On multi-adjoint concept lattices: definition and representation theorem

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Abstract. Several fuzzifications of formal concept analysis have been proposed to deal with uncertainty or incomplete information. In this paper, we focus on the new paradigm of multi-adjoint concept lattices which embeds different fuzzy extensions of concept lattices, our main result being the representation theorem of this paradigm. As a consequence of this theorem, the representation theorems of the other paradigms can be proved more directly. Moreover, the multi-adjoint paradigm enriches the language providing greater flexibility to the user.

Keywords: concept lattices, multi-adjoint lattices, Galois connection, implication triples.

1 Introduction

The study of reasoning methods under uncertainty, imprecise data or incomplete information has shown to be an important topic in the recent years. Most of the current research areas are receiving this message and it is frequent to see *fuzzified* versions of several well-known standard structures. In this paper, we focus on the area of formal concept analysis and, specifically, on the generalization of the classical definition of concept lattice to the fuzzy case.

A number of different approaches have been proposed to generalize the classical concept lattices given by Ganter and Wille [6] allowing some uncertainty in data. One of these approaches was proposed by Burusco and Fuentes-González [3] where fuzzy concept lattices were first presented, and later further developed by Pollandt [13]. Non-commutative fuzzy logic was considered in the context of concept lattices and similarity by Georgescu and Popescu [7]. This approach, consisting in generalizing the equality relation and considering an alternative similarity relation, underlies in the recent work of Bělohlávek [2], which considered L-equalities to extend the fuzzy concept lattice. His approach was extended in an asymmetric way, although only for the case of classical equality ($L = \{0, 1\}$) by Krajči, who introduced the so-called generalized concepts lattices in [9, 10].

In the context of general logical frameworks, a recent approach so-called multi-adjoint has been recently introduced and is receiving considerable attention [8, 12]. The multi-adjoint framework was originated as a generalization of several non-classical logic programming frameworks, its semantic structure is the multi-adjoint lattice, in which a lattice is considered together with several conjunctors and implications making up adjoint pairs.

With the idea of providing a general framework in which the different approaches stated above could be conveniently accommodated, we have to work in a general non-commutative environment; this naturally leads to the consideration of adjoint triples, also called pre-implication triples [1] or bi-residuated structures [11] as the main building blocks of our multi-adjoint concept lattices.

The main result introduced here, apart from the introduction of multi-adjoint concept lattices, is its representation theorem on the multi-adjoint concept lattices, which gives equivalent conditions in a complete lattice in order to be isomorphic to a multi-adjoint concept lattice. This theorem can be instantiated to the above mentioned paradigms and provide a much more easier proof. We also present an example which shows that the multi-adjoint framework is more expressive that the generalized framework.

The plan of this paper is the following: in Section 2 we recall the basics about Galois connection and the notion of multi-adjoint concept lattice is introduced, in Section 3 contains the proof of the representation theorem; in Section 4 an example of the multi-adjoint framework is presented; the paper ends with some conclusions and prospects for future work.

2 Multi-adjoint concept lattice

A basic notion in formal concept analysis is that of *Galois connection*, we start this section recalling a result which proves that each Galois connection has an associated complete lattice, called *Galois lattice* or *concept lattice*.

Definition 1. *Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be posets, and ${}^\downarrow\colon P_1 \to P_2$, ${}^\uparrow\colon P_2 \to P_1$ applications, the pair $({}^\uparrow, {}^\downarrow)$ forms a* Galois connection *between P_1 and P_2 if and only if:*

1. *${}^\uparrow$ and ${}^\downarrow$ are decreasing.*
2. *$x \leq_1 x^{\downarrow\uparrow}$ for all $x \in P_1$.*
3. *$y \leq_2 y^{\uparrow\downarrow}$ for all $y \in P_2$.*

If P_1 and P_2 are complete lattices then the following theorem can be established, see [5]:

Theorem 1. *Let $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ be complete lattices, $({}^\uparrow, {}^\downarrow)$ a Galois connection between L_1, L_2 and $\mathcal{C} = \{\langle x, y\rangle \mid x^\uparrow = y, x = y^\downarrow; x \in L_1, y \in L_2\}$ then $\mathcal{C}$ is a complete lattice, where*

$$\bigwedge_{i\in I}\langle x_i, y_i\rangle = \langle \bigwedge_{i\in I} x_i, (\bigvee_{i\in I} y_i)^{\downarrow\uparrow}\rangle \quad \textit{and} \quad \bigvee_{i\in I}\langle x_i, y_i\rangle = \langle (\bigvee_{i\in I} x_i)^{\uparrow\downarrow}, \bigwedge_{i\in I} y_i\rangle$$

We will use this theorem in order to prove that our construction of multi-adjoint concept lattices actually leads to a complete lattice.

Firstly, a generalization of multi-adjoint lattices is introduced in order to admit different sorts, in which we allow non-commutative conjunctors as in [1, 7, 11]. To begin with, the adjoint pairs are generalized into adjoint triples, the basic blocks of multi-adjoint concept lattices, as follows:

Definition 2. *Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets and $\&\colon P_1 \times P_2 \longrightarrow P_3$, $\swarrow\colon P_3 \times P_2 \longrightarrow P_1$, $\nwarrow\colon P_3 \times P_1 \longrightarrow P_2$ be applications, then $(\&, \swarrow, \nwarrow)$, is a* adjoint triple *with respect to P_1, P_2, P_3 if:*

- *$\&$ is increasing in both arguments.*
- *$\swarrow$ and $\nwarrow$ are increasing in the first argument and decreasing in the second.*
- *$x \leq_1 z \swarrow y$ iff $x \,\&\, y \leq_3 z$ iff $y \leq_2 z \nwarrow x$, where $x \in P_1, y \in P_2$ and $z \in P_3$.*

This last property is known as *adjoint property* and generalises the *modus ponens* rule in a non-commutative multi-valued setting. Notice that no boundary condition is required, in difference to the usual definition of multi-adjoint lattice [12] or implication triples [1].

In order to introduce a Galois connection which generalizes that given in the classical case, the usual motivation underlying the multi-adjoint framework [8, 12] is applied to that of adjoint triples, and leads to the following definition of multi-adjoint frame.

Definition 3. *A* multi-adjoint frame *$\mathcal{L}$ is a tuple*

$$(L_1, L_2, P, \preceq_1, \preceq_2, \leq, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$$

where L_i are complete lattices and P is a poset, and such that $(\&_i, \swarrow^i, \nwarrow_i)$ is an adjoint triple with respect to L_1, L_2, P for all $i = 1, \dots, n$.

A multi-adjoint frame as above will be denoted as $(L_1, L_2, P, \&_1, \dots, \&_n)$, for short. It is convenient to note that, in principle, L_1, L_2 and P could be simply posets, the reason to consider complete lattices is that multi-adjoint frames will used as the underlying lattice on which the operations will be made; hence, general joins and meets are required.

A *context* for a given frame will mean a tuple (A, B, R, σ) defined as below where, following the usual terminology, A is to be considered as a set of attributes and B as a set of objects.

Definition 4. *A* context *for a given frame $(L_1, L_2, P, \&_1, \dots, \&_n)$ is a tuple (A, B, R, σ) such that A and B are non-empty sets, R is a P-fuzzy relation $R\colon A \times B \longrightarrow P$ and σ is a mapping which associates any object in B (or attribute in A) with some particular adjoint triple in the frame, that is, $\sigma\colon B \to \{1, \dots, n\}$ (or $\sigma\colon A \to \{1, \dots, n\}$).*

The fact that in a multi-adjoint context each object (or attribute) has an associated implication is interesting in that subgroups with different degrees of preference can be established in a convenient way; however, a complete study of this possibility is outside the scope of this paper. From now on, we will consider in the context the association $\sigma\colon B \to \{1, \dots, n\}$.

Now, given a frame and a context for that frame, the following mappings ${}^{\uparrow_\sigma}\colon L_2^B \longrightarrow L_1^A$ and ${}^{\downarrow^\sigma}\colon L_1^A \longrightarrow L_2^B$ can be defined:

$$g^{\uparrow_\sigma}(a) = \inf\{R(a, b) \swarrow^{\sigma(b)} g(b) \mid b \in B\}$$
$$f^{\downarrow^\sigma}(b) = \inf\{R(a, b) \nwarrow_{\sigma(b)} f(a) \mid a \in A\}$$

Notice that these mappings generalise those given in [3, 10] and, as proved below, generate a Galois connection.

Proposition 1. *Given a multi-adjoint frame* $(L_1, L_2, P, \&_1, \dots, \&_n)$ *and a context* (A, B, R, σ)*, the pair* $({}^{\uparrow_\sigma}, {}^{\downarrow^\sigma})$ *is a Galois connection between* L_1^A *and* L_2^B*.*

Proof. From now on, to improve readability, we will write $({}^{\uparrow}, {}^{\downarrow})$ instead of $({}^{\uparrow_\sigma}, {}^{\downarrow^\sigma})$ and $\swarrow^b$, $\nwarrow_b$ instead of $\swarrow^{\sigma(b)}$, $\nwarrow_{\sigma(b)}$.

By definition, we have to prove that:

1. ${}^{\uparrow}$ and ${}^{\downarrow}$ are decreasing.
   This is trivial since the implications are decreasing in the second argument.
2. $g \le g^{\uparrow\downarrow}$ for all $g \in L_2^B$,
   Given $a \in A$ and $b \in B$ the next chain of inequalities holds because of the adjoint property:

$$g^{\uparrow}(a) = \inf\{R(a, b') \swarrow^{b'} g(b') \mid b' \in B\} \preceq_2 R(a, b) \swarrow^{b} g(b)$$
$$g^{\uparrow}(a) \,\&_b\, g(b) \preceq_2 R(a, b)$$
$$g(b) \preceq_2 R(a, b) \nwarrow_b g^{\uparrow}(a)$$

   As the inequality above holds for all $a \in A$, by using the infimum property, it can be obtained that

$$g(b) \preceq_2 \inf\{R(a, b) \nwarrow_b g^{\uparrow}(a) \mid a \in A\} = g^{\uparrow\downarrow}(b)$$

3. $f \le f^{\downarrow\uparrow}$ for all $f \in L_1^A$.
   The proof is similar. □

Now, a *concept* is a pair $\langle g, f\rangle$ satisfying that $g \in L_2^B$, $f \in L_1^A$ and that $g^{\uparrow} = f$ and $f^{\downarrow} = g$; with $({}^{\uparrow}, {}^{\downarrow})$ being the Galois connection defined above.

Definition 5. *The* multi-adjoint concept lattice *associated to a multi-adjoint frame* $(L_1, L_2, P, \&_1, \dots, \&_n)$ *and a context* (A, B, R, σ) *is the set of concepts:*

$$\mathcal{M} = \{\langle g, f\rangle \mid g \in L_2^B, f \in L_1^A \textit{ and } g^{\uparrow} = f, f^{\downarrow} = g\}$$

with the ordering $\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle$ *if and only if* $g_1 \preceq g_2$ *(equivalently* $f_1 \preceq f_2$*).*

Note that, by Theorem 1, the poset $(\mathcal{M}, \preceq)$ defined above is a complete lattice, since the arrows $({}^{\uparrow}, {}^{\downarrow})$ form a Galois connection between the complete lattices L_1^A and L_2^B.[1]

3 The representation theorem

An extension of the representation (or fundamental) theorem on the classical concept lattice [6] for the multi-adjoint framework is presented below. The result is similar to those given in previous extensions of the classical concept lattices, but in this general framework the proof is simpler. To begin with, we need to introduce some definitions and preliminary results.

[1] In the rest of the paper we will assume a fixed multi-adjoint frame and context.

Definition 6. *Given a set A, a poset P with bottom element $\bot$, and elements $a \in A$, $x \in P$, the characteristic mapping $@_a^x\colon A \to P$ is defined as:*

$$@_a^x(a') = \begin{cases} x, & \textit{if } a' = a \\ \bot, & \textit{otherwise} \end{cases}$$

The following lemma gives a technical property which will be needed later.

Lemma 1. *In the concept lattice $(\mathcal{M}, \preceq)$, given $a \in A$, $b \in B$, $x \in L_1$ and $y \in L_2$, the following equalities hold:*

$$@_a^{x\downarrow}(b') = R(a, b') \nwarrow_{b'} x \qquad \textit{for all} \quad b' \in B$$
$$@_b^{y\uparrow}(a') = R(a', b) \swarrow^{b} y \qquad \textit{for all} \quad a' \in A$$

Proof. By definition of $@_a^{x\downarrow}$:

$$@_a^{x\downarrow}(b') = \inf\{R(a', b') \nwarrow_{b'} @_a^x(a') \mid a' \in A\} = R(a, b') \nwarrow_{b'} x$$

where the last inequality follows because $R(a', b) \nwarrow_b \bot_1 = \top_2$ (this fact is a consequence of the adjoint property, since $\bot_1 \preceq_1 R(a', b) \swarrow^{b} \top_2$).

The other equality follows similarly. $\square$

The following definitions introduce properties which will be used in the statement of Prop. 2.

Definition 7. *Given a complete lattice L, a subset $K \subseteq L$ is infimum-dense (resp. supremum-dense) if and only if for all $x \in L$ there exists $K' \subseteq K$ such that $x = \inf(K')$ (resp. $x = \sup(K')$).*

Definition 8. *Let $(\mathcal{M}, \preceq)$ be a multi-adjoint concept lattice, $(V, \sqsubseteq)$ a complete lattice and $\alpha\colon A \times L_1 \to V$, $\beta\colon B \times L_2 \to V$ two maps. We say that β* is (V, R)-related with *α if we have that:*

1a) $\alpha[A \times L_1]$ is infimum-dense;
1b) $\beta[B \times L_2]$ is supremum-dense; and
2) for each $a \in A$, $b \in B$, $x \in L_1$ and $y \in L_2$:

$$\beta(b, y) \sqsubseteq \alpha(a, x) \quad \textit{if and only if} \quad x \,\&_b\, y \leq R(a, b)$$

Proposition 2. *Given a multi-adjoint concept lattice $(\mathcal{M}, \preceq)$, a complete lattice $(V, \sqsubseteq)$ and two maps $f \in L_1^A$, $g \in L_2^B$, if there exist two applications $\beta\colon B \times L_2 \to V$, $\alpha\colon A \times L_1 \to V$, where β is (V, R)-related with α we have that:*

1. *β is increasing in the second argument.*
2. *α is decreasing in the second argument.*
3. *$g^\uparrow(a) = \sup\{x \in L_1 \mid v_g \sqsubseteq \alpha(a, x)\}$, where $v_g = \sup\{\beta(b, g(b)) \mid b \in B\}$.*
4. *$f^\downarrow(b) = \sup\{y \in L_2 \mid \beta(b, y) \sqsubseteq v_f\}$, where $v_f = \inf\{\alpha(a, f(a)) \mid a \in A\}$.*
5. *If $g_v(b) = \sup\{y \in L_2 \mid \beta(b, y) \sqsubseteq v\}$, then $\sup\{\beta(b, g_v(b)) \mid b \in B\} = v$.*
6. *If $f_v(a) = \sup\{x \in L_1 \mid v \sqsubseteq \alpha(a, x)\}$, then $\sup\{\alpha(a, f_v(a)) \mid a \in A\} = v$.*

Proof. We give the proofs for items 1, 3 and 5, since the others are similar.
1. Let $y_1 \preceq_2 y_2 \in L_2$, as $\beta(b, y_2) \in V$ and $\alpha[A \times L_1]$ is infimum-dense there exists a set of indices Λ and $K = \{(a_j, x_j) \mid j \in \Lambda\} \subseteq A \times L_1$ such that $\beta(b, y_2) = \inf\{\alpha(a_j, x_j) \mid j \in \Lambda\}$, so $\beta(b, y_2) \sqsubseteq \alpha(a_j, x_j)$ for all $j \in \Lambda$. Now, by Def. 8 property 2, it follows that $x_j \mathbin{\&_b} y_2 \le R(a_j, b)$ for all j and, as $y_1 \preceq_2 y_2$,

$$x_j \mathbin{\&_b} y_1 \le x_j \mathbin{\&_b} y_2 \le R(a_j, b) \text{ for all } j$$

Therefore, $\beta(b, y_1) \sqsubseteq \alpha(a_j, x_j)$ for all j and, as $\beta(b, y_2)$ is the infimum, $\beta(b, y_1) \sqsubseteq \beta(b, y_2)$, so β is increasing in the second argument.
3. Given $x \in L_1$, by the adjoint property the inequality $x \preceq_1 R(a, b) \swarrow^b g(b)$ is equivalent to $x \mathbin{\&_b} g(b) \le R(a, b)$ which is also equivalent, by Def. 8 property 2, to $\beta(b, g(b)) \sqsubseteq \alpha(a, x)$ for all $b \in B$, therefore by the supremum property

$$v_g = \sup\{\beta(b, g(b)) \mid b \in B\} \sqsubseteq \alpha(a, x)$$

Thus, we obtain the equality of the sets:

$$\{x \in L_1 \mid x \preceq_1 R(a, b) \swarrow^b g(b) \text{ for all } b \in B\} = \{x \in L_1 \mid v_g \sqsubseteq \alpha(a, x)\}$$

Therefore:

$$\begin{aligned} g^{\uparrow}(a) &= \inf\{R(a, b) \swarrow^b g(b) \mid b \in B\} \\ &\overset{(*)}{=} \sup\{x \in L_1 \mid x \preceq_1 R(a, b) \swarrow^b g(b) \text{ for all } b \in B\} \\ &= \sup\{x \in L_1 \mid v_g \sqsubseteq \alpha(a, x)\} \end{aligned}$$

where $(*)$ is given from the adjoint property.
5. Firstly we will show that, for any $v \in V$, $\sup\{\beta(b, g_v(b)) \mid b \in B\} \sqsubseteq v$, and let us write $Y_b = \{y \in L_2 \mid \beta(b, y) \sqsubseteq v\}$ for any $b \in B$, so that $g_v(b) = \sup Y_b$.

Given $v \in V$, as $\alpha[A \times L_1]$ is infimum-dense, there is a set of indices Λ and $K = \{(a_j, x_j) \mid j \in \Lambda\} \subseteq A \times L_1$ such that $v = \inf\{\alpha(a_j, x_j) \mid j \in \Lambda\}$.

If $Y_b = \varnothing$, then $g_v(b) = \bot_2$ and we have the next chain of equivalences:

$$g_v(b) \preceq_2 R(a_j, b) \nwarrow_b x_j \text{ iff } x_j \mathbin{\&_b} g_v(b) \le R(a_j, b) \text{ iff } \beta(b, g_v(b)) \sqsubseteq \alpha(a_j, x_j) \quad (1)$$

Otherwise, if Y_b is non-empty, then, by Def. 8 property 2, we have for all $j \in \Lambda$ and $y \in Y_b$:

$$\beta(b, y) \sqsubseteq v \sqsubseteq \alpha(a_j, x_j) \text{ iff } x_j \mathbin{\&_b} y \le R(a_j, b) \text{ iff } y \preceq_2 R(a_j, b) \nwarrow_b x_j$$

by computing the supremum on y, we get to $g_v(b) = \sup Y_b \preceq_2 R(a_j, b) \nwarrow_b x_j$, and then the rest of equivalences in (1) apply.

Recalling that $v = \inf\{\alpha(a_j, x_j) \mid j \in \Lambda\}$ we obtain than $\beta(b, g_v(b)) \sqsubseteq v$ for all $b \in B$. Finally, taking supremum on the left hand side, we get

$$\sup\{\beta(b, g_v(b)) \mid b \in B\} \sqsubseteq v$$

For the other inequality, as $\beta[B \times L_2]$ is supremum-dense we have that $v = \sup\{\beta(b_j, y_j) \mid (b_j, y_j) \in A \times L_2, j \in \Lambda'\}$. Then, for any $j \in \Lambda'$ we have that $y_j \in Y_{b_j}$ and, moreover, $y_j \preceq_2 \sup Y_{b_j} = g_v(b_j)$. Since β is increasing in the second argument, by item 1, we obtain:

$$\beta(b_j, y_j) \sqsubseteq \beta(b_j, g_v(b_j)) \sqsubseteq \sup\{\beta(b_j, g_v(b_j)) \mid j \in \Lambda\} \sqsubseteq \sup\{\beta(b, g_v(b)) \mid b \in B\}$$

As v is the supremum on j of $\beta(b_j, y_j)$, we get $v \sqsubseteq \sup\{\beta(b, g_v(b)) \mid b \in B\}$. $\square$

We can now state and prove the representation theorem for multi-adjoint concept lattices.

Theorem 2. *Given a complete lattice* $(V, \sqsubseteq)$ *and a multi-adjoint concept lattice* $(\mathcal{M}, \preceq)$*, we have that* V *is isomorphic to* $\mathcal{M}$ *if and only if there exist applications* $\alpha\colon A \times L_1 \to V$*,* $\beta\colon B \times L_2 \to V$ *such that* β *is* (V, R)*-related to* α*.*

Proof. Given an isomorphism $\varphi\colon \mathcal{M} \to V$, the mappings $\alpha\colon A \times L_1 \to V$ and $\beta\colon B \times L_2 \to V$ can be naturally defined, for every $a \in A$, $b \in B$, $x \in L_1$ and $y \in L_2$, as follows:

$$\alpha(a, x) = \varphi(\langle @_a^{x\downarrow}, @_a^{x\downarrow\uparrow} \rangle) \qquad \beta(b, y) = \varphi(\langle @_b^{y\uparrow\downarrow}, @_b^{y\uparrow} \rangle)$$

Let us prove that β is (V, R)-related to α:

Firstly, let us show that $\alpha[A \times L_1]$ is infimum-dense. By definition, we have to prove that given $v \in V$ there exists $K \subseteq A \times L_1$ such that $v = \inf(\alpha[K])$.

If $\varphi^{-1}(v) = \langle g, f \rangle \in \mathcal{M}$, we define $K = \{(a, f(a)) \mid a \in A\} \subseteq A \times L_1$. Since φ is an isomorphism, it is sufficient to prove that

$$\langle g, f \rangle = \inf\{\langle @_a^{f(a)\downarrow}, @_a^{f(a)\downarrow\uparrow} \rangle \mid a \in A\}$$

Let us prove, for instance, that $g(b) = \inf\{@_a^{f(a)\downarrow}(b) \mid a \in A\}$. By Lemma 1, we have that $@_a^{f(a)\downarrow}(b) = R(a, b) \nwarrow_b f(a)$, thus

$$\inf\{@_a^{f(a)\downarrow}(b) \mid a \in A\} = \inf\{R(a, b) \nwarrow_b f(a) \mid a \in A\} = f^{\downarrow}(b) = g(b)$$

Similarly, we can prove that $\beta[B \times L_2]$ is supremum-dense.

It only remains to prove that given $a \in A$, $b \in B$, $x \in L_1$ and $y \in L_2$, we have that $\beta(b, y) \sqsubseteq \alpha(a, x)$ iff $x \mathbin{\&_b} y \leq R(a, b)$.

For the direct implication, as φ is order-preserving and reflecting, we have that $\beta(b, y) \sqsubseteq \alpha(a, x)$ is equivalent to $\langle @_b^{y\uparrow\downarrow}, @_b^{y\uparrow} \rangle \leq \langle @_a^{x\downarrow}, @_a^{x\downarrow\uparrow} \rangle$ and, in particular, to $@_b^{y\uparrow\downarrow} \leq @_a^{x\downarrow}$. From the properties of Galois connection, Lemma 1, and the adjoint property we obtain the following chain:

$$y = @_b^{y}(b) \preceq_2 @_b^{y\uparrow\downarrow}(b) \preceq_2 @_a^{x\downarrow}(b) = R(a, b) \nwarrow_b x \quad \text{iff} \quad x \mathbin{\&_b} y \leq R(a, b)$$

For the other implication, it is sufficient to prove that $@_a^{x} \leq @_b^{y\uparrow}$ as this is equivalent to $@_b^{y\uparrow\downarrow} \leq @_a^{x\downarrow}$ which finally implies $\beta(b, y) \sqsubseteq \alpha(a, x)$, from the definition of α and β, and φ order-preserving.

But this is clear because, if $a' \in A$ with $a' \neq a$, then $@^x_a(a') \preceq_1 @^{y\uparrow}_b(a')$ holds because $@^x_a(a') = \bot_1$. If $a' = a$, as $x \,\&_b\, y \leq R(a,b)$ applying the adjoint property and Lemma 1 we obtain that:

$$@^x_a(a) = x \preceq_1 R(a,b) \swarrow^b y = @^{y\uparrow}_b(a)$$

Now, conversely, assume we have mappings $\alpha\colon A \times L_1 \to V$, $\beta\colon B \times L_2 \to V$ where β is (V,R)-related to α, and let us construct an isomorphism $\varphi\colon \mathcal{M} \to V$. We define the mapping φ for every $\langle g,f\rangle \in \mathcal{M}$ as follows:

$$\varphi(\langle g,f\rangle) = \sup\{\beta(b,g(b)) \mid b \in B\}$$

To prove that it is a lattice isomorphism we introduce another mapping $\psi\colon V \to \mathcal{M}$ which is the inverse mapping of φ.

The mapping ψ is defined for each $v \in V$ as $\psi(v) = \langle g_v, f_v\rangle$, where, for each $b \in B$ and $a \in A$, $g_v(b)$ and $f_v(a)$ are defined as in Proposition 2. This proposition shows that ψ is well-defined as well, that is, $\langle g_v, f_v\rangle$ is a concept. The argument is as follows:

$${g_v}^\uparrow(a) = \sup\{x \in L_1 \mid v_{g_v} \sqsubseteq \alpha(a,x)\} = \sup\{x \in L_1 \mid v \sqsubseteq \alpha(a,x)\} = f_v$$

where the first equality is obtained from item 3 and, from item 5 we have the other equality because $v_{g_v} = \sup\{\beta(b, g_v(b)) \mid b \in B\} = v$. The equality $f_v^\downarrow = g_v$ is proved analogously.

To prove the equality $\psi(\varphi(\langle g,f\rangle)) = \langle g,f\rangle$, it is sufficient to prove that $f = f_{v_\varphi}$, where $v_\varphi = \varphi(\langle g,f\rangle)$, but this follows from Proposition 2 (item 3) since $v_g = \sup\{\beta(b,g(b)) \mid b \in B\} = \varphi(\langle g,f\rangle) = v_\varphi$ and

$$g^\uparrow(a) = \sup\{x \in L_1 \mid v_g \sqsubseteq \alpha(a,x)\}$$

The other composition gives the identity as well, that is, $v = \varphi(\psi(v)) = \varphi(\langle g_v, f_v\rangle) = \sup\{\beta(b, g_v(b)) \mid b \in B\}$ for all $v \in V$, as an application of item 5 of Proposition 2.

To finish the proof it is sufficient to prove that φ it is order-preserving, since any order-preserving bijection between lattice is a lattice isomorphism, see [5]. Given $\langle g_1, f_1\rangle$, $\langle g_2, f_2\rangle$ in $\mathcal{M}$ with $\langle g_1, f_1\rangle \leq \langle g_2, f_2\rangle$, we have that $g_1 \leq g_2$ and therefore $\beta(b, g_1(b)) \sqsubseteq \beta(b, g_2(b))$ for all $b \in B$, since β is increasing in the second argument. Thus, by definition of φ, we obtain that:

$$\varphi(\langle g_1, f_1\rangle) \sqsubseteq \varphi(\langle g_2, f_2\rangle)$$

□

This theorem can be shown to embed the corresponding ones given in [2, 3, 9].Regarding an improvement of a previous representation theorem: let us notice that, in Proposition 2 it is proved directly that the function α is decreasing and β is increasing in their second argument, hence these hypotheses, which are explicitly required for the representation theorem of [9], can be dropped.

Let us finish this section with a further proposition which relates the behaviour of the mappings α and β.

Proposition 3. *Given a multi-adjoint concept lattice* $(\mathcal{M}, \preceq)$*, a concept* $\langle g, f\rangle \in \mathcal{M}$ *and two mappings* $\beta\colon B \times L_2 \to \mathcal{M}$*,* $\alpha\colon A \times L_1 \to \mathcal{M}$*, where* β *is* $(\mathcal{M}, R)$*-related to* α*, we have that:*

$$\sup\{\beta(b, g(b)) \mid b \in B\} = \inf\{\alpha(a, f(a)) \mid a \in A\}$$

Proof. Given $a \in A$, we have that

$$f(a) = g^{\uparrow}(a) = \inf\{R(a, b) \swarrow^b g(b) \mid b \in B\}$$

then $f(a) \preceq_1 R(a, b) \swarrow^b g(b)$ for all $b \in B$ and applying the adjoint property and Property 2 we have that $\beta(b, g(b)) \sqsubseteq \alpha(a, f(a))$ for all $b \in B$. Therefore if we apply the supremum and infimum properties we obtain the inequality:

$$\sup\{\beta(b, g(b)) \mid b \in B\} \sqsubseteq \inf\{\alpha(a, f(a)) \mid a \in A\}$$

Let $v_\beta = \sup\{\beta(b, g(b)) \mid b \in B\} \in V$ be, as $\alpha[A \times L_1]$ is infimum-dense there exists a set of indices Λ and $K = \{(a_j, x_j) \mid j \in \Lambda\} \subseteq A \times L_1$ such that $v_\beta = \inf\{\alpha(a_j, x_j) \mid j \in \Lambda\}$ and, for all $j \in \Lambda$ and $b \in B$, we have that $\beta(b, g(b)) \sqsubseteq \alpha(a_j, x_j)$ which leads us, from Property 2, to $x_j \preceq_1 R(a_j, b) \swarrow^b g(b)$ and, using that $f = g^{\uparrow}$, to $x_j \preceq_1 f(a_j)$ for all $j \in \Lambda$. Hence we have the following chain which provides the required equality:

$$\begin{aligned} v_\beta = \sup\{\beta(b, g(b)) \mid b \in B\} &\sqsubseteq \inf\{\alpha(a, f(a)) \mid a \in A\} \\ &\sqsubseteq \inf\{\alpha(a_j, f(a_j)) \mid j \in \Lambda\} \\ &\overset{(*)}{\sqsubseteq} \inf\{\alpha(a_j, x_j) \mid j \in \Lambda\} \\ &= v_\beta \end{aligned}$$

where $(*)$ holds because $x_j \preceq_1 f(a_j)$ for all $j \in \Lambda$ and α is decreasing in the second argument. $\square$

4 A toy example

Now, we apply the language capabilities of the multi-adjoint concept lattices in an example introduced by Umbreit [14] and used by [4]. Furthermore, in the multi-adjoint concept lattice framework the use can express in a better way his necessities.

Example 1. Let $([0,1], [0,1], [0,1], \leq, \leq, \leq, \&_G, \&_L)$ be the multi-adjoint frame where $\&_G$ and $\&_L$ are the commutative Gödel and Łukasiewicz conjunctors respectively, so the residuated implications are defined as:

$$b \nwarrow_L a = b \swarrow^L a = \min\{1, 1 + b - a\}$$
$$b \nwarrow_G a = b \swarrow^G a = \begin{cases} 1, & \text{if } b \geq a; \\ b, & \text{otherwise.} \end{cases}$$

The different contexts considered later have the same set of objects and attributes:

$$A = \{\text{warm}, \text{cold}, \text{poor in rain}, \text{calm wind}\}$$
$$B = \{\text{Mon}, \text{Tue}, \text{Wed}, \text{Thu}, \text{Fri}, \text{Sat}, \text{Sun}\}$$

R	warm	cold	poor in rain	calm wind
Mon	0.5	0.5	1	1
Tue	1	0	1	1
Wed	0.5	0.5	0	0
Thu	0.5	0.5	1	0
Fri	0	1	0	0
Sat	0	1	0.5	0
Sun	0	1	1	1

Fig. 1. Table Example 1.

and identical relationship $R\colon A \times B \to P$, which is defined in Fig. 1.

Now, if we consider the contexts (A, B, R, σ_1), (A, B, R, σ_2), where $\sigma_1(b) = \&_G$ and $\sigma_2(b) = \&_L$ for every $b \in B$ we can check that we obtain the same result as [4]. We can see this in the concrete example of the problem of *walking time*, that is defined in [14] as a day of the week not much warm or cold and with no rain, so the fuzzy notion can be expressed by the fuzzy subset $f\colon A \to [0, 1]$ defined as:

$$f(\text{warm}) = 0.5,\ f(\text{cold}) = 0.5,\ f(\text{poor in rain}) = 1,\ f(\text{calm wind}) = 0.5$$

and represented as: $f = \{\text{warm}/0.5, \text{cold}/0.5, \text{poor in rain}/1, \text{calm wind}/0.5\}$. A multi-adjoint concept which represents the situation given by f is required.

With the first context we have that

$$\begin{aligned} f^{\downarrow}(\text{Mon}) &= \inf\{R(a, \text{Mon}) \nwarrow_G f(a) \colon a \in A\} \\ &= \inf\{0.5 \nwarrow_G 0.5, 0.5 \nwarrow_G 0.5, 1 \nwarrow_G 1, 1 \nwarrow_G 0.5\} \\ &= 1 \end{aligned}$$

Doing the same for the other days the value 0 is obtained.

In a similar way $f^{\downarrow\uparrow}$ is calculated:

$$\begin{aligned} f^{\downarrow\uparrow}&(\text{warm}) = \\ &= \inf\{R(\text{warm}, b) \swarrow^G f^{\downarrow}(b) \colon b \in B\} \\ &= \inf\{0.5 \swarrow^G 1, 1 \swarrow^G 0, 0.5 \swarrow^G 0, 0.5 \swarrow^G 0, 0 \swarrow^G 0, 0 \swarrow^G 0, 0 \swarrow^G 0\} \\ &= 0.5 \end{aligned}$$

If the same is done for the other attributes we have that $f^{\downarrow\uparrow}(\text{cold}) = 0.5$ and that $f^{\downarrow\uparrow}(\text{poor in rain}) = f^{\downarrow\uparrow}(\text{calm wind}) = 1$. So, the best days for walking time (with definition given above) is Monday while the others are bad days.

If we use the second context we obtain that:

$$\begin{aligned} f^{\downarrow} &= \{\text{Mon}/1, \text{Tue}/0.5, \text{Wed}/0, \text{Thu}/0.5, \text{Fri}/0, \text{Sat}/0.5, \text{Sun}/0.5\} \\ f^{\downarrow\uparrow} &= \{\text{warm}/0.5, \text{cold}/0.5, \text{poor in rain}/1, \text{calm wind}/0.5\} \end{aligned}$$

In this case the best day is also Monday, but Tuesday, Thursday, Saturday and Sunday are good ones, while Wednesday and Friday are bad ones. Hence, as stated above, the concepts obtained $\langle f^{\downarrow}, f^{\downarrow\uparrow}\rangle$ are the same as in [4].

However, we can consider a multi-adjoint context where we can adapt the definition of *walking time* in order to consider some restriction in the objects (or attributes). Given the context (A, B, R, σ_3), where $\sigma_3(b) = \&_G$ for every $b \in B_1$ and $\sigma_3(b) = \&_L$ for every $b \in B_2$, where $B_1 = \{\text{Mon}, \text{Tue}, \text{Wed}, \text{Thu}, \text{Fri}\}$ and $B_2 = \{\text{Sat}, \text{Sun}\}$, we can think in the problem of *walking time* as besides the considerations above, better at weekends, obtaining in this case the next results:

$$\begin{aligned} f^{\downarrow}(b_1) &= \inf\{R(a, b_1) \nwarrow_G f(a) \colon a \in A\} \quad \text{ for } b_1 \in B_1 \\ f^{\downarrow}(b_2) &= \inf\{R(a, b_2) \nwarrow_L f(a) \colon a \in A\} \quad \text{ for } b_2 \in B_2 \end{aligned}$$

hence $f^{\downarrow} = \{\text{Mon}/1, \text{Tue}/0, \text{Wed}/0, \text{Thu}/0, \text{Fri}/0, \text{Sat}/0.5, \text{Sun}/0.5\}$. We make the same for $f^{\downarrow\uparrow}$ taking into account the relationship between objects and implications:

$$\begin{aligned} & f^{\downarrow\uparrow}(\text{warm}) = \\ & = \inf(\{R(\text{warm}, b_1) \swarrow^G f^{\downarrow}(b_1) \colon b_1 \in B_1\} \cup \{R(\text{warm}, b_2) \swarrow^L f^{\downarrow}(b_2) \colon b_2 \in B_2\} \\ & = 0.5 \end{aligned}$$

Doing the same for the other attributes:

$$f^{\downarrow\uparrow} = \{\text{warm}/0.5, \text{cold}/0.5, \text{poor in rain}/1, \text{calm wind}/0.5\}$$

Now, though the user prefers weekends, Monday is still the best day, but now, Saturday and Sunday are better days than the others. Remind that fuzzy notions related to the attributes can be given, for example *weather in weekends* can be studied, represented by the fuzzy set:

$$g = \{\text{Mon}/0, \text{Tue}/0, \text{Wed}/0, \text{Thu}/0, \text{Fri}/0, \text{Sat}/1, \text{Sun}/1\}$$

and fixed the attention in the attributes ‘warm’ and ‘poor in rain’, considering different implications, that is, the context could be (A, B, R, τ) where τ is defined as:

$$\tau(\text{warm}) = \tau(\text{poor in rain}) = \&_L \ ; \ \tau(\text{cold}) = \tau(\text{calm wind}) = \&_G$$

5 Conclusions and Future Work

Multi-adjoint concept lattices have been introduced as a generalization of different existing approaches to fuzzified and/or generalized versions of the classical concept lattice. One of the interesting features is that in a multi-adjoint context each object (or attribute) has an associated implication and, thus, subgroups with different degrees of preference can be easily established; this is one topic of future work.

The representation theorem for multi-adjoint concept lattices has been shown by taking advantage of the relationship between Galois connections and concept lattices given in [5]. This fact shows that the “concepts” defined in [2, 3, 9] form a complete lattice without having to rely on the particular definitions of the Galois connections.

The multi-adjoint concept lattice embeds the generalized concept lattice [10] and, as a consequence, other different fuzzy extensions of the classical concept lattice [6], such as the fuzzy concepts of [3] and of [2] for the case of $\{0, 1\}$-equality.

Continuing with the comparison of the multi-adjoint frame with other fuzzy approaches, one future work would be to study the relationship between the concepts given in [7]. Another point to take into account is the introduction of L-equalities to completely embed the fuzzy concept lattice of [2].

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