A measure of contradiction based on the notion of N-weak-contradiction

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***Abstract*—In this work we elaborate on the notion of contradiction between fuzzy sets introduced by Trillas et al in a fuzzy logic context. Our approach is parametric in that the operator used to define contradiction is rather a variable than a constant introduced prior to the analysis of contradiction. We give several motivations to consider weaker operators than the usual involutive negations, and obtain some preliminary results which validate this proposal.**

I. Introduction

Contradictions appear in all branch of knowledge; from pure Science to History through Law. Consider just in Physics the twin paradox in the special relativity theory; in History the Gospels of John and the other three Gospels of the New Testament; and in Law the description of an event given by two lawyers, one of the defense and another of the accuser. Moreover, it is interesting to note that contradictions entail usually new knowledge. A clear example of that feature is that in Science usually new theories arise from contradictions between the current theory and new discovered facts; for instance the consequences of Michelson-Morley experiment, which contradicted the existence of ether, led to the development of relativity.

From a pure computer science point of view, contradictions have been already considered in different topics. Actually, there exists currently a number of articles in the literature studying contradictions in text mining [4] and in medical databases [1]. In the theoretical area, [2] defines various *measures of contradiction* and, if we establish a link between contradiction and inconsistency, in [3], [9], [7] the reader can find several *measures of inconsistency* under different logics.

The measure of contradiction we define in this paper is based on the notion of N-contradiction presented in [10] in the fuzzy framework. Roughly, the idea underlying in the notion of N-contradiction is that two fuzzy sets A and B are N-contradictory if the logic formula $A \to \neg B$ holds (where the symbol $\neg$ represents the negation logic operator associated to N). If the implication $\to$ is residuated, then the satisfiability of the formula above is equivalent to saying that $A(x) \leq \neg B(x)$. However, in this paper we do not consider literally the definition given in [10] but a weaker one. The motivation of that consideration is given in Section II, together with some results.

In Section III we define the measure of contradiction $\mathcal{C}$. Moreover, we prove that if the universe $\mathcal{U}$ is finite or the membership functions of the fuzzy sets are continuous, the measure $\mathcal{C}$ satisfies the following properties:

- symmetry; i.e $\mathcal{C}(A,B) = \mathcal{C}(B,A)$.
- antitonicity; i.e. $A_1 \leq A_2$ implies $\mathcal{C}(A_1,B) \geq \mathcal{C}(A_2,B)$.
- $\mathcal{C}(A,B) = 0$ iff there is $x \in \mathcal{U}$ such that $A(x) = 1$ and $B(x) = 1$.
- $\mathcal{C}(A,B) = 1$ iff $A(x) = 1$ implies $B(x) = 0$ for all $x \in \mathcal{U}$.

At the end of the paper we present also some conclusions and future lines of work. Finally, let us recall some well-known notions in order to make this paper as self-contained as possible. A fuzzy set A is a pair $(\mathcal{U}, \mu_A)$ where $\mathcal{U}$ is a set (called the universe of A) and μ_A is a mapping from $\mathcal{U}$ to $[0,1]$ (called the membership function of A). Note that a fuzzy set is fully determined by its membership function. Hence, for the sake of clarity, we use the same notation for fuzzy sets and membership functions (i.e $A(x) = \mu_A(x)$) and the universe is omitted if whenever it does not generate misunderstandings.

II. N-weak-contradiction

As we stated in the introduction, the measure of contradiction defined in this paper is based on a slight generalization of the notion of N-contradiction given by [10]. In this section we present such a generalization, provide some aspects motivating the generalization and give some results used in further sections. Let us begin by giving the definition of f-weak-contradictory sets.

Definition 1: Let A and B be two fuzzy sets and let $f\colon [0,1] \to [0,1]$ be an antitonic mapping such that $f(0) = 1$. We say that A *is f-weak-contradictory w.r.t.* B if the inequality $A(x) \leq f(B(x))$ holds for all $x \in \mathcal{U}$.

The idea underlying the notion of f-weak-contradiction is to manage contradictory information provided by two fuzzy sets through a mapping f. Fixed a mapping f and a value of $B(x)$ (for some $x \in \mathcal{U}$), the f-weak-contradiction determines an upper boud on the value of $A(x)$. Note that as f is antitonic, the greater the value $B(x)$, the smaller the upper bound and then also the smaller the value $A(x)$. Moreover, as $f(0) = 1$, we have that if $B(x) = 0$ then there is no restriction on the value of $A(x)$. Note that the restriction depends strongly on the mapping chosen, thus somehow, different mappings f determine also "different kinds of contradictions".

The difference of our approach w.r.t. [10] is related to the conditions imposed on mappings f used to determine the

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kind of contradiction. Specifically, [10] considers involutive negations; i.e antitonic mappings $N\colon [0,1] \to [0,1]$ such that $N(0) = 1$, $N(1) = 0$ and verifying that $N(N(x)) = x$ for all $x \in [0,1]$. Our approach weakens these requirements because of the following motivations:

- **Removing the symmetry imposed by involutions.** Considering involutive negations involves a "strong" symmetry. That is because if A and B are two fuzzy sets such that A is N-contradictory w.r.t. B (with N involutive), then necessarily B is N-contradictory w.r.t. A as well. This feature entails that for all element in the universe, if a degree α of A contradicts a degree β of B, then necessarily a degree β of A contradicts a degree α of A. In our opinion, there are some cases where, although the degree α of A contradicts a degree β of B, the degree β in A does not contradict a degree α of A (i.e. they can coexist) as the following example shows:

  Example 1: Let us consider a situation based on control systems. In a factory there are two gas tanks A and B and, by some requirement, the pressure of both tanks must be equal; however, we can only control the pressure of tank A. Moreover, by safety reasons, if the pressure in A is high enough, there is an upper limit on the pressure gain in A. For instance, if $P_A(t)$ and $P_B(t)$ are two fuzzy sets denoting the pressure in A and B respectively at time t, then the pressure put in A could be given by a fuzzy set of the following form:

  $$I_A(t) = \min\{P_B(t), 2 - 2 \cdot P_A(t)\}$$

  Then, if we assume that the control system is working well, the values $P_A(t) = 0.9$ and $I_A(t) = 0.5$ represent a clear contradiction since those values cannot be given by the control system (i.e. be given by the formula which defines the value of I_A). However, the values $P_A(t) = 0.5$ and $I_A(t) = 0.9$ do not represent any contradiction since if $P_B(t) = 0.9$, the formula above explains the value of $I_A(t)$. As a result, in this context, the symmetry imposed by the use of involutive negations needs not hold. $\square$

  The example above is really simple but also enough to motivate, at least, the consideration of arbitrary negations in the definition of N-contradiction. Moreover, it is convenient to recall that the notion of *coherence* presented in [6] allows arbitrary negations in its definition and is closely related to N-contradiction (see [5]).
- **Mappings f as variables.** In our point of view, the contradiction between two fuzzy sets cannot be explained *a priori* by fixing a specific f-contradiction. That is, we do not fix a mapping f in order to study whether two fuzzy sets are f-contradictory. Contrariwise, we believe the mapping f must be fixed *a posteriori*. That is, given two fuzzy sets A and B, we study which is the best f-weak-contradiction to represent the contradiction between A and B. So, we consider mappings f as variables instead of fixed elements. Therefore, the structure of the set of mappings considered to define f-contradictions is now crucial.
- **Duality instead of *"formal"* symmetry imposed by involutions.** We recall that in the first motivational item we have removed the symmetry imposed by involutions. However, everybody has in mind that the idea of contradiction is, somehow, symmetric. That idea can be roughly described by: if A is "contradictory" w.r.t. B then B must be, somehow, also "contradictory" w.r.t. A as well; although possibly under different f-weak-contradictions. Unfortunately, considering only negation operators is not enough to guarantee that feature, since there are fuzzy sets A and B such that A is N-weak-contradictory w.r.t. B but B is not $\overline{N}$-weak-contradictory w.r.t. A for any negation $\overline{N}$. Therefore, we have two options: either imposing more restrictions on negation operators or reducing them. We have chosen the latter approach in order to achieve a structure of complete lattice in the set of mappings used to define f-weak-contradictions. Specifically, we substitute the requirement $f(1) = 0$ (imposed on negation operators) by $f(1) \in [0,1)$. Under such consideration we can obtain Proposition 5 (to be introduced later in this section), which relates the weak-contradiction of A w.r.t. B with the weak-contradiction of B w.r.t. A. Hence, from a formal point of view, the idea of symmetry is changed by duality. It is worth mentioning, that in Section III we show that the degree of both kinds of contradictions is the same. The two types of contradiction given in Example 1 are related below:

  Example 2: It is not difficult to check that, by assuming that the control system works good, P_A is f_1-weak-contradictory w.r.t. I_A and I_A is f_2-weak-contradictory w.r.t. P_A where:

  $$f_1(x) = \begin{cases} 1 & \text{if } x \leq \frac{1}{2} \\ 2 - 2 \cdot x & \text{otherwise} \end{cases}$$

  $$f_2(x) = \begin{cases} \frac{3}{2} - \frac{1}{2} \cdot x & \text{if } x \leq \frac{1}{2} \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

  Note that although f_1 defines a negation operator, f_2 does not; actually, it is not difficult to check that there is no negation N satisfying that I_A is N-weak-contradictory w.r.t. P_A.
- **A final requirement.** The last point concerns only practical motivations (practical from a theoretical point of view). Note that up to now, we have motivated the use of antitonic mappings $f\colon [0,1] \to [0,1]$ verifying that $f(0) = 1$ and $f(1) \neq 1$. This family of mappings (denoted by Ω) forms a lattice by considering the pointwise ordering; but does not define a complete lattice, since

  $$\sup(\Omega) = f^{\top}(x) = 1 \quad \text{for all } x \in [0,1]$$

  does not belong to Ω. Including $f^{\top}$ in Ω has two consequences. On the one hand, it gives to Ω the structure of complete lattice (useful feature as we show in Section III). On the other hand, the use of $f^{\top}$ to define the $f^{\top}$-weak-contradiction allows us to consider the non-contradiction as a special case of weak-contradiction. Hence every pair of fuzzy sets can be considered contradictory up to some degree; that feature is useful in practice.

In the rest of this section we provide some theoretical results on the notion of f-weak-contradictory fuzzy sets to be used later. The first result shows that the f-weak-contradiction is preserved by considering subsets of f-weak-contradictory fuzzy sets.

Proposition 1: Let A, B, C and D be four fuzzy sets such that $A \leq C$ and $B \leq D$. If A is f-weak-contradictory w.r.t. B then C is f-weak-contradictory w.r.t. D for any mapping f.

As we explained in the motivational items above, the mappings f used to defined the f-weak-contradiction can be considered as variables. The following result provides a relationship between f-weak-contradictory fuzzy sets w.r.t. different mappings. Let us recall that Ω denotes the set of antitonic mappings $f\colon [0,1] \to [0,1]$ such that $f(0) = 1$ and $f(1) \neq 1$.

Proposition 2: Let A and B be two fuzzy sets and let f_1 and f_2 be two mappings in Ω such that $f_1 \leq f_2$. If A is f_1-weak-contradictory w.r.t. B then A is f_2-weak-contradictory w.r.t. B as well.

From the previous result we can consider the mapping f in Definition 1 as a degree. Note that if A is f-weak-contradictory w.r.t. B, then as a consequence of the proposition above, A is g-weak-contradictory w.r.t. B for all $f \leq g$. Thus, the lesser the mapping f, the more "kinds of contradictions of A w.r.t. B" hold. In other words: the lesser the mapping, the greatest the contradiction. Hence the mappings

$$f^{\top}(x) = \sup(\Omega)(x) = 1 \qquad \text{and}$$

$$f^{\perp}(x) = \inf(\Omega)(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

determine the weakest and strongest degree of f-weak-contradiction, respectively. The following corollaries show some basic properties on these two extremal cases of f-weak-contradiction.

Corollary 1: Let A and B be two fuzzy sets such that A is $f_{\perp}$-weak-contradictiory w.r.t. B. Then A is f-weak-contradictory w.r.t. B for all $f \in \Omega$.

Corollary 2: Let A and B be two fuzzy sets, then A is $f^{\top}$-weak-contradictory w.r.t. B.

The result above states that every pair of fuzzy sets is, at least, non- contradictory. The following two results determine the structure of $f^{\perp}$-weak-contradictory and $f^{\top}$-weak-contradictory fuzzy sets, respectively.

Proposition 3: Let A and B be two fuzzy sets. A is $f_{\perp}$-weak-contradictiory w.r.t. B if and only if $B(x) > 0$ implies $A(x) = 0$ for all $x \in \mathcal{U}$.

Proposition 4: Let A and B be two fuzzy sets. The only f-weak-contradiction of A w.r.t. B is the $f^{\top}$-weak-contradiction if and only if there exists a sequence $\{x_i\}_{i\in\mathbb{N}} \in \mathcal{U}$ such that $B(x_i) = 1$ for all $x_i \in \mathcal{U}$ and $\lim A(x_i) = 1$.

Proof: Let us assume firstly that there exists a sequence $\{x_i\}_{i\in\mathbb{N}} \in \mathcal{U}$ such that $B(x_i) = 1$ for all $x_i \in \mathcal{U}$ and $\lim A(x_i) = 1$. Then the only mapping $f \in \Omega$ satisfying the inequality $A(x_i) \leq f(B(x_i)) = f(1)$ for all x_i is $f^{\top}$.

On the other hand, let us consider that all sequence $\{x_i\}_{i\in\mathbb{N}} \in \mathcal{U}$ such that $B(x_i) = 1$ for all $x_i \in \mathcal{U}$ implies $\lim A(x_i) \neq 1$. Consider the mapping

$$f(x) = \begin{cases} \sup\{(A(x)\colon x \in \mathcal{U} \text{ and } B(x) = 1\} & \text{if } x = 1 \\ 1 & \text{otherwise} \end{cases}$$

Note that $f \in \Omega$, that $f \neq f^{\top}$ (since $\sup\{(A(x)\colon x \in \mathcal{U}$ and $B(x) = 1\} \neq 1$ straightforwardly by hypothesis) and that A is f-weak-contradictory w.r.t. B (straightforward). ■

Corollary 3: Let A and B be two fuzzy sets defined on a finite universe and such that the only f-weak-contradiction of A w.r.t. B is the $f^{\top}$-weak-contradiction. Then, there exists $x \in \mathcal{U}$ such that $A(x) = B(x) = 1$.

The final result is related to the duality introduced in the motivational items above.

Proposition 5: Let A and B be two fuzzy sets such that A is f-weak-contradictory w.r.t. B. Then B is $\overline{f}$-weak-contradictory w.r.t. A, where $\overline{f}$ is defined by:

$$\overline{f}(z) = \begin{cases} 1 & \text{if } x = 0 \\ \sup\{x \in [0,1] \text{ such that } f(x) \geq z\} & \text{otherwise} \end{cases}$$

Note that if the mapping f used in the result above is bijective, then $\overline{f} = f^{-1}$. Note also, that such result establishes a way to permute the fuzzy sets involved in f-weak-contradiction.

Proof: Obviously, $\overline{f}$ is well-defined and belongs to Ω. Let us show that $B(x) \leq \overline{f}(A(x))$ for all $x \in \mathcal{U}$. If $f(B(x)) = 0$, then as A is f-weak-contradictory w.r.t. B we have that $A(x) \leq f(B(x)) = 0$; thus in that case $A(x) = 0$ as well. Therefore, as $\overline{f}(A(x)) = \overline{f}(0) = 1$, the inequality holds trivially. Let us assume now that $f(B(x)) \neq 0$. Then, by definition of $\overline{f}$, we have that

$$\overline{f}(f(B(x))) = \sup\{x \in [0,1] \text{ such that } f(x) \geq f(B(x))\} \geq B(x).$$

Then, by using also that $A(x) \leq f(B(x))$ we have that:

$$B(x) \leq \overline{f}(f(B(x))) \leq \overline{f}(A(x))$$

■

III. A MEASURE OF CONTRADICTION

In this section we introduce a measure to determine how contradictory two fuzzy sets are. The idea underlying such measure is the more f such that two fuzzy sets A and B are f-weak-contradictory, the more contradiction between A and B. Hence the contradiction between two fuzzy sets A and B can be characterized by a subset of mappings of Ω. That is, by considering the set:

$$\mathcal{F}(A,B) = \big\{ f \in \Omega \mid A \text{ is } f\text{-weak-contradictory w.r.t. } B \big\} \tag{1}$$

So the idea is to measure the contradiction between two fuzzy sets A and B by measuring the set $\mathcal{F}(A,B)$. Note that $\mathcal{F}(A,B)$ is not empty thanks to Proposition 2. In what follows, we show that the set $\mathcal{F}(A,B)$ can be characterized by its minimum element; so measuring $\mathcal{F}(A,B)$ is equivalent to measure such element. Note that we can guarantee the existence of the infimum of $\mathcal{F}(A,B)$ thanks to the complete latice structure of Ω. Hence we only have two prove that, effectively, $\inf(\mathcal{F}(A,B)) \in \mathcal{F}(A,B)$.

Proposition 6: Let A and B be two fuzzy sets and let $\{f_i\}$ be a subset of mappings of Ω. If A is f_i-weak-contradictory w.r.t. B for any f_i, then A is $\inf\{f_i\}$-weak-contradictory w.r.t. B.

Proof: It is straightforward to check that $f = \inf\{f_i\}$ is given by $f(x) = \inf\{f_i(x)\}$ for all $x \in [0,1]$. Moreover, as A is f_i-weak-contradictory w.r.t. B for any f_i then $A(u) \le f_i(B(u))$ for all $u \in \mathcal{U}$. That implies that $A(u) \le \inf\{f_i(B(u)))\} = f(B(u))$ for all $u \in \mathcal{U}$. So A is f-weak-contradictory w.r.t. B. ■

As a straightforward consequence of the above result, we have that the infimum of $\mathcal{F}(A,B)$ is indeed a minimum. In other words, for all fuzzy sets A and B, there exists the least mapping $f \in \Omega$ verifying that A is f-weak-contradictory w.r.t. B. Moreover, thanks to Proposition 2, we can characterize the set $\mathcal{F}(A,B)$ as follows:

Corollary 4: Let A and B be two fuzzy sets, then:

$$\mathcal{F}(A,B) = \{f \in \Omega \text{ such that } f_{A,B} \le f\}$$

where $f_{A,B} = \min\{\mathcal{F}(A,B)\}$

Therefore, as we previously announced, measuring $\mathcal{F}(A,B)$ is equivalent to measure the least mapping $f \in \Omega$ verifying that A is f-weak-contradictory w.r.t. B. And without any doubt, the best way to measure a mapping defined from $[0,1]$ to $[0,1]$ is by definite integrals. So we define the measure of contradiction between two fuzzy sets as follows.

Definition 2: Let A and B be two fuzzy sets, let $f_{A,B}$ be the least element of $\mathcal{F}(A,B)$. *The measure of contradiction between A and B* is defined by:

$$\mathcal{C}(A,B) = 1 - \int_0^1 f_{A,B}(x)dx$$

Some remarks about the "measure" $\mathcal{C}$:

- Note that $\mathcal{C}$ is well-defined for all pair of fuzzy sets A and B, since the mapping $f_{A,B}$ is always antitonic and the Lebesgue's integral of antitonic functions is always well-defined.
- The definite integral appears negated in the formula of $\mathcal{C}(A,B)$ since the lesser the mapping $f_{A,B}$, the more contradictory A w.r.t. B is (i.e the greater $\mathcal{F}(A,B)$ is). On the other hand, the constant 1 is included because $\int_0^1 f^{\top}(x)dx$ (the definite integral corresponding to the mapping which determines the minimal degree of contradiction) is equal to 1. Hence, $\mathcal{C}$ is always positive, $\mathcal{C}(A,B)=0$ means that there is no contradiction between A and B and $\mathcal{C}(A,B) = 1$ means that A is f-weak-contradictory w.r.t. B for all $f \in \Omega$.
- A priori, the measure $\mathcal{C}(A,B)$ is not symmetric. However, below we show that, although the contradiction of A w.r.t. B is not necessarily the same as the contradiction of B w.r.t. A (in terms of the sets of weak-contradictions $\mathcal{F}(A,B)$ and $\mathcal{F}(B,A)$), the measure of contradiction coincides in both cases. In other words, we show that $\mathcal{C}$ is symmetric; i.e. $\mathcal{C}(A,B) = \mathcal{C}(B,A)$ for all fuzzy set A and B.

First of all, let us show that the term "measure of contradiction" makes sense, since $\mathcal{C}$ is antitonic.

Proposition 7: Let A, B, C and D be four fuzzy sets such that $A \le C$ and $B \le D$. Then $\mathcal{C}(A,B) \ge \mathcal{C}(C,D)$.

Proof: By using Proposition 1 we have that:

$$\mathcal{F}(A,B) \subseteq \mathcal{F}(C,D)$$

and by Corollary 4, that is equivalent to $f_{A,B} \le f_{C,D}$. Now the inequality $\mathcal{C}(A,B) \ge \mathcal{C}(C,D)$ is straightforward. ■

To prove the final statement given in the remarks is necessary to provide the algebraic structure of $f_{A,B}$ (i.e. the least element of $\mathcal{F}(A,B)$). Let A and B two fuzzy sets, and let us denote by $Im(A \times B)$ the set of pairs $(A(x), B(x))$ for $x \in \mathcal{U}$.

Lemma 1: Let A and B be two fuzzy sets. The operator $f_{A,B}\colon [0,1] \to [0,1]$ defined by

$$f_{A,B}(x) = \begin{cases} 1 & \text{if } x = 0 \\ \sup\limits_{x \le \gamma}\{\delta \in [0,1] \mid (\gamma,\delta) \in Im(A \times B)\} & \text{otherwise} \end{cases} \tag{2}$$

is the minimum of $\mathcal{F}(A,B)$.

The proof of the lemma above is similar to the proof of the least negation conserving coherence presented in [8]. Hereafter, $f_{A,B}$ denotes the least element of $\mathcal{F}(A,B)$. The structure of $f_{A,B}$ given in Lemma 1 allows us to prove the symmetry of the measure $\mathcal{C}$.

Theorem 1: Let A and B be two fuzzy sets. Then $\mathcal{C}(B,A) = \mathcal{C}(A,B)$.

Proof: Note that $\mathcal{C}(A,B)$ and $\mathcal{C}(B,A)$ determine the areas of the sets:

$$S_{A,B} = \{(x,y) \mid y \le f_{A,B}(x)\}$$

$$S_{B,A} = \{(x,y) \mid y \le f_{B,A}(x)\}$$

respectively. Hence, if we show that $S_{A,B}$ is the reflection of $S_{B,A}$ w.r.t. the line $r \equiv x = y$ then both areas are equal and the proof ends (since reflections do no modify areas). So, we only have to prove the following equality:

$$\begin{aligned} \tau_r(S_{A,B}) = \{(x,y) \mid x \le f_{A,B}(y)\} = \\ = \{(x,y) \mid y \le f_{B,A}(x)\} = S_{B,A} \end{aligned}$$

where $\tau_r(S_{A,B})$ denotes the reflection of $S_{A,B}$ w.r.t. the line r. Let us begin by showing $\tau_r(S_{A,B}) \subseteq S_{B,A}$. Consider $(x,y) \in \tau_r(S_{A,B})$ with $y \ne 0$ (the case $y = 0$ is straightforward), then $x \le \sup_{y \le \gamma}\{\delta \in [0,1] \mid (\gamma,\delta) \in Im(A \times B)\}$. Therefore, there exists $(\gamma,\delta) \in Im(A \times B)$ such that $x \le \delta$ and $y \le \gamma$. Hence, both equalities implies that $y \le \sup_{x \le \delta}\{\gamma \in [0,1] \mid (\gamma,\delta) \in Im(A \times B)\} = f_{B,A}(x)$; as we want to prove. The another inclusion is similar. ■

The measure of contradiction $\mathcal{C}(A,B)$ is a value between 0 and 1.

Proposition 8: Let A and B be two fuzzy sets, then $\mathcal{C}(A,B) \in [0,1]$

Proof: Follows from the fact that $\int_0^1 f dx \in [0,1]$ for all $f \in \Omega$. ■

The following proposition presents other interesting properties of the measure $\mathcal{C}$. Specifically, the result characterizes the extremal cases $\mathcal{C}(A,B) = 0$ and $\mathcal{C}(A,B) = 1$.

Proposition 9: Let A and B be two fuzzy sets. Then:

- $\mathcal{C}(A,B) = 1$ if and only if $B(x) > 0$ implies $A(x) = 0$ for all $x \in \mathcal{U}$.
- $\mathcal{C}(A,B) = 0$ if and only if there exists a sequence $\{x_i\}_{i \in \mathbb{N}} \subseteq \mathcal{U}$ such that $\lim A(x_i) = \lim B(x_i) = 1$.

Proof: Let us begin by proving the first item. It is easy to check that $\mathcal{C}(A,B)=1$ if and only if $\int_0^1 f_{A,B}dx=0$, and the only $f\in\Omega$ such that $\int_0^1 f dx=0$ is $f^{\perp}$. Therefore necessarily $f_{A,B}=f^{\perp}$. On the other hand, by Proposition 3 we have that $f_{A,B}=f^{\perp}$ is equivalent to the statement $B(x)>0$ implies $A(x)=0$ for all $x\in\mathcal{U}$.

Let us continue by proving the second item. Note that $\mathcal{C}(A,B)=0$ if and only if $\int_0^1 f_{A,B}dx=1$, and it is easy to check that the set of $f\in\Omega$ satisfying that $\int_0^1 f dx=1$ is given by $\{f_\alpha\}_{\alpha\in[0,1]}$ where:

$$f_\alpha(x)=\begin{cases}\alpha & \text{if } x=1\\ 1 & \text{otherwise}\end{cases}$$

Note that if $\alpha=1$ then $f_\alpha=f^{\top}$. Let $\{\beta_i\}_{i\in\mathbb{N}}$ be a sequence of elements in $[0,1]$ such that $0\neq\beta_i\neq 1$ for all $i\in\mathbb{N}$ and $\lim\beta_1=1$. Then, by using that $f_{A,B}$ is given by Lemma 1:

$$f_{A,B}(x)=\begin{cases}1 & \text{if } x=0\\ \sup\limits_{x\leq\gamma}\{\delta\in[0,1]\mid(\gamma,\delta)\in Im(A\times B)\} & \text{otherwise}\end{cases}$$

and that $f_{A,B}=f_\alpha$ for some $\alpha\in[0,1]$, we have that:

$$f_{A,B}(\beta_i)=\sup_{\beta_i\leq\gamma}\{\delta\in[0,1]\mid(\gamma,\delta)\in Im(A\times B)\}=1$$

So, for each $\beta_i\neq 0$ there exists a sequence $\{x_j^{\beta_i}\}$ of elements in $\mathcal{U}$ such that $\lim A(x_j^{\beta_i})=1$ and that $B(x_j^{\beta_i})\geq\beta_i$ for all $j\in\mathbb{N}$. Hence, as $\lim\beta_i=1$, we can construct a general sequence $\{x_k\}$ of elements in $\mathcal{U}$ such that $\lim A(x_k)=\lim B(x_k)=1$.

To prove the converse we only have to take into account that if such sequence exists, then necessarily $f_{A,B}=f_\alpha$ for some $\alpha\in[0,1]$. ∎

The proposition above entails some interesting corollaries for the case $\mathcal{C}(A,B)=0$.

Corollary 5: Let A and B be two fuzzy sets. If there exists an element $x\in\mathcal{U}$ such that $A(x)=B(x)=1$ then $\mathcal{C}(A,B)=0$.

The converse of the result above is achieved if the universe considered is finite or if the membership functions are continuous.

Corollary 6: Let A and B be two fuzzy sets defined on a finite universe $\mathcal{U}$. Then: $\mathcal{C}(A,B)=0$ if and only if there exists an element $x\in\mathcal{U}$ such that $A(x)=B(x)=1$.

The following result is interesting when the universe considered is a bounded subset of $\mathbb{R}^n$ and the fuzzy sets are defined by using continuous membership functions.

Corollary 7: Let $\mathcal{U}$ be a bounded set of $\mathbb{R}$ and let A and B be two fuzzy sets defined on $\mathcal{U}$ with continuous membership functions. Then, $\mathcal{C}(A,B)=0$ if and only if there exists an element $x\in\mathcal{U}$ such that $A(x)=B(x)=1$.

Proof: Note that one implication is a direct consequence of Corollary 5. Let us assume that $\mathcal{C}(A,B)=0$. By Proposition 9, we know that there exists a sequence $\{x_i\}_{i\in\mathbb{N}}\subseteq\mathcal{U}$ such that $\lim A(x_i)=\lim B(x_i)=1$. Let us show that there exists an element $x\in\{x_i\}_{i\in\mathbb{N}}$ such that $A(x)=B(x)=1$. As $\mathcal{U}$ is bounded, then $\{x_i\}_{i\in\mathbb{N}}$ is bounded as well. Thus we can ensure there exist a convergent subsequence of $\{x_i\}_{i\in\mathbb{N}}$; let us denote such subsequence by $\{y_i\}_{i\in\mathbb{N}}$. Let us show that $x=\lim y_i$ is the searched element. On the one hand, let us show that $x\in\mathcal{U}$. As $[0,1]$ is a closed set of $\mathbb{R}$ and the membership function of A is continuous, the set $\mathcal{U}=A^{-1}([0,1])$ is closed. Moreover, as $\mathcal{U}$ is bounded by hypothesis, $\mathcal{U}$ is necessarily a compact of $\mathbb{R}$. Therefore, as $\{y_i\}_{i\in\mathbb{N}}\subseteq\mathcal{U}$ then $x=\lim y_i\in\mathcal{U}$. On the other hand, by continuity we have:

$$A(x)=A(\lim y_i)=\lim A(y_i)=1$$

Similarly we can prove that $B(x)=1$. ∎

IV. Conclusion and future work

The main contribution of this paper is the presentation of a measure of contradiction between fuzzy sets considered on the basis of the notion of f-weak-contradiction. The notion of f-weak-contradiciton has been introduced and motivated by considering the mappings f's as degrees. Moreover, results concerning with the notion of f-weak-contradiction has been presented. With respect to the measure of contradiction $\mathcal{C}(A,B)$ we have motivated its definition by the idea of considering the least mapping $f\in\Omega$ verifying that A is f-weak-contradictory w.r.t. B and we have proved four interesting properties, namely: antitonicity; symmetry; and characterizations of the two extremal cases (both given by Proposition 9).

A number of things remain to be done as future work. On the one hand, a comparison between the measure of contradiction defined in this paper and that defined in [2] should be done. On the other hand, more properties for $\mathcal{C}$ must be studied (for instance the continuity) and other measures of contradiction can be defined following the same idea that followed here to define $\mathcal{C}$ (for instance measuring $f_{A,B}$ without using the Lebesgue's integral).

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