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RESEARCH ARTICLE

Fuzzy congruence relations on nd-groupoids

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In this work we introduce the notion of fuzzy congruence relation on an nd-groupoid and study conditions on the nd-groupoid which guarantee a complete lattice structure on the set of fuzzy congruence relations. The study of these conditions allowed to construct a counterexample to the statement that the set of fuzzy congruences on a hypergroupoid is a complete lattice.

Keywords: fuzzy congruence relation, nd-groupoid, hypergroupoid

MSC codes: 04A72, 06B10, 20N02, 26E25

1. Introduction

The systematic generalization of crisp concepts to the fuzzy case has proven to be an important theoretical tool for the development of new methods of reasoning under uncertainty, imprecision and lack of information.

Regarding the generalization level, it is important to note that the definition of fuzzy sets originally presented as mappings with codomain $[0, 1]$, was soon replaced by more general structures, for instance a complete lattice, as in the L -fuzzy sets introduced by Goguen [8].

This paper continues previous work [4, 5] which is aimed at investigating L -fuzzy sets where L has the structure of a multilattice, a structure introduced in [2] and later recovered for use in other contexts, both theoretical and applied [10, 13].

Roughly speaking, a multilattice is an algebraic structure in which the restrictions imposed on a lattice, namely, the “existence of least elements in the sets of upper bounds and greatest elements in the sets of lower bounds” are relaxed to the “existence of minimals and maximals, respectively, in the corresponding sets of bounds”. Attending to this informal description, the main difference that one notices when working with multilattices is that the operators which compute suprema and infima are no longer single-valued, since there may be several multi-suprema or multi-infima, or may be none, see Figure 1. This immediately leads to the theory of hyperstructures, that is, algebras whose operations are set-valued.

If A is a non-empty set and H is a family of set-valued operations on A , the ordered pair (A, H) is called a hyperalgebra (or multialgebra, or polyalgebra). The study of hyperalgebras was originated in 1934 when Marty introduced the so-called hypergroups in [12]. Since then, a number of papers have been published on this topic, focussing essentially on special types of hyperalgebras (such as hypergroups, hyperrings, hyperfields, vector hyperspaces, boolean hyperalgebras, ...)

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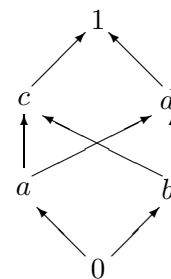


Figure 1. A complete multilattice which is not a lattice.

and guided, sometimes by purely theoretical motivations and sometimes because of their applications in other areas.

In this paper, we will focus on the most general hyperstructures, namely hypergroupoids and nd-groupoids. Our interest in these structures arises from the fact that, in a multilattice, the operators which compute the multi-suprema and multi-infima provide precisely the structure of nd-groupoids or, if we have for granted that at least a multi-supremum always exists, a hypergroupoid. Actually, some of the results will be stated just in terms of multiseuilattices.

Several papers have investigated the structure of the set of fuzzy congruences on different algebraic structures [1, 6, 7, 15, 17]; and in [4, 5] we initiated our research in this direction. Specifically, we focused on the theory of (crisp) congruences on a multilattice and on an nd-groupoid, as a necessary step before studying the fuzzy congruences on multilattices and the multilattice-based generalization of the concept of L -fuzzy congruence. In this paper, we study the notion of fuzzy congruence relation on nd-groupoids.

The fact that the structure of nd-groupoid is simpler than that of a multilattice does not necessarily mean that the theory is simpler as well. We will show that, in general, the set of fuzzy congruences on an nd-groupoid is not a lattice unless we assume some extra properties.

The structure of the paper is as follows: in Section 2, the preliminary definitions and concepts related to nd-groupoids and fuzzy congruences are introduced; then, in Section 3, based on the definition of fuzzy congruence on a hypergroupoid given in [1], we introduce our generalization to the context of nd-groupoids, and prove that a fuzzy relation ρ is a fuzzy congruence relation on an nd-groupoid A if and only if its power extension $\widehat{\rho}$ is a fuzzy congruence relation on the powerset groupoid 2^A . Section 4 is devoted to the study of the lattice structure of fuzzy congruence relations, the two main results being that, (1) contrariwise to what is stated in [1], Theorem 3.14, the set of fuzzy congruences on A , $FCon(A)$, is not always a lattice, and (2) sufficient conditions to prove that $FCon(A)$ is a complete lattice. In the final section, we draw some conclusions and present prospects of future work.

2. Preliminaries

We can find in the literature the definition of a hypergroupoid as a non-empty set endowed with a hyperoperation $* : A \times A \rightarrow 2^A \setminus \{\emptyset\}$. However, we are interested in a generalization of hypergroupoid that we will call non-deterministic groupoid (nd-groupoid, for short) which also considers the empty set as possible image of the hyperoperation.

DEFINITION 2.1 An nd-groupoid $(A, *)$ is defined by an nd-operation $* : A \times A \rightarrow 2^A$ on a non-empty set A . The induced power groupoid is defined as $(2^A, *)$ where

1 the operation is given by $X * Y = \{x * y \mid x \in X, y \in Y\}$ for all $X, Y \subseteq A$.

2
 3 Notice that the definition allows the assignment of the empty set to a pair of
 4 elements, that is $a * b = \emptyset$. This mere fact, albeit simple, represents an important
 5 difference with hypergroupoids, as it will be explained later.

6 The following notational conventions will be used hereafter:

- 7
 8 • Multiplicative notation; thus, the symbol of the nd-operation will be omitted.
 9 • If $a \in A$ and $X \subseteq A$, we will denote $aX = \{ax \mid x \in X\}$ and $XA = \{xa \mid x \in X\}$.
 10 In particular, $a\emptyset = \emptyset a = \emptyset$.
 11 • When the result of the nd-operation is a singleton, we will often omit the braces.

12
 13 As stated in the introduction, our interest in extending the concept of hyper-
 14 groupoid is justified by the algebraic characterization of multilattices and multi-
 15 semilattices, since the operators for multi-suprema and multi-infima are both
 16 examples of nd-operators.

17 With this idea in mind, we introduce below the extension to the framework of
 18 nd-groupoids of some well-known properties. Assume that (A, \cdot) is an nd-groupoid:

- 19
 20 • **Idempotency:** $aa = a$ for all $a \in A$.
 21 • **Commutativity:** $ab = ba$ for all $a, b \in A$.
 22 • **Left m-associativity:** if $ab = b$, then $(ab)c \subseteq a(bc)$, for all $a, b, c \in A$.
 23 • **Right m-associativity:** if $bc = c$, then $a(bc) \subseteq (ab)c$, for all $a, b, c \in A$.
 24 • **m-associativity:** if it is left and right m-associative.

25
 26 Note that the prefix ‘m-’ has its origin in the concept of multilattice.

27 We will focus our interest on the binary relation usually named *natural ordering*,
 28 which is defined by

29
 30
 31
$$a \leq b \text{ if and only if } ab = b$$

32
 33 Although, in general, this relation is not an ordering, the properties above guar-
 34 antee that the relation just defined is an ordering. Specifically, it is reflexive if
 35 the nd-groupoid is idempotent, the relation is antisymmetric if the nd-groupoid is
 36 commutative and, finally, it is transitive if the nd-groupoid is m-associative.

37 The two following properties of nd-groupoids, named **comparability** properties,
 38 have an important role in multilattice theory:

- 39
 40 • C_1 : $c \in ab$ implies that $a \leq c$ and $b \leq c$.
 41 • C_2 : $c, d \in ab$ and $c \leq d$ imply that $c = d$.

42
 43 Similarly to lattice theory, we can define algebraically the concept of multise-
 44 milattice as an nd-groupoid that satisfies idempotency, commutativity, m-associativity
 45 and comparability laws. The ordered and the algebraic definitions of multise-
 46 milattice can be proved to be equivalent simply by considering $a \cdot b = \text{multisup}\{a, b\}$
 47 and \leq being the natural ordering (see [11], Theorem 2.11).

48 Since our aim is to extend the results about fuzzy congruences to nd-groupoids
 49 and multise-
 50 milattices, let us recall some notions about the concepts that we will
 51 generalize.

52
 53 **DEFINITION 2.2** (Zadeh, [18]) *Let A be a non-empty set. A fuzzy relation ρ on A*
 54 *is a fuzzy subset of $A \times A$ (i.e. ρ is a function from $A \times A$ to $[0, 1]$). ρ is reflexive*
 55 *in A if $\rho(x, x) = 1$ for all $x \in A$, ρ is symmetric in A if $\rho(x, y) = \rho(y, x)$ for all*
 56 *$x, y \in A$, finally, ρ is transitive if*

57
 58
$$\sup_{z \in A} \min \{ \rho(x, z), \rho(z, y) \} \leq \rho(x, y) \text{ for all } x, y \in A$$

1 A fuzzy equivalence relation is a reflexive, symmetric and transitive fuzzy relation.

2 Since a fuzzy relation in a non-empty set A is a fuzzy subset of $A \times A$, we can
3 define the inclusion, union and intersection of fuzzy relations as follows:

- 4
- 5 • $\rho \subseteq \sigma$ if $\rho(x, y) \leq \sigma(x, y)$ for all $x, y \in A$,
 - 6 • $\bigcup_{i \in \Lambda} \rho_i(x, y) = \sup_{i \in \Lambda} \rho_i(x, y)$ for all $x, y \in A$ and
 - 7 • $\bigcap_{i \in \Lambda} \rho_i(x, y) = \inf_{i \in \Lambda} \rho_i(x, y)$ for all $x, y \in A$.
- 8
9
10
11

12 DEFINITION 2.3 (Zadeh, [18]) Let A be a non-empty set and ρ and σ two fuzzy
13 relations in A . Then we define the sup-min composition of ρ and σ as:

$$14 (\rho \circ \sigma)(x, y) = \sup_{z \in A} \min \{ \rho(x, z), \sigma(z, y) \} \text{ for all } x, y \in A$$

15
16 It is easy to prove that the sup-min composition of two fuzzy relations is asso-
17 ciative. Moreover, a fuzzy relation ρ is transitive on A if $\rho \circ \rho \subseteq \rho$.

18 Let $FEq(A)$ be the set of fuzzy equivalence relations on a non-empty set A .
19 Murali proved in [14] that $(FEq(A), \subseteq)$ is a complete lattice where the meet is
20 the intersection and the join is the transitive closure of the union.

21 Finally, let us introduce the definition of fuzzy congruence on a groupoid.

22 DEFINITION 2.4 Let ρ be a fuzzy relation on a groupoid (G, \cdot) ; we say that ρ is right
23 compatible with the operation if $\rho(ac, bc) \geq \rho(a, b)$ for all $a, b, c \in G$; similarly,
24 ρ is said to be left compatible if $\rho(ca, cb) \geq \rho(a, b)$ for all $a, b, c \in G$. A fuzzy
25 congruence on G is a left and right compatible fuzzy equivalence relation.

3. Fuzzy congruence relations on nd-groupoids

26 Regarding the extension of the definition of fuzzy congruence to the non-
27 deterministic case, the following definition of compatibility, in the case of an un-
28 derlying hypergroupoid, was introduced by Bakhshi and Borzooei in [1].

29 DEFINITION 3.1 Let (A, \cdot) be an nd-groupoid. Then a fuzzy relation ρ on A is said
30 to be right (left) compatible if for all $x \in ac$ ($x \in ca$) there exists $y \in bc$ ($y \in cb$)
31 and for all $y \in bc$ ($y \in cb$) there exists $x \in ac$ ($x \in ca$) such that $\rho(x, y) \geq \rho(a, b)$,
32 for all $a, b, c \in A$ and compatible if it is both fuzzy right and left compatible.

33 This definition explicitly uses the fact that the images of the hyperoperator
34 are non-empty. Thus, we propose an alternative definition which generalizes the
35 previous one and adequately handles the empty images.

36 As a previous step to the consideration of fuzzy congruence relations on an nd-
37 groupoid, let us note that it is possible to extend any fuzzy relation on a set A
38 to its powerset 2^A ; this construction leads to the definition of an operator $\hat{\rho}$ from
39 the set $FR(A)$ of fuzzy relations on A to the set $FR(2^A)$ of fuzzy relations on 2^A .
40 Namely, given a fuzzy relation $\rho : A \times A \rightarrow [0, 1]$, its power extension is a fuzzy
41 relation $\hat{\rho} : 2^A \times 2^A \rightarrow [0, 1]$ defined by

$$42 \hat{\rho}(X, Y) = \left(\bigwedge_{x \in X} \bigvee_{y \in Y} \rho(x, y) \right) \wedge \left(\bigwedge_{y \in Y} \bigvee_{x \in X} \rho(x, y) \right)$$

43 where \bigvee and \bigwedge denotes the supremum and the infimum in the unit interval, re-
44 spectively.

1 Notice that $\widehat{\rho}(\emptyset, X) = \widehat{\rho}(X, \emptyset) = 0$, for all non-empty $X \subseteq A$, $\widehat{\rho}(\emptyset, \emptyset) = 1$ and
 2 $\widehat{\rho}(\{a\}, \{b\}) = \rho(a, b)$, for all $a, b \in A$.

3 With this power extension of a fuzzy relation, the definition of fuzzy congruence
 4 relation on an nd-groupoid (A, \cdot) follows exactly the one for the deterministic case:
 5 a fuzzy equivalence relation that satisfies
 6

$$7 \quad \widehat{\rho}(ac, bc) \geq \rho(a, b) \text{ and } \widehat{\rho}(ca, cb) \geq \rho(a, b), \text{ for all } a, b, c \in A \quad (1)$$

8 It is easy to check that a fuzzy relation that is compatible with \cdot (in the sense
 9 of Definition 3.1) satisfies Condition (1) but, in general, both concepts are not
 10 equivalent as the following example shows:

11 *Example 3.2* Let $A = [0, 1]$ be the hypergroupoid endowed with the hyperoperation
 12 $a * b := (0, 1)$ and consider the fuzzy relation $\rho(a, b) = 1 - ab$. Observe that

$$13 \quad \begin{aligned} 14 \quad \widehat{\rho}(a * c, b * c) &= \left(\bigwedge_{x \in (0,1)} \bigvee_{y \in (0,1)} (1 - xy) \right) \wedge \left(\bigwedge_{y \in (0,1)} \bigvee_{x \in (0,1)} (1 - xy) \right) = \\ 15 \quad &= \left(\bigwedge_{x \in (0,1)} 1 \right) \wedge \left(\bigwedge_{y \in (0,1)} 1 \right) = 1 \geq \rho(a, b) \end{aligned}$$

16 for all $a, b, c \in A$. However, for all $x \in 0 * c$ and $y \in b * c$, we have $\rho(x, y) <$
 17 $\rho(0, b) = 1$ because otherwise, we would have either $x = 0$ or $y = 0$ contradicting
 18 that $x, y \in (0, 1)$. Thus, ρ is not compatible with the hyperoperation $*$. \square

19 Once we have introduced the power extension of a fuzzy relation, in order to use
 20 the above condition to define the concept of fuzzy congruence relation, we study
 21 the behaviour of the operator $\widehat{\rho}$ wrt the properties of reflexivity, simmetry and
 22 transitivity.

23 **THEOREM 3.3** *Let ρ be a fuzzy relation in a non-empty set A and let $\widehat{\rho}$ be its power*
 24 *extension as defined above. ρ is a fuzzy equivalence relation if and only if so is $\widehat{\rho}$.*

25 *Proof* For the cases of reflexivity and antisymmetry it is just a matter of routine
 26 calculation. Let us concentrate on transitivity.

27 Under the assumption that ρ is transitive on A , it is sufficient to prove that, for
 28 all $X, Y, Z, \subseteq A$, we have that $\widehat{\rho}(X, Y) \wedge \widehat{\rho}(Y, Z) \leq \widehat{\rho}(X, Z)$, in order to ensure that
 29 $\widehat{\rho}$ is transitive.

$$30 \quad \begin{aligned} 31 \quad \widehat{\rho}(X, Y) \wedge \widehat{\rho}(Y, Z) &= \\ 32 \quad &= \bigwedge_{x \in X} \bigvee_{y \in Y} \rho(x, y) \wedge \bigwedge_{y \in Y} \bigvee_{x \in X} \rho(x, y) \wedge \bigwedge_{y \in Y} \bigvee_{z \in Z} \rho(y, z) \wedge \bigwedge_{z \in Z} \bigvee_{y \in Y} \rho(y, z) \\ 33 \quad &= \left[\bigwedge_{x \in X} \bigvee_{y \in Y} \rho(x, y) \wedge \bigwedge_{y \in Y} \bigvee_{z \in Z} \rho(y, z) \right] \wedge \left[\bigwedge_{z \in Z} \bigvee_{y \in Y} \rho(y, z) \wedge \bigwedge_{y \in Y} \bigvee_{x \in X} \rho(x, y) \right] \end{aligned}$$

34 Now, by idempotency and distributivity, we have that $\widehat{\rho}(X, Y) \wedge \widehat{\rho}(Y, Z)$ equals

$$35 \quad \left[\bigwedge_{x \in X} \bigvee_{y \in Y} \left(\rho(x, y) \wedge \bigwedge_{y' \in Y} \bigvee_{z \in Z} \rho(y', z) \right) \right] \wedge \left[\bigwedge_{z \in Z} \bigvee_{y \in Y} \left(\rho(y, z) \wedge \bigwedge_{y' \in Y} \bigvee_{x \in X} \rho(x, y') \right) \right]$$

As $\bigwedge_{y' \in Y} \bigvee_{z \in Z} \rho(y', z) \leq \bigvee_{z \in Z} \rho(y, z)$, for all $y \in Y$, we have that

$$\begin{aligned} \widehat{\rho}(X, Y) \wedge \widehat{\rho}(Y, Z) &\leq \\ &\leq \left[\bigwedge_{x \in X} \bigvee_{y \in Y} \left(\rho(x, y) \wedge \bigvee_{z \in Z} \rho(y, z) \right) \right] \wedge \left[\bigwedge_{z \in Z} \bigvee_{y \in Y} \left(\rho(y, z) \wedge \bigvee_{x \in X} \rho(x, y) \right) \right] \\ &= \left[\bigwedge_{x \in X} \bigvee_{y \in Y} \bigvee_{z \in Z} \left(\rho(x, y) \wedge \rho(y, z) \right) \right] \wedge \left[\bigwedge_{z \in Z} \bigvee_{y \in Y} \bigvee_{x \in X} \left(\rho(y, z) \wedge \rho(x, y) \right) \right] \\ &= \left[\bigwedge_{x \in X} \bigvee_{z \in Z} \bigvee_{y \in Y} \left(\rho(x, y) \wedge \rho(y, z) \right) \right] \wedge \left[\bigwedge_{z \in Z} \bigvee_{x \in X} \bigvee_{y \in Y} \left(\rho(x, y) \wedge \rho(y, z) \right) \right] \end{aligned}$$

Since ρ is transitive, $\rho(x, z) \geq \bigvee_{y \in Y} (\rho(x, y) \wedge \rho(y, z))$, and so

$$\widehat{\rho}(X, Y) \wedge \widehat{\rho}(Y, Z) \leq \left[\bigwedge_{x \in X} \bigvee_{z \in Z} \rho(x, z) \right] \wedge \left[\bigwedge_{z \in Z} \bigvee_{x \in X} \rho(x, z) \right] = \widehat{\rho}(X, Z)$$

and as this is true for all $Y \subseteq A$ (including $Y = \emptyset$), we have that

$$\bigvee_{Y \in 2^A} (\widehat{\rho}(X, Y) \wedge \widehat{\rho}(Y, Z)) \leq \widehat{\rho}(X, Z)$$

and so $\widehat{\rho}$ is transitive.

Conversely, if $\widehat{\rho}$ is transitive, then

$$\begin{aligned} \rho(a, b) = \widehat{\rho}(\{a\}, \{b\}) &\geq \bigvee_{Z \in 2^A} (\widehat{\rho}(\{a\}, Z) \wedge \widehat{\rho}(Z, \{b\})) \\ &\geq \bigvee_{\{z\} \in 2^A} (\widehat{\rho}(\{a\}, \{z\}) \wedge \widehat{\rho}(\{z\}, \{b\})) = \bigvee_{z \in A} (\rho(a, z) \wedge \rho(z, b)) \end{aligned}$$

■

Summarizing the previous considerations we can state the following definition and theorem.

DEFINITION 3.4 A fuzzy equivalence relation ρ on an nd-groupoid (A, \cdot) is said to be a right (resp. left) congruence relation if $\widehat{\rho}(ac, bc) \geq \rho(a, b)$ (resp. $\widehat{\rho}(ca, cb) \geq \rho(a, b)$) for all $a, b, c \in A$. A fuzzy relation is said to be a congruence relation if it is a left and right congruence relation.

Notice that, henceforth, in order to avoid repetitions, we will only concentrate on the right versions of properties.

THEOREM 3.5 Let ρ be a fuzzy relation on an nd-groupoid (A, \cdot) . Then, ρ is a fuzzy congruence relation if and only if $\widehat{\rho}$ is a fuzzy congruence relation in the induced power groupoid $(2^A, \cdot)$.

Proof By Theorem 3.3, we only need to prove the compatibility with the nd-operation. If $\widehat{\rho}$ is a congruence in $(2^A, \cdot)$ then, for all $a, b, c \in A$,

$$\widehat{\rho}(ac, bc) = \widehat{\rho}(\{a\}\{c\}, \{b\}\{c\}) \geq \widehat{\rho}(\{a\}, \{b\}) = \rho(a, b)$$

Conversely, for all $X, Y, Z \in 2^A$,

$$\begin{aligned} \widehat{\rho}(XZ, YZ) &= \left(\bigwedge_{xz \in XZ} \bigvee_{yz' \in YZ} \rho(xz, yz') \right) \wedge \left(\bigwedge_{yz' \in YZ} \bigvee_{xz \in XZ} \rho(xz, yz') \right) \\ &\geq \left(\bigwedge_{xz \in XZ} \bigvee_{y \in Y} \rho(xz, yz) \right) \wedge \left(\bigwedge_{yz \in YZ} \bigvee_{x \in X} \rho(xz, yz) \right) \\ &\geq \left(\bigwedge_{x \in X} \bigvee_{y \in Y} \rho(x, y) \right) \wedge \left(\bigwedge_{y \in Y} \bigvee_{x \in X} \rho(x, y) \right) = \widehat{\rho}(X, Y) \end{aligned}$$

■

The sup property, introduced in Definition 3.6 below, guarantees the equivalence between our definition of fuzzy congruence relation and the one given in [1].

DEFINITION 3.6 *Let A be a non-empty set and ρ a fuzzy relation on A . We say that ρ satisfies the right (resp. left) sup property if for all $a \in A$ and for all non-empty $X \subseteq A$, there exists $y_0 \in X$ (resp. $x_0 \in X$) such that $\sup_{y \in X} \rho(a, y) = \rho(a, y_0)$ (resp. $\sup_{x \in X} \rho(x, a) = \rho(x_0, a)$).*

LEMMA 3.7 *Let ρ be a fuzzy equivalence relation on an nd -groupoid (A, \cdot) which satisfies sup property. Then, ρ is a fuzzy congruence relation if and only if ρ is compatible with the nd -operation.*

Proof Let us suppose that ρ is compatible. Let $x \in ac$, then there exists $y \in bc$ such that $\rho(x, y) \geq \rho(a, b)$. So $\bigvee_{y \in bc} \rho(x, y) \geq \rho(a, b)$ and $\bigwedge_{x \in ac} \bigvee_{y \in bc} \rho(x, y) \geq \rho(a, b)$.

Analogously $\bigwedge_{y \in bc} \bigvee_{x \in ac} \rho(x, y) \geq \rho(a, b)$ therefore

$$\widehat{\rho}(ac, bc) \geq \rho(a, b)$$

Notice that the sup property is not required in this implication.

For the converse, we only check the first condition of Definition 3.1 because the other ones follows the same scheme. If ρ is a fuzzy congruence relation, then $\widehat{\rho}(ac, bc) \geq \rho(a, b)$. In particular $\bigwedge_{x \in ac} \bigvee_{y \in bc} \rho(x, y) \geq \rho(a, b)$. By the sup property, for

all $x \in ac$ there exists $y_0 \in bc$ such that $\bigvee_{y \in bc} \rho(x, y) = \rho(x, y_0)$. Since $\bigwedge_{x \in ac} \rho(x, y_0) \leq \rho(x, y_0)$, we obtain $\rho(a, b) \leq \rho(x, y_0)$

■

4. On the lattice structure of fuzzy congruence relations

In the previous section, we introduced the map $\widehat{\cdot}: FR(A) \rightarrow FR(2^A)$ and proved that $\rho \in FR(A)$ is a fuzzy equivalence relation if and only if $\widehat{\rho}$ is a fuzzy equivalence relation. Let us now consider this map on $FCon(A)$, the subset of $FEq(A)$ consisting of the fuzzy congruence relations. First, notice that Theorem 3.5 guarantees that $\widehat{\cdot}: FCon(A) \rightarrow FCon(2^A)$ is well defined.

In the crisp case, Murali proved in [15] that the set of fuzzy congruence relations on a groupoid X is a complete sublattice of the set of all fuzzy equivalence relations. This result might suggest that the lattice structure of $FCon(2^A)$ can be projected on $FCon(A)$, via the map $\widehat{\cdot}$. However, although $\widehat{\rho}$ is injective, since $\widehat{\rho}(\{a\}, \{b\}) =$

$\rho(a, b)$, for all $a, b \in A$, it is not surjective. If it were surjective, then for all $\Theta \in FCon(2^A)$ the following equality would hold

$$\Theta(X, Y) = \left(\bigwedge_{x \in X} \bigvee_{y \in Y} \Theta(\{x\}, \{y\}) \right) \wedge \left(\bigwedge_{y \in Y} \bigvee_{x \in X} \Theta(\{x\}, \{y\}) \right)$$

but, in general, this is not the case, as the following example shows.

Example 4.1 Let (A, \cdot) be the nd-groupoid with $A = \{a, b\}$ and $x \cdot y = \{a\}$, for all $x, y \in A$. Let Θ be the reflexive and symmetric fuzzy relation on 2^A given by $\Theta(\{a\}, \{b\}) = 1$; $\Theta(\{a\}, A) = \Theta(\{b\}, A) = 1/2$ and $\Theta(\emptyset, \{a\}) = \Theta(\emptyset, \{b\}) = \Theta(\emptyset, A) = 0$. It is routine calculation to check that Θ is transitive and, in fact, is a congruence relation, however

$$\begin{aligned} \left(\bigwedge_{a \in \{a\}} \bigvee_{y \in A} \Theta(\{a\}, \{y\}) \right) \wedge \left(\bigwedge_{y \in A} \bigvee_{a \in \{a\}} \Theta(\{a\}, \{y\}) \right) = \\ \left(\bigvee_{y \in A} \Theta(\{a\}, \{y\}) \right) \wedge \left(\bigwedge_{y \in A} \Theta(\{a\}, \{y\}) \right) = \bigwedge_{y \in A} \Theta(\{a\}, \{y\}) = 1 \end{aligned}$$

but $\Theta(\{a\}, A) = \frac{1}{2} \neq 1$. □

Under the additional assumption of commutativity with respect to the usual composition of binary relations, Bakhshi and Borzooei [1], stated that the set of all fuzzy congruence relations on a hypergroupoid (H, \cdot) is a complete lattice. The following example proves that this result is not true even in the crisp case and, thus, it can not be true in a fuzzy framework either.

Example 4.2 Let H be the set $\{a, b, c, u_0, u_1, v_0, v_1\}$ provided with a commutative hyperoperation $*$ which is defined as follows:

$a*a = a*b = b*b = \{a, b\}$; $a*c = \{u_0, u_1\}$; $b*c = \{v_0, v_1\}$ and $x*y = \{c\}$, elsewhere

Consider $R, S : H \times H \rightarrow \{0, 1\}$ two binary relations, where R is the least equivalence relation containing $\{(a, b), (u_0, v_0), (u_1, v_1)\}$ and S the least equivalence relation containing $\{(a, b), (u_0, v_1), (u_1, v_0)\}$.

It is not difficult to check that R and S commute; moreover, easy but tedious calculations show that R and S are compatible with the hyperoperation $*$ (i.e. they are congruence relations). However, the only candidate for the meet of R and S , that is, the intersection $R \cap S$, is not a congruence relation because $a(R \cap S)b$ and for $u_0 \in a * c$ there is no element $x \in b * c$ such that $u_0(R \cap S)x$.

For the benefit of the reader, a pictorial representation of all the relations involved in this example are shown in Figure 2. □

An obvious consequence of the previous counterexample is the convenience of investigating conditions on the nd-groupoid (or hypergroupoid) that guarantee the lattice structure on $FCon(A)$.

The following result is an immediate consequence from the definition of fuzzy congruence relation.

LEMMA 4.3 *Let ρ be a fuzzy congruence relation in an idempotent nd-groupoid (A, \cdot) . If $\rho(a, b) > 0$ then $ab \neq \emptyset$. Moreover, $\rho(a, b) = \rho(a, c) \wedge \rho(c, b)$ for all $c \in ab$.*

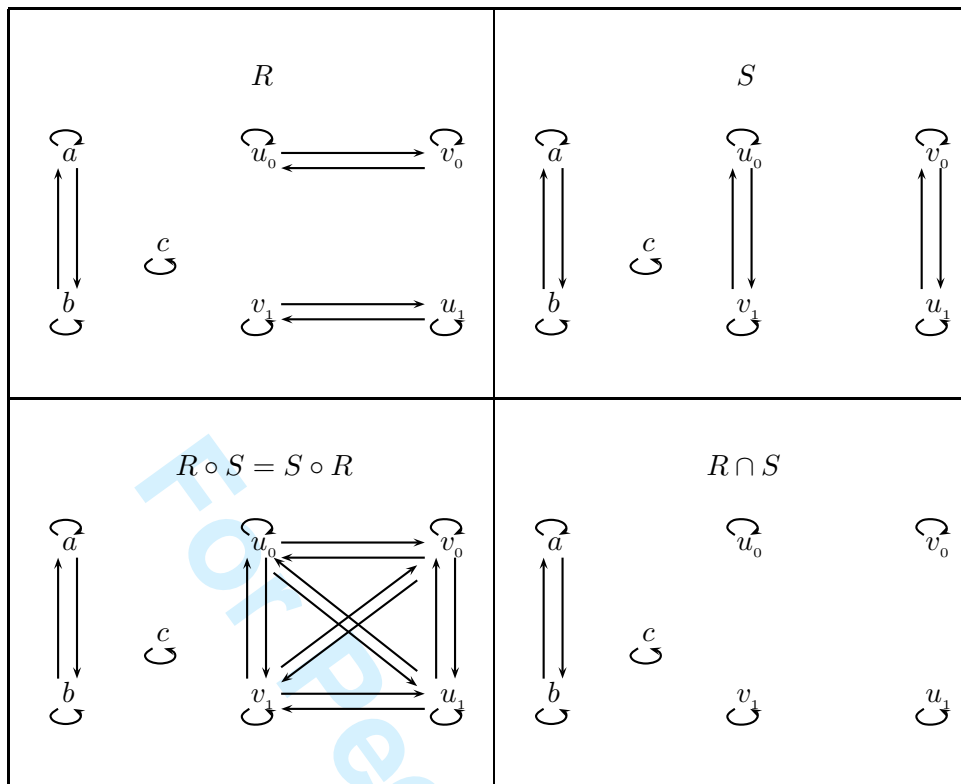


Figure 2.

Proof Given $a, b \in A$ such that $\rho(a, b) > 0$, we have that $ab \neq \emptyset$ because

$$0 < \rho(a, b) \leq \widehat{\rho}(aa, ab) = \widehat{\rho}(a, ab)$$

Let us now consider $c \in ab$. By transitivity $\rho(a, b) \geq \rho(a, c) \wedge \rho(c, b)$. On the other hand, as $\widehat{\rho}(a, ab) \geq \rho(a, b)$, we have in particular $\rho(a, c) \geq \rho(a, b)$; analogously $\widehat{\rho}(ab, b) \geq \rho(a, b)$ which implies that $\rho(c, b) \geq \rho(a, b)$. Thus

$$\rho(a, c) \wedge \rho(c, b) \geq \rho(a, b)$$

As a result, $\rho(a, b) = \rho(a, c) \wedge \rho(c, b)$. ■

THEOREM 4.4 *Let (A, \cdot) be an nd -groupoid satisfying idempotency and property C_1 , and let ρ be a fuzzy equivalence relation. Then ρ is a congruence relation if and only if the following condition holds:*

$$\text{For all } a, b, c \in A \text{ with } a \leq b \text{ we have that } \widehat{\rho}(ac, bc) \geq \rho(a, b) \quad (2)$$

Proof The necessity is obvious, thus we will just prove the sufficiency.

If $\rho(a, b) > 0$ then, by Lemma 4.3, there exists $z \in ab$ such that $\rho(a, b) = \rho(a, z) \wedge \rho(z, b)$. Property C_1 ensures that $a \leq z$ and $b \leq z$ and then, by Condition (2) and symmetry, $\widehat{\rho}(ac, zc) \geq \rho(a, z)$ and $\widehat{\rho}(zc, bc) \geq \rho(z, b)$. Now, by transitivity,

$$\widehat{\rho}(ac, bc) \geq \bigvee_{X \in 2^A} (\widehat{\rho}(ac, X) \wedge \widehat{\rho}(X, bc)) \geq \widehat{\rho}(ac, zc) \wedge \widehat{\rho}(zc, bc) \geq \rho(a, z) \wedge \rho(z, b) = \rho(a, b)$$

■

From now on, we focus on the search of properties that ensure Condition (2) of the previous theorem.

PROPOSITION 4.5 *Let (A, \cdot) be a commutative, m-associative nd-groupoid satisfying both comparability properties, ρ be a fuzzy congruence relation and $a, b, c \in A$. If $a \leq b$, $w \in ac$ and $z \in bc$ with $w \leq z$ then $\rho(w, z) \geq \rho(a, b)$.*

Proof Consider $w \in ac$ such that $wz = z$ (i.e., $w \leq z$). By C_1 , we have that $a \leq w$ ($w = aw$) and, by ρ being a fuzzy congruence relation

$$\rho(a, b) \leq \widehat{\rho}(aw, bw) \leq \bigwedge_{y \in bw} \bigvee_{x \in aw} \rho(x, y) = \bigwedge_{y \in bw} \rho(w, y)$$

As a result, it is sufficient to prove that $z \in bw$.

By using $b \leq z$ (which holds by C_1 and the general hypothesis $z \in bc$) and $w \leq z$, and m-associativity, we can write $z = bz = b(wz) \subseteq (bw)z$, that is, $z \in (bw)z$; therefore, there exists $z' \in bw$ such that $z = z'z$, that is, $z' \leq z$.

A similar application of m-associativity, based on the inequalities $b \leq z'$ and $c \leq w \leq z'$ (which follow from C_1 and transitivity), shows the existence of $z'' \in bc$ satisfying $z'' \leq z'$ and therefore $z'' \leq z$. Now, recalling that z also belongs to bc by general hypothesis, the comparability property C_2 leads to $z'' = z$. As a result of this equality, we have that $z \leq z'$ and $z' \leq z$. By commutativity, the relation \leq is antisymmetric and, hence, $z = z' \in bw$. ■

Recall that the general idea is that, given $a \leq b$, to prove that $\widehat{\rho}(ac, bc) \geq \rho(a, b)$. The previous proposition ensures the inequality $\rho(w, z) \geq \rho(a, b)$ for elements $w \in ac$ and $z \in bc$ such that $w \leq z$. Now, in order to obtain the inequality for $\widehat{\rho}$, one has to start from $z \in bc$ and show the existence of the suitable $w \in ac$, and vice versa.

PROPOSITION 4.6 *Let (A, \cdot) be an m-associative nd-groupoid that satisfies C_1 and, for $a, b, c \in A$, consider $a \leq b$ and $z \in bc$. Then there exists $w \in ac$ such that $w \leq z$.*

Proof By hypothesis $a \leq b$ and, by C_1 , since $z \in bc$, we obtain $b \leq z$. Now, as the nd-operation \cdot is m-associative, the relation \leq is transitive and, therefore, $a \leq z$.

Applying C_1 again on $z \in bc$ leads to $c \leq z$; by m-associativity, $z = az = a(cz) \subseteq (ac)z$. In particular, we have that $z \in (ac)z$ and this implies the existence of $w \in ac$ such that $z = wz$, that is, $w \leq z$. ■

Next, we concentrate on the converse, that is, beginning with an element in ac , find suitable elements in bc so that the congruence holds. This is based on the property of m-distributivity introduced below:

DEFINITION 4.7 *An nd-operation in a set A is said to be **m-distributive** when, for all $a, b, c \in A$, if $a \leq b$ and $w \in ac$, then $bw \cap bc \neq \emptyset$.*

It is convenient to remark that m-distributivity arose in the context of multilatitices [4], although we will not work with this algebraic structure in this paper.

PROPOSITION 4.8 *Let (A, \cdot) be an m-distributive nd-groupoid that satisfies C_1 and consider $a, b, c \in A$. If $a \leq b$ and $w \in ac$ then there exists $z \in bc$ such that $w \leq z$.*

Proof By m-distributivity, from $a \leq b$ and $w \in ac$, we obtain that there exists $z \in bw \cap bc$. Now, $w \leq z$ holds by C_1 . ■

Now, we have all the required properties and propositions needed in order to face the main goal of this paper, namely, to prove that under certain circumstances the

set of congruences on an nd-groupoid is a complete lattice.

Our proof of the complete lattice structure of the set of fuzzy congruences on an nd-groupoid is based on Theorem 4.4 and propositions 4.5, 4.6 and 4.8. If we summarize all the required hypotheses, we have that the nd-groupoid has to be an m-distributive multisemilattice.

THEOREM 4.9 *The set of the fuzzy congruence relations, $FCon(M)$, in an m-distributive multisemilattice M , is a sublattice of $FEq(M)$ and, moreover is a complete lattice wrt the fuzzy inclusion ordering.*

Proof Let $\{\rho_i\}_{i \in \Lambda}$ be a set of fuzzy congruence relations in M , consider ρ_{\cap} to be their intersection.

From Theorem 4.4 we have just to check that, every $a, b, c \in M$ with $a \leq b$ satisfy that $\widehat{\rho_{\cap}}(ac, bc) \geq \rho_{\cap}(a, b)$.

From Proposition 4.6, if $z \in bc$ then there exists $w \in ac$ such that $w \leq z$ and, then, Proposition 4.8 implies $\rho_i(w, z) \geq \rho_i(a, b)$ for all $i \in \Lambda$. So,

$$\bigvee_{x \in ac} \rho_{\cap}(x, z) \geq \bigvee_{\substack{w \leq z \\ w \in ac}} \rho_{\cap}(w, z) = \bigvee_{\substack{w \leq z \\ w \in ac}} \bigwedge_{i \in \Lambda} \rho_i(w, z) \geq \bigvee_{\substack{w \leq z \\ w \in ac}} \bigwedge_{i \in \Lambda} \rho_i(a, b) = \rho_{\cap}(a, b)$$

Analogously, from Proposition 4.5 and Proposition 4.8, if $w \in ac$ then there exists $z \in bc$ such that $w \leq z$ and

$$\bigvee_{y \in bc} \rho_{\cap}(w, y) \geq \bigvee_{\substack{z \geq w \\ z \in bc}} \rho_{\cap}(w, z) = \bigvee_{\substack{z \geq w \\ z \in bc}} \bigwedge_{i \in \Lambda} \rho_i(w, z) \geq \bigvee_{\substack{z \geq w \\ z \in bc}} \bigwedge_{i \in \Lambda} \rho_i(a, b) = \rho_{\cap}(a, b)$$

Therefore, $\widehat{\rho_{\cap}}(ac, bc) \geq \rho_{\cap}(a, b)$.

The proof for the transitive closure of union follows by a routine calculation. ■

5. Conclusions and future work

Starting with the usual notion of fuzzy congruence relation in a groupoid, we have introduced the definition of fuzzy congruence relation in an nd-groupoid by means of the power extension of the relation to the powerset of the carrier. Our definition is proved to be an adequate generalization of that introduced by Bakhshi and Borzooei in [1]. Moreover, contrariwise to their claim, we have proved that, if (A, \cdot) is a hypergroupoid (and thus an nd-groupoid), the set of fuzzy congruences on A , with the usual operations for infimum and supremum is not necessarily a lattice.

As a consequence of this negative result, we investigated conditions on the nd-groupoid so that we can guarantee the structure of complete lattice of its set of fuzzy congruences. Such conditions are those of an m-distributive multisemilattice.

As future work on this research line, our plan is to keep investigating new or analogue results concerning congruences on generalized algebraic structures, specially in a non-deterministic sense; in this topic, it seems to be important to study the so-called power structures from a universal standpoint [3, 9]. We will also focus on the corresponding fuzzifications of concepts such as ideal, closure systems and homomorphisms over nd-structures, in the line of [16].

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