B \gamma)\rangle\}.$$

Finally, given two formal context C_1, C_2, we will consider a completely L-ordered set based on the on the set of L-Chu correspondences between both contexts. This definition is original and does not follow from Bělohlávek's work.

Definition 18. *Given two L-fuzzy contexts $\langle B_i, A_i, r_i\rangle$ for $i \in \{1, 2\}$ we define $\big\langle\langle L\text{-}\mathrm{ChuCors}, \approx_2\rangle, \preceq_2\big\rangle$, where*

$$\begin{aligned}
\varphi_1 \approx_2 \varphi_2 &= \bigwedge_{o_1\in B_1} \bigwedge_{a_2\in A_2} (\uparrow_2 (\varphi_{2L}(o_1))(a_2) \leftrightarrow \uparrow_2 (\varphi_{1L}(o_1))(a_2)) \\
&= \bigwedge_{o_1\in B_1} \bigwedge_{a_2\in A_2} (\downarrow_1 (\varphi_{2R}(a_2))(o_1) \leftrightarrow \downarrow_1 (\varphi_{1R}(a_2))(o_1)) \\
&= \bigwedge_{o_1\in B_1} \bigwedge_{a_2\in A_2} (\beta_{\varphi_2}(o_1)(a_2) \leftrightarrow \beta_{\varphi_1}(o_1)(a_2))
\end{aligned}$$

$$\begin{aligned}
\varphi_1 \preceq_2 \varphi_2 &= \bigwedge_{o_1\in B_1} \bigwedge_{a_2\in A_2} (\uparrow_2 (\varphi_{2L}(o_1))(a_2) \to \uparrow_2 (\varphi_{1L}(o_1))(a_2)) \\
&= \bigwedge_{o_1\in B_1} \bigwedge_{a_2\in A_2} (\downarrow_1 (\varphi_{2R}(a_2))(o_1) \to \downarrow_1 (\varphi_{1R}(a_2))(o_1)) \\
&= \bigwedge_{o_1\in B_1} \bigwedge_{a_2\in A_2} (\beta_{\varphi_2}(o_1)(a_2) \to \beta_{\varphi_1}(o_1)(a_2))
\end{aligned}$$

3 The internal Hom functor

It is noticeable the existence of an internal Hom functor between L-fuzzy formal contexts. The construction is based on the definition below, which extends one given by Mori in [15]:

Definition 19. *Given two L-fuzzy contexts $C_i = \langle B_i, A_i, r_i \rangle$ for $i \in \{1,2\}$ a new formal L-fuzzy context $C_1 \multimap C_2$ is defined as $\langle L\text{-ChuCors}(C_1,C_2), B_1 \times A_2, r^\star \rangle$ where the mapping $r^\star \colon B_1 \times A_2 \to L$ is given by*

$$r^\star(\varphi, (o_1, a_2)) = \uparrow_2 (\varphi_L(o_1))(a_2) = \downarrow_1 (\varphi_r(a_2))(o_1)$$

Theorem 4. *Consider two L-fuzzy contexts $C_i = \langle B_i, A_i, r_i \rangle$ for $i \in \{1,2\}$, then there is an isomorphism*

$$\big\langle \langle L\text{-}FCL(C_1 \multimap C_2), \approx_1 \rangle, \preceq_1 \big\rangle \quad \cong \quad \big\langle \langle L\text{-ChuCors}(C_1,C_2), \approx_2 \rangle, \preceq_2 \big\rangle.$$

Proof. Consider an arbitrary concept $\langle \Phi, \beta \rangle$, where $\Phi \in L^{L\text{-ChuCors}}$ and $\beta \in L^{B_1 \times A_2}$, then

$$\begin{aligned}
\beta(o_1)(a_2) &= \uparrow^\star (\Phi)(o_1, a_2) \\
&= \bigwedge_{\varphi \in L\text{-ChuCors}(C_1,C_2)} (\Phi(\varphi) \to r^\star(\varphi, (o_1, a_2))) \\
&= \bigwedge_{\varphi} (\Phi(\varphi) \to \uparrow_2 (\varphi_L(o_1))(a_2)) \\
&= \bigwedge_{\varphi} (\Phi(\varphi) \to \bigwedge_{o_2 \in B_2} (\varphi_L(o_1) \to r_2(o_2, a_2))) \\
&= \bigwedge_{o_2 \in B_2} \bigwedge_{\varphi} (\Phi(\varphi) \to (\varphi_L(o_1) \to r_2(o_2, a_2))) \\
&= \bigwedge_{o_2 \in B_2} \bigwedge_{\varphi} ((\Phi(\varphi) \otimes \varphi_L(o_1)) \to r_2(o_2, a_2)) \\
&= \bigwedge_{o_2 \in B_2} (\bigvee_{\varphi} (\Phi(\varphi) \otimes \varphi_L(o_1)) \to r_2(o_2, a_2)) \\
&= \bigwedge_{o_2 \in B_2} ((\bigcup \Phi)_L(o_1)(o_2) \to r_2(o_2, a_2)) \\
&= \uparrow_2 ((\bigcup \Phi)_L(o_1))(a_2)
\end{aligned}$$

Similarly we obtain:

$$\begin{aligned}
\beta^t(a_2)(o_1) &= \uparrow^\star (\Phi)(o_1, a_2) \\
&\dots \\
&= \bigwedge_{\varphi} (\Phi(\varphi) \to \downarrow_1 (\varphi_R(a_2))(o_1)) \\
&\dots \\
&= \bigwedge_{a_1 \in A_1} (\bigvee_{\varphi} (\Phi(\varphi) \otimes \varphi_R(a_2)(a_1)) \to r_1(o_1, a_1)) \\
&= \downarrow_1 ((\bigcup \Phi)_R(a_2))(o_1)
\end{aligned}$$

Now, as we have seen that $\beta \in L^{B_1 \times A_2}$ is closed in $C_1 \multimap C_2$, then β is in L-Bonds(C_1, C_2).

Every bond $\beta \in L$-Bonds(C_1, C_2) is closed in $C_1 \multimap C_2$, because of the following chain of equalities:

$$\begin{aligned}\beta(o_1)(a_2) &= \uparrow_2 (\varphi_\beta(o_1))(a_2) = r^\star(\varphi_\beta, (o_1, a_2)) \\ &= 1 \to r^\star(\varphi_\beta, (o_1, a_2)) \\ &= \bigwedge_{\varphi} (\chi_{\varphi_\beta}(\varphi) \to r^\star(\varphi_\beta, (o_1, a_2))) \\ &= \uparrow^\star (\chi_{\varphi_\beta})(o_1, a_2)\end{aligned}$$

As a result we obtain that there is a bijection between L-ChuCors(C_1, C_2) and L-$FCL(C_1 \multimap C_2)$.

Let $\langle \Phi_i, \beta_i \rangle$ for $i \in \{1, 2\}$ be two concepts of $C_1 \multimap C_2$, then

$$\langle \Phi_1, \beta_1 \rangle \preceq_1 \langle \Phi_2, \beta_2 \rangle = \bigwedge_{o_1 \in B_1} \bigwedge_{a_2 \in A_2} (\beta_2(o_1)(a_2) \to \beta_1(o_1)(a_2)) = \varphi_{\beta_1} \preceq_2 \varphi_{\beta_2}$$

Similarly for the L-equalities $\approx_i$. □

Theorem 5. *Let $C = \langle B, A, r \rangle$ be an arbitrary L-context. Then there is an isomorphism between L-ordered sets*

$$\big\langle \langle L\text{-}FCL(C), \approx_1 \rangle, \preceq_1 \big\rangle \quad \cong \quad \big\langle \langle L\text{-ChuCors}(\perp, C), \approx_2 \rangle, \preceq_2 \big\rangle$$

such that $\perp = \langle \{\diamond\}, L, \lambda \rangle$, where $\lambda(\diamond, l) = l$, for any $l \in L$.

Proof. Let $\varphi \in L$-ChuCors$(\perp, C)$ be an arbitrary L-Chu correspondence. Then $\varphi_L : \{\diamond\} \to L^B$ and $\varphi_R : A \to L^L$ where $\varphi_L(\diamond)$ is closed in C and $\varphi_R(a)$ is closed in $\perp$ for any $a \in A$. It means that every left side of any Chu correspondence from $\perp$ to C is an object part of some concept of C.

Now let $\langle f, g \rangle$ be an arbitrary concept of C. Then we can construct the L-Chu correspondence from $\perp$ to C. $\varphi_L(\diamond) = f$. From Lemma 7 we know that

$$\begin{aligned}\varphi_R(a) &= \uparrow_\lambda (\varphi_L^*(\downarrow (\chi_a))) = \uparrow_\lambda (\bigwedge_{o \in B} (\varphi_L(\diamond)(o) \to r(o, a))) \\ &= \uparrow_\lambda (\bigwedge_{o \in B} (f(o) \to r(o, a))) = \uparrow_\lambda (\uparrow (f)(a)) = \uparrow_\lambda (g(a))\end{aligned}$$

Hence φ_R will assign a closed L-set in $\perp$ to every $a \in A$. And with any closed $g \in L^A$ there will be a new L-set from L^L such that $\varphi_R(a)(l) = \uparrow_\lambda (g(a))(l) = (l \to g(a))$.

Consider new two L-concepts $\langle f_1, g_1 \rangle$, $\langle f_2, g_2 \rangle$ of context C and two L-Chu correspondences φ_{f_1} and φ_{f_2} assigned to the concepts. Then

$$\begin{aligned}\langle f_1, g_1 \rangle \preceq_1 \langle f_2, g_2 \rangle &= \bigwedge_{a \in A} (g_2(a) \to g_1(a)) = \bigwedge_{a \in A} (\uparrow (f_2)(a) \to \uparrow (f_1)(a)) \\ &= \bigwedge_{a \in A} (\uparrow (\varphi_{f_2})(a) \to \uparrow (\varphi_{f_1})(a)) = \varphi_{f_1} \preceq_2 \varphi_{f_2}.\end{aligned}$$

The equality $\langle f_1, g_1 \rangle \approx_1 \langle f_2, g_2 \rangle = \varphi_{f_1} \approx_2 \varphi_{f_2}$ can be proved similarly. □

Corollary 1. *For any L-concept $C = \langle B, A, r\rangle$ there is an isomorphism between L-ordered sets*

$$\big\langle\langle L\text{-}FCL(C), \approx_1\rangle, \preceq_1\big\rangle \quad\cong\quad \big\langle\langle L\text{-}FCL(\bot \multimap C), \approx_1\rangle, \preceq_1\big\rangle.$$

We have that $\bot \multimap C = \langle L\text{-ChuCors}(\bot, C), \{\diamond\}\times A, r_C\rangle$, where $r_C(\varphi, (\diamond, a)) = \uparrow(\varphi_L(\diamond))(a)$, but because of the previous isomorphisms we can consider $\bot \multimap C$ as $\langle L\text{-}FCL(C), A, r_C\rangle$, where $r_C(\langle f, g\rangle, a) = g(a)$, for any concept $\langle f, g\rangle$ of C and for any attribute $a \in A$.

Now consider an arbitrary $\gamma \in L^{L\text{-}FCL(C)}$.

$$\begin{aligned}
\uparrow_C(\gamma)(a) &= \bigwedge_{\langle f',g'\rangle\in L\text{-}FCL(C)} (\gamma(\langle f', g'\rangle) \to r_C(\langle f', g'\rangle, a))\\
&= \bigwedge_{\langle f',g'\rangle} (\gamma(\langle f', g'\rangle) \to g'(a))\\
&= \bigwedge_{\langle f',g'\rangle} (\gamma(\langle f', g'\rangle) \to \uparrow(f')(a))\\
&= \bigwedge_{\langle f',g'\rangle} (\gamma(\langle f', g'\rangle) \to \bigwedge_{o\in B}(f'(o) \to r(o, a)))\\
&= \bigwedge_{o\in B}\bigwedge_{\langle f',g'\rangle} (\gamma(\langle f', g'\rangle) \to (f'(o) \to r(o, a)))\\
&= \bigwedge_{o\in B}\bigwedge_{\langle f',g'\rangle} ((\gamma(\langle f', g'\rangle) \otimes f'(o)) \to r(o, a)))\\
&= \bigwedge_{o\in B}(\bigvee_{\langle f',g'\rangle} (\gamma(\langle f', g'\rangle) \otimes f'(o)) \to r(o, a))\\
&= \bigwedge_{o\in B}(\bigcup_B \gamma) \to r(o, a))\\
&= \uparrow(\bigcup_B \gamma)(a)
\end{aligned}$$

Then $\langle\downarrow(\uparrow_C(\gamma)), \uparrow_C(\gamma)\rangle = \langle\downarrow\uparrow(\bigcup_B \gamma), \uparrow(\bigcup_B \gamma)\rangle$ the concept of C is the only element of ${}^1\sup(\gamma)$ from Bělohlávek's theorem 3.

4 Galois functor from *L*-ChuCors to the category of semilattices *Slat*

Let us start with the following technical lemma.

Lemma 10. *Let $C_i = \langle B_i, A_i, r_i\rangle$ be an L-fuzzy formal context for $i \in \{1, 2\}$ and $\varphi \in L$-ChuCors(C_1, C_2). Assign $b \in L^{B_1\times A_2}$ as a new L-relation defined by an L-bond β_φ, namely, $b(o_1, a_2) = \beta_\varphi(o_1)(a_2)$, for all $o_1 \in B_1$ and $a_2 \in A_2$. Finally, consider the (up- and down-) arrow mappings $\uparrow_b, \downarrow_b$ defined on the relation b. For all $f \in L^{B_1}$, $g \in L^{A_2}$ holds*

$$\uparrow_2(\varphi_{L*}(f)) = \uparrow_b(f) \text{ and } \downarrow_1(\varphi_{R*}(g)) = \downarrow_b(g).$$

Proof.

$$
\begin{aligned}
\downarrow_1(\varphi_{R*}(g)) &= \bigwedge_{a_1\in A_1}(\varphi_{R*}(g)\to r_1(o_1,a_1))\\
&= \bigwedge_{a_1\in A_1}(\bigvee_{a_2\in A_2}(\varphi_R(a_2)(a_1)\otimes g(a_2)))\to r_1(o_1,a_1))\\
&= \bigwedge_{a_1\in A_1}\bigwedge_{a_2\in A_2}((\varphi_R(a_2)(a_1)\otimes g(a_2))\to r_1(o_1,a_1))\\
&= \bigwedge_{a_1\in A_1}\bigwedge_{a_2\in A_2}((g(a_2)\otimes\varphi_R(a_2)(a_1))\to r_1(o_1,a_1))\\
&= \bigwedge_{a_1\in A_1}\bigwedge_{a_2\in A_2}(g(a_2)\to(\varphi_R(a_2)(a_1)\to r_1(o_1,a_1)))\\
&= \bigwedge_{a_2\in A_2}(g(a_2)\to\bigwedge_{a_1\in A_1}(\varphi_R(a_2)(a_1)\to r_1(o_1,a_1)))\\
&= \bigwedge_{a_2\in A_2}(g(a_2)\to\downarrow_1(\varphi_R(a_2))(o_1))\\
&= \bigwedge_{a_2\in A_2}(g(a_2)\to\beta_\varphi(o_1)(a_2))\\
&= \bigwedge_{a_2\in A_2}(g(a_2)\to b(o_1,a_2))\\
&=\downarrow_b(g)(o_1)
\end{aligned}
$$

The second equation can be proved similarly. □

Lemma 11. *For all $f\in L^{B_1}$ and $g\in L^{A_2}$ holds*

$$f\le\downarrow_1(\varphi_{R*}(g))\quad\Leftrightarrow\quad g\le\uparrow_2(\varphi_{L*}(f)).$$

Proof.

$\Leftarrow$ Let us assume $g\le\uparrow_2(\varphi_{L*}(f))$, then

$$
\begin{aligned}
\downarrow_1(\varphi_{R*}(g)) &= \downarrow_b(g)\\
&= \bigwedge_{a_2\in A_2}(g(a_2)\to b(o_1,a_2))
\end{aligned}
$$

by hypothesis

$$\ge\bigwedge_{a_2\in A_2}(\uparrow_2(\varphi_{L*}(f))(a_2)\to b(o_1,a_2))$$

from Lemma 10

$$
\begin{aligned}
&= \bigwedge_{a_2\in A_2}(\uparrow_b(f)(a_2)\to b(o_1,a_2))\\
&= \downarrow_b\uparrow_b(f)(o_1)\ge f(o_1)
\end{aligned}
$$

$\Rightarrow$ Similar. $\square$

Proposition 1. *For all $f \in L^{B_1}$ closed in C_1 and $g \in L^{A_2}$ closed in C_2, the following equivalence holds*

$$\langle f, \uparrow_1 (f) \rangle \leq \langle \downarrow_1 (\varphi_{R*}(g)), \uparrow_1\downarrow_1 (\varphi_{R*}(g)) \rangle \Leftrightarrow$$
$$\Leftrightarrow \langle \downarrow_2\uparrow_2 (\varphi_{L*}(f)), \uparrow_2 (\varphi_{L*}(f)) \rangle \leq \langle \downarrow_2 (g), g \rangle$$

Proof. The equivalence above can be rewritten as

$$f \leq \downarrow_1 (\varphi_{R*}(g)) \quad \Leftrightarrow \quad g \leq \uparrow_2 (\varphi_{L*}(f)),$$

which holds from Lemma 11. $\square$

Given an L-fuzzy formal context C, let us assume the existence of a mapping Gal such that $Gal(C)$ is a complete lattice of formal concepts in C.

Definition 20. *Let $C_i = \langle B_i, A_i, r_i \rangle$ be an L-formal context for $i \in \{1,2\}$, $\varphi \in L\text{-ChuCors}(C_1, C_2)$ and $\langle f, \uparrow_1 (f) \rangle, \langle \downarrow_2 (g), g \rangle$ be L-concepts from $Gal(C_1)$ or $Gal(C_2)$ respectively. Define $\varphi_\vee : Gal(C_1) \to Gal(C_2)$ by*

$$\varphi_\vee(\langle f, \uparrow_1 (f) \rangle) = \langle \downarrow_2\uparrow_2 (\varphi_{L*}(f)), \uparrow_2 (\varphi_{L*}(f)) \rangle$$

and $\varphi_\wedge : Gal(C_2) \to Gal(C_1)$ by

$$\varphi_\wedge(\langle \downarrow_2 (g), g \rangle) = \langle \downarrow_1 (\varphi_{R*}(g)), \uparrow_1\downarrow_1 (\varphi_{R*}(g)) \rangle.$$

Lemma 12. *Let $C_i = \langle B_i, A_i, r_i \rangle$ be an L-context for $i \in \{1,2\}$. For every $\varphi \in L\text{-ChuCors}(C_1, C_2)$ holds*

1. *$\varphi_\vee$ is supremum-preserving,*
2. *$\varphi_\wedge$ is infimum-preserving.*

Proof. 1. From Lemma 8

$$\varphi_\vee(\langle \downarrow_1\uparrow_1 (\bigvee_{i \in I} f_i), \bigwedge_{i \in I} \uparrow_1(f_i) \rangle) =$$
$$= \langle \downarrow_2\uparrow_2 (\varphi_{L*}(\downarrow_1\uparrow_1 (\bigvee_{i \in I} f_i))), \uparrow_2 (\varphi_{L*}(\downarrow_1\uparrow_1 (\bigvee_{i \in I} f_i))) \rangle$$
$$= \langle \downarrow_2\uparrow_2 (\bigvee_{i \in I} \varphi_{L*}(f_i))(a_2), \bigwedge_{i \in I} \uparrow_2 (\varphi_{L*}(f_i))(a_2) \rangle$$

Hence, $\varphi_\vee$ assigns to the join $\langle \downarrow_1\uparrow_1 (\bigvee_{i \in I} f_i), \bigwedge_{i \in I} \uparrow_1 (f_i) \rangle$ in $L\text{-}FCL(C_1)$, a join of L-concepts $\langle \downarrow_2\uparrow_2 (\bigvee_{i \in I} \varphi_{L*}(f_i))(a_2), \bigwedge_{i \in I} \uparrow_2 (\varphi_{L*}(f_i))(a_2) \rangle$ from $L\text{-}FCL(C_2)$.

2. Similarly to Lemma 8 it can be proved that $\varphi_\wedge$ preserves meets, since it assigns to a meet of L-concepts $\langle \bigwedge_{i \in I} \downarrow_2 (g_i), \uparrow_2\downarrow_2 (\bigvee_{i \in I} g_i) \rangle$ from $L\text{-}FCL(C_2)$ the meet of L-concepts $\langle \bigwedge_{i \in I} \downarrow_1 (\varphi_{R*}(g_i)), \uparrow_1\downarrow_1 (\bigvee_{i \in I} g_i) \rangle$ from $L\text{-}FCL(C_1)$. $\square$

We are in position to define the Galois functor announced at the beginning of this section

Definition 21. *Define a mapping* Gal*, which to every* L*-context* C *assigns the complete lattice* $\langle L\text{-}FCL(C), \leq \rangle$ *of* L*-concepts of* C*, and to every* L*-Chu correspondence from* L-ChuCors(C_1, C_2) *between two* L*-contexts* C_1 *and* C_2 *assigns a supremum-preserving mapping* $\varphi_\vee$ *between* $L\text{-}FCL(C_1)$ *and* $L\text{-}FCL(C_2)$*.*

Theorem 6. Gal *is a functor from category* L-ChuCors *to category* Slat*.*

Proof. Given an arbitrary L-context, $C = \langle B, A, r \rangle$, by definition $Gal(C)$ is a complete concept lattice, in particular, a semilattice.

Recall that, for the identity arrow of category L-ChuCors, the following equalities hold for all $f \in L^B$ and all $g \in L^A$ $\uparrow(\iota_{l*}(f)) = \uparrow(f)$ and $\downarrow(\iota_{r*}(g)) = \downarrow(g)$.

Now, consider $\langle f, \uparrow(f) \rangle \in L\text{-}FCL(C)$,

$$\iota_{C*}(\langle f, \uparrow(f) \rangle) = \langle \downarrow\uparrow(\iota_{l*}(f)), \uparrow(\iota_{l*}(f)) \rangle = \langle \downarrow\uparrow(f), \uparrow(f) \rangle = \langle f, \uparrow(f) \rangle$$

Hence Gal assigns to an identity arrow of L-ChuCors an identity arrow of Slat, i.e. $Gal(\iota_C) = \iota_{Gal(C)}$.

Let $C_i = \langle B_i, A_i, r_i \rangle$ be an L-context for all $i = \{1, 2, 3\}$, and for all $j \in \{1, 2\}$ let $\varphi_j \in L$-ChuCors(C_j, C_{j+1}). Consider $\langle f, \uparrow(f) \rangle \in L\text{-}FCL(C_1)$

$$\begin{aligned}
(\varphi_2 \circ \varphi_1)_\vee(\langle f, \uparrow_1(f) \rangle) &= \langle \downarrow_3\uparrow_3((\varphi_2 \circ \varphi_1)_{L*}(f)), \uparrow_3((\varphi_2 \circ \varphi_1)_{L*}(f)) \rangle \\
&= \langle \downarrow_3\uparrow_3\downarrow_3\uparrow_3(\varphi_{2L*}(\varphi_{1L*}(f))), \uparrow_3\downarrow_3\uparrow_3(\varphi_{2L*}(\varphi_{1L*}(f))) \rangle \\
&= \langle \downarrow_3\uparrow_3(\varphi_{2L*}(\varphi_{1L*}(f))), \uparrow_3(\varphi_{2L*}(\varphi_{1L*}(f))) \rangle \\
&= \langle \downarrow_3\uparrow_3(\varphi_{2L*}(\downarrow_2\uparrow_2(\varphi_{1L*}(f)))), \uparrow_3(\varphi_{2L*}(\downarrow_2\uparrow_2(\varphi_{1L*}(f)))) \rangle \\
&= \varphi_{2\vee}(\langle \downarrow_2\uparrow_2(\varphi_{1L*}(f)), \uparrow_2(\varphi_{1L*}(f)) \rangle) \\
&= \varphi_{2\vee}(\varphi_{1\vee}(\langle f, \uparrow_1(f) \rangle))
\end{aligned}$$

Hence $Gal(\varphi_2 \circ \varphi_1) = (\varphi_2 \circ \varphi_1)_\vee = \varphi_{2\vee} \circ \varphi_{1\vee} = Gal(\varphi_2) \circ Gal(\varphi_1)$

So $Gal : L$-ChuCors $\to$ Slat is a functor. ☐

5 Galois functor from *L*-ChuCors to *L*-Slat

The results in the two sections above are merged here in order to extend the definition of the previously introduced functor.

Lemma 13. *For any two arbitrary* L*-contexts* C_1 *and* C_2 *there is an isomorphism*

$$\Big\langle \langle L\text{-ChuCors}(C_1, C_2), \approx_2 \rangle, \preceq_2 \Big\rangle \;\cong\; \Big\langle \langle L\text{-ChuCors}(\bot \multimap C_1, \bot \multimap C_2), \approx_2 \rangle, \preceq_2 \Big\rangle$$

Proof. Consider $\varphi \in L$-ChuCors(C_1, C_2). Now by Lemma 7 we can construct an L-Chu correspondence $\overline{\varphi} \in L$-ChuCors$(\bot \multimap C_1, \bot \multimap C_2)$ such that $\overline{\varphi}_R : A_2 \to L^{A_1}$ and $\overline{\varphi}_L : L\text{-}FCL(C_1) \to L^{L\text{-}FCL(C_2)}$ in the following way:

– $\overline{\varphi}_R = \varphi_R$
– $\overline{\varphi}_L(\langle f_1, g_1\rangle) = \downarrow_{C_2} (\overline{\varphi}^*_R(\uparrow_{C_1} (\chi_{\langle f_1,g_1\rangle}))) = \downarrow_{C_2} (\overline{\varphi}^*_R(g_1))) = \downarrow_{C_2} (\varphi^*_R(g_1)))$

Conversely, given an L-Chu correspondence $\overline{\varphi} \in L$-ChuCors$(\perp \multimap C_1, \perp \multimap C_2)$ then we can construct $\varphi \in L$-ChuCors(C_1, C_2) as follows:

– $\varphi_R = \overline{\varphi}_R$
– $\varphi_L(o) = \downarrow_2 (\varphi^*_R(\uparrow_1 (\chi_o))) = \downarrow_2 (\varphi^*_R(\uparrow_1 (\chi_o))) = \downarrow_2 (\overline{\varphi}^*_R(\uparrow_1 (\chi_o)))$ for any object $o \in B_1$

For any pair $\varphi_1, \varphi_2 \in L$-ChuCors$(C_1, C_2)$ we have

$$\begin{aligned}\varphi_1 \preceq_2 \varphi_2 &= \bigwedge_{o_1\in B_1} \bigwedge_{a_2\in A_2} (\downarrow_1 (\varphi_{2R}(a_2))(o_1) \to \downarrow_1 (\varphi_{1R}(a_2))(o_1)) \\ &= \bigwedge_{o_1\in B_1} \bigwedge_{a_2\in A_2} (\downarrow_1 (\overline{\varphi}_{2R}(a_2))(o_1) \to \downarrow_1 (\overline{\varphi}_{1R}(a_2))(o_1)) \\ &= \overline{\varphi}_1 \preceq_2 \overline{\varphi}_2\end{aligned}$$

Similarly for $\approx_2$. □

Now we can create a mapping that assigns, to every L-Chu correspondence $\varphi \in L$-ChuCors(C_1, C_2), a supremum preserving mapping between completely L-ordered sets $\left\langle\langle L\text{-}FCL(C_1), \approx_1\rangle, \preceq_1 \right\rangle$ and $\left\langle\langle L\text{-}FCL(C_2), \approx_1\rangle, \preceq_1 \right\rangle$ in the following way: Let γ be an arbitrary L-set of concepts $\gamma \in L^{L\text{-}FCL(C_1)}$. Now we will use the same construction as in the previous section, but for $\overline{\varphi}$.

$$\overline{\varphi}_\vee(\langle\downarrow_{C_1}\uparrow_{C_1} (\gamma), \uparrow_{C_1} (\gamma)\rangle) = \langle\downarrow_{C_2}\uparrow_{C_2} (\overline{\varphi}_{L*}(\gamma)), \uparrow_{C_2} (\overline{\varphi}_{L*}(\gamma))\rangle$$

From previous results we know that ${}^1\sup(\gamma) = \uparrow_{C_1} (\gamma)$ and ${}^1\sup(\overline{\varphi}_{L*}(\gamma)) = \uparrow_{C_2} (\overline{\varphi}_{L*}(\gamma))$, so then mapping $\overline{\varphi}_\vee$ is supremum preserving .

Now we will create a Galois functor L-Gal from L-ChuCors to L-Slat, the category of supremum preserving mappings between completely L-ordered sets, in following way:

– Given an L-fuzzy context C, L-$Gal(C)$ will be the completely L-ordered set $\left\langle\langle L\text{-}FCL(C), \approx_1\rangle, \preceq_1 \right\rangle$
– to every $\varphi \in L$-ChuCors(C_1, C_2), L-$Gal(\varphi)$ will be the supremum preserving mapping $\overline{\varphi}_\vee$

As the construction is the same as in the previous section about Galois functor, we can state that the mapping L-Gal : L-ChuCors $\to$ L-Slat is a functor from category L-ChuCors to L-Slat.

6 Conclusions and future work

We have presented some interesting and useful properties of the category L-ChuCors of L-Chu correspondences between formal L-fuzzy contexts. Specifically, we have introduced a functor between L-ChuCors and a category of supremum preserving mappings between completely L-ordered sets.

As future work, we plan to continue the study of the functor L-Gal, and consider its possible fullness and/or faithfullness.

References

1. M. Barr, Ch. Wells, *Category theory for computing science*, 2nd ed., Prentice Hall International (UK) Ltd., 1995.
2. R. Bělohlávek, Fuzzy concepts and conceptual structures: induced similarities, *Joint Conference on Information Sciences*, pp.179–182, 1998.
3. R. Bělohlávek, Lattices of fixed points of fuzzy Galois connections, *Mathematical Logic Quartely*, vol.47, no.1, pp.111–116, 2001.
4. R. Bělohlávek, Concept lattices and order in fuzzy logic, *Annals of Pure and Applied Logic* 128(1-3):277–298, 2004.
5. R. Bělohlávek, Lattice-type fuzzy order is uniquely given by its 1-cut: proof and consequences, *Fuzzy Sets and Systems*, vol.143, pp.447–458, 2004.
6. B. Davey and H. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, second edition, 2002.
7. B. Ganter and R. Wille, *Formal concept analysis*, Springer–Verlag, 1999.
8. B. Ganter, Relational Galois connections, in Proc. Intl Conf on Formal Concept Analysis, *Lecture Notes in Computer Science* 4390:1–17, 2007.
9. P. Hájek, *Metamathematics of fuzzy logic*, Kluwer Academic Publisher, 2001.
10. S. Krajči, Every concept lattice with hedges is isomorphic to some generalized concept lattice, In *Proc Intl Ws of Concept Lattices and Applications*, 1–9, 2005.
11. S. Krajči, A generalized concept lattice, *Logic Journal of the IGPL*, vol.13, no.5, pp.543–550, 2005.
12. O. Kridlo and M. Ojeda-Aciego, On the L-fuzzy generalization of Chu correspondences, *International Journal of Computer Mathematics*, 2011, iFirst version at `http://dx.doi.org/10.1080/00207160903494147`.
13. O. Kridlo, S. Krajči and M. Ojeda-Aciego, The category of L-Chu correspondences and the structure of L-bonds', Submitted, 2011.
14. M. Krötzsch, P. Hitzler, G-Q. Zhang, Morphisms in Context. In Proc of Intl Conf on Computational Science, *Lecture Notes in Computer Science* 3516:223–237, 2005.
15. H. Mori, Chu Correspondences, *Hokkaido Matematical Journal* 37:147–214, 2008.
16. H. Mori, Functorial properties of Formal Concept Analysis, In Proc Intl Conf on Conceptual Structures, *Lect Notes in Computer Science* 4604:505–508, 2007.
17. G.Q. Zhang, Chu spaces, concept lattices, and domains, *Electronic Notes in Theoretical Computer Science* 83 (2004).