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Congruence relations on some hyperstructures

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Abstract In this work we study the structure of the set of congruences on several hyperstructures with one and two (hyper-)operations. On the one hand, we show sufficient conditions guaranteeing that the set of congruences of an nd-groupoid is a complete lattice (which, in turn, is a sublattice of the lattice of equivalence relations on the nd-groupoid). On the other hand, we focus on the study of the congruences on a multilattice; specifically, we prove that the set of congruences on an m-distributive multilattice forms a complete lattice and, moreover, show that the classical relationship between homomorphisms and congruences can be adequately adapted to work with multilattices under suitable restrictions.

Keywords Lattice theory; Non-determinism; Multisemilattice

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1 Introduction

The problem of providing suitable fuzzifications of crisp concepts is an important topic which attracts the attention of a number of researchers. Since the inception of fuzzy sets and fuzzy logic, there have been approaches to consider underlying sets of truth-values more general than the unit interval; for instance, consider the L-fuzzy sets introduced by Goguen [6], where L is a complete lattice.

This paper is part of a research line aimed at investigating M-fuzzy sets where M has the structure of a multilattice. The concepts of ordered and algebraic multilattice were introduced by Benado [2] in 1954. Roughly speaking, a multilattice is a poset in which the restrictions imposed on a (complete) lattice, namely, the “existence of least upper bounds and greatest lower bounds” (suprema/infima) are relaxed to the “existence of minimal upper bounds and maximal lower bounds” (multi-suprema/multi-infima).

Much more recently, Cordero *et al.* [3,8,9] proposed an alternative algebraic definition of multilattice which is more closely related to that of lattice, allowing for natural definitions of related structures such that multisemilattices and, in addition, is better suited for applications. For instance, Medina *et al.* [11] developed a general approach to fuzzy logic programming based on a multilattice as underlying set of truth-values for the logic.

Several papers have been published about the lattice of fuzzy congruences on different algebraic structures [1, 4, 5, 12, 13], and in this paper we initiate research in this direction. First, we focus on the theory of congruences on a multilattice, as this is a necessary step prior to considering the multilattice-based generalization of the concept of L-fuzzy congruence.

Attending to the description given above, the main difference that one notices when working with multilattices is that the operators which compute multi-suprema and multi-infima are no longer single-valued, since there may be several multi-suprema or multi-infima, or may be none. This immediately leads to the theory of hyperstructures, that is, set-valued operations.

Hyperstructure theory was introduced in [10] when Marty defined hypergroups, began to analyse their properties and applied them to groups, rational fractions and algebraic functions. Nowadays, a number of different hyperstructures are widely studied from the theoretical point of view and for their use in applied mathematics and artificial intelligence.

We are interested in the most general hyperstructures, namely hypergroupoids and nd-groupoids. In a nutshell, let us recall that a hypergroupoid is simply a binary operator $H \times H \to \mathcal{P}(H) \smallsetminus \{\varnothing\}$, whereas a non-deterministic groupoid (nd-groupoid, for short) is a binary operator $H \times H \to \mathcal{P}(H)$, that is, a hypergroupoid in which the restriction of the codomain being the non-empty set is dropped. Notice that the operators which compute the multi-suprema and multi-infima in a poset provide precisely nd-groupoids or, if we have for granted that at least a multi-supremum always exists, a hypergroupoid.

It is well-known that, for every set A, the set of equivalence relations on A, $Eq(A)$, with the inclusion ordering (in the powerset of $A \times A$) is a complete lattice in which the infimum is the meet and the supremum is the transitive closure of the join. In this paper, we concentrate on the structure of the set of congruences on an hypergroupoid or nd-groupoid.

The first part of the paper concentrates on finding conditions granting that the set of congruences on a hypergroupoid or nd-groupoid is a complete lattice. Later, we concentrate on multilattices, which can now be seen as sets with two interlaced hyperoperations verifying some properties. We will search for suitable extensions of the results on nd-groupoids and hypergroupoids to the framework of multilattices in which we adhere to the development presented in Grätzer's General Lattice Theory [7]. After some preliminary results, we prove that the set of congruences on a multilattice which satisfies a sort of distributivity, that we have called m-distributivity, forms a complete lattice. We also show that the classical relationship between homomorphisms and congruences can be adequately adapted to work with multilattices.

2 Nd-groupoids, multisemilattices, multilattices

The concept of multilattice [2] was introduced by Benado in 1954, and is an extension of the concept of lattice by means the so-called multi-suprema and multi-infima.

Given $(M, \leq)$ a partially ordered set (henceforth poset) and $B \subseteq M$, a *multi-supremum of* B is a minimal element of the set of upper bounds of B and $\mathrm{multisup}(B)$ denote the set of multi-suprema of B. Dually, we define the *multi-infima*.

We continue with the concept of ordered multisemilattice as an adequate notion which generalizes that of ordered semilattice.

Definition 1 (Multisemilattice) A poset $(M, \leq)$ is said to be a *multisemilattice* if it satisfies that, for all $a, b, x \in M$ with $a \leq x$ and $b \leq x$, there exists $z \in \mathrm{multisup}(\{a, b\})$ such that $z \leq x$.

Similarly to what happens in the theory of lattices, a poset $(M, \leq)$ is said to be a *multilattice* if it is a multisemilattice and so is its dual $(M, \geq)$.

Note that the definition is consistent with the existence of two incomparable elements *without* any multi-supremum or multi-infimum. In other words, $\mathrm{multisup}(\{a, b\})$ and $\mathrm{multinf}(\{a, b\})$ can be empty.

In the concept of ordered multisemilattice, minimal upper bounds (multi-suprema) play the role of least upper bounds in a lattice (analogously for the dual). The main difference that one notices is that the operators which compute multi-suprema are not single-valued, since there may be several multisuprema or may be none.

By abstracting out the algebraic properties of the multi-suprema (or multi-infima) we obtain a more general hyperstructure: the hypergroupoid that is, a set H together with a set-valued operation $H \times H \to \mathcal{P}(H) \smallsetminus \{\varnothing\}$. Notice, however, that the output of the multisup operator can be empty, therefore, it is convenient to drop the non-emptiness restriction in the codomain of the hyperoperation.

Definition 2 (Nd-groupoid) A *non-deterministic* (nd, for short) *groupoid* is a pair $(A, \star)$ consisting of a non-empty set A and a mapping $\star\colon A \times A \to \mathcal{P}(A)$.

Notation: Given an nd-groupoid $(A, \star)$, we will use the following conventions:

- We will use multiplicative notation and, thus, the operation is omitted.
- If $a \in A$ and $X \subseteq A$ $aX = \{ax \mid x \in X\}$ and $Xa = \{xa \mid x \in X\}$. In particular, $a\varnothing = \varnothing a = \varnothing$.
- When the result of the nd-operation is a singleton, we will often omit the braces.

Similarly to lattice theory, if we define $a \vee b = \mathrm{multisup}(\{a, b\})$ it is possible to give an algebraic version of multisemilattices as pairs $(M, \vee)$ where M is a non-empty set and $\vee$ satisfies suitable conditions.

In order to show this alternative definition, we start by lifting some properties to the non-deterministic case.

– **Idempotency**: $aa = a$ for all $a \in A$.

– **Commutativity**: $ab = ba$ for all $a, b \in A$.

– **Left m-associativity**: if $ab = b$, then $(ab)c \subseteq a(bc)$ for all $a, b, c \in A$.

– **Right m-associativity**: if $bc = c$, then $a(bc) \subseteq (ab)c$, for all $a, b, c \in A$.

– **m-associativity**: if it is left and right m-associative.

We will focus our interest in the binary relation usually named *natural ordering*, which is defined by

$$a \leq b \text{ if and only if } ab = b$$

In general, this relation is not an ordering. However, it is reflexive if the nd-groupoid is idempotent; it is antisymmetric if the nd-groupoid is commutative; finally, it is transitive if the nd-groupoid is m-associative.

The two following properties of nd-groupoids, called *comparability* have an important role in multilattice theory:

– C_1: $c \in ab$ implies that $a \leq c$ and $b \leq c$.

– C_2: $c, d \in ab$ and $c \leq d$ imply that $c = d$.

Now, we can give the algebraic definition of a *multisemilattice* as an nd-groupoid that satisfies idempotency, commutativity, m-associativity and comparability laws. Both definitions of multisemilattice can be proved to be equivalent [9].

3 Congruence relations on nd-groupoids

The definition of congruence relation on an nd-groupoid can be viewed both as the set-valued generalization of the congruences of lattice theory and as the crisp case of the fuzzy congruence given in [1], which we recall below.

Definition 3 (Fuzzy equivalence relation [14]) Let H be a nonempty set. A *fuzzy relation* ρ on H is a fuzzy subset of $H \times H$ (i.e. ρ is a function from $H \times H$ to $[0,1]$). Now, ρ is said to be a *fuzzy equivalence relation* if

1. $\rho(x,x) = \sup_{y,z \in H} \rho(y,z)$ (fuzzy reflexivity),
2. $\rho(y,x) = \rho(x,y)$ (fuzzy symmetry),
3. $\rho(x,y) \geq \sup_{z \in H} \min(\rho(x,z), \rho(z,y))$ (fuzzy transitivity).

Definition 4 (Fuzzy congruence relation [1]) Let $(H, \cdot)$ be a hypergroupoid and ρ a fuzzy relation on H.

1. ρ is *fuzzy left* (right) *compatible* if for all $u \in ax$ ($u \in xa$) there exists $v \in ay$ ($v \in ya$) and for all $v \in ay$ ($v \in ya$) there exists $u \in ax$ ($u \in xa$) such that $\rho(u,v) \geq \rho(x,y)$, for all $x, y, a \in H$ and *fuzzy compatible* if it is both fuzzy left and right compatible.
2. ρ is a *fuzzy* (left, right) *congruence* relation on H if it is a fuzzy (left, right) compatible equivalence relation on H.

Under the additional assumption of commutativity with respect to the usual composition of binary relations, Bakhshi and Borzooei [1], apparently proved that the set of all fuzzy congruence relations on an hypergroupoid $(H, \cdot)$ is a complete lattice where, in particular, the infimum of two congruences is its intersection. The following example proves that, without additional hypotheses, the statement is false even in a crisp setting.

Example 1 Let H be the set $\{a, b, c, u_0, u_1, v_0, v_1\}$ provided with a commutative hyperoperation $\cdot$ which is defined as follows:

$$aa = ab = bb = \{a, b\}; \quad ac = \{u_0, u_1\};$$

$$bc = \{v_0, v_1\} \quad \text{and } xy = \{c\}, \quad \text{elsewhere}$$

Consider $R, S : H \times H \to \{0, 1\}$ two binary relations, where R is the least equivalence relation containing $\{(a, b), (u_0, v_0), (u_1, v_1)\}$ and S the least equivalence relation containing $\{(a, b), (u_0, v_1), (u_1, v_0)\}$. A simple (but tedious) check shows that $R \circ S = S \circ R$ and they are both compatible with the hyperoperation $\cdot$, therefore, both R and S are congruence relations. However, one can check that the intersection $R \cap S$ is not a congruence relation. $\square$

The previous example motivated the search for a sufficient condition which granted the structure of complete lattice for the set of congruences on a hypergroupoid and, by extension, on an nd-groupoid.

Definition 5 (Congruence) Let $(A, \star)$ be an nd-groupoid. An equivalence relation $\equiv$ defined on A is said to be a *congruence* if $a \equiv b$ and $c \in A$ implies that

- for all $x \in ac$ there exists $y \in bc$ such that $x \equiv y$
- for all $x \in ca$ there exists $y \in cb$ such that $x \equiv y$

Notation: Let $\mathcal{R}$ be a binary relation in A and $X, Y \subseteq A$ then $X \, \mathcal{R} \, Y$ denotes that, for all $x \in X$, there exists $y \in Y$ such that $x\mathcal{R}y$ and for all $y \in Y$ there exists $x \in X$ such that $x\mathcal{R}y$.

With this notation, we can state the definition of congruence on nd-groupoids more in the line of the definition on groupoids; that is, an equivalence relation $\equiv$ defined on an nd-groupoid $(A, \star)$ is said to be a *congruence* if $a \equiv b$ and $c \in A$ implies that $ac \equiv bc$ and $ca \equiv cb$.

The following lemma is an immediate consequence from the definition.

Lemma 1 *Let $(A, \star)$ be an idempotent nd-groupoid and $\equiv$ be a congruence relation. If $a \equiv b$ then $\varnothing \neq ab \equiv a$.*

Theorem 1 *Let $(A, \star)$ be an nd-groupoid satisfying idempotency and property C_1, and let $\equiv$ be an equivalence relation. Then, $\equiv$ is a congruence relation if and only if the following condition holds:*

$$\text{for all } a, b, c \in A, \text{ if } a \leq b \text{ and } a \equiv b, \text{ then } ac \equiv bc \text{ and } ca \equiv cb.$$

Proof The necessity is obvious, thus we will just prove the sufficiency.

If $a \equiv b$ then, by Lemma 1, there exists $z \in ab$ such that $a \equiv z \equiv b$. Property C_1 ensures that $a \leq z$ and $b \leq z$ and then, by the condition, $ac \equiv zc \equiv bc$. □

In the rest of the paper, we focus on the search of properties that ensure the condition of the previous theorem. In order to avoid repetitions, we will only concentrate on the right versions of the properties.

Proposition 1 *Let $(A, \star)$ be a commutative, m-associative nd-groupoid satisfying both comparability properties, $\equiv$ be a congruence relation and $a, b, c \in A$. If $a \leq b$, $w \in ac$ and $z \in bc$ with $w \leq z$, then $w \equiv z$.*

Proof Consider $w \in ac$ such that $wz = z$ (i.e., $w \leq z$). By C_1 we have that $a \leq w$ and $w = aw \equiv bw$. Now, it is sufficient to prove that $z \in bw$.

Since $b \leq z$ and $w \leq z$, by m-associativity, $z = bz = b(wz) \subseteq (bw)z$, that is, $z \in (bw)z$. Consider $z' \in bw$ such that $z' \leq z$. By C_1, $b \leq z'$ and $c \leq w \leq z'$. Once again, m-associativity ensures that there exists $z'' \in bc$ such that $z'' \leq z'$ and therefore $z'' \leq z$. Now, by C_2, $z'' = z$. From $z \leq z'$ and $z' \leq z$, by commutativity, the relation $\leq$ is antisymmetric and, hence, $z = z' \in bw$. □

In order to obtain that $ac \equiv bc$, one has to start from $z \in bc$ and show the existence of the suitable $w \in ac$ and vice versa.

Proposition 2 *Let $(A, \star)$ be an m-associative nd-groupoid that satisfies C_1 and, for $a, b, c \in A$, consider $a \leq b$ and $z \in bc$. Then, there exists $w \in ac$ such that $w \leq z$.*

Proof By hypothesis $a \leq b$ and, by C_1, since $z \in bc$, we obtain $b \leq z$. Therefore $a \leq z$ because, by m-associativity of the nd-operation $\star$, the relation $\leq$ is transitive.

Now, again by C_1, since $z \in bc$, $c \leq z$ and, by m-associativity, $z = az = a(cz) \subseteq (ac)z$. In particular, we have that $z \in (ac)z$, this implies the existence of $w \in ac$ such that $z = wz$, that is, $w \leq z$. $\square$

Definition 6 (m-distributivity) An nd-operation $\star$ in a set A is said to be *m-distributive* when, for all $a, b, c \in A$, if $a \leq b$ and $w \in ac$, then $bw \cap bc \neq \varnothing$.

The name of this property is justified in the context of multilattice theory (that is the reason of using the 'm' prefix) as we will show later. This generalization of distributivity will be proved to be a sufficient condition for the set of congruence relations being a complete lattice.

Proposition 3 *Let $(M, \star)$ be an m-distributive nd-groupoid that satisfies property C_1 and $a, b, c \in M$. If $a \leq b$ and $w \in ac$ then there exists $z \in bc$ such that $w \leq z$.*

Proof By m-distributivity, from $a \leq b$ and $w \in ac$, we obtain that there exists $z \in bw \cap bc$. Now, by C_1 $w \leq z$. $\square$

Now, we have all the required properties and lemmas needed in order to face the first goal of this paper, namely, to prove that the set of congruences of an nd-groupoid is a complete lattice.

Theorem 2 *The set of the congruence relations in an m-distributive multisemilattice M, $Con(M)$, is a sublattice of $Eq(M)$ and, moreover is a complete lattice wrt the inclusion ordering.*

Proof Let $\{\equiv_i\}_{i\in\Lambda}$ be a set of congruence relations in M, consider $\equiv_\cap$ to be their intersection.

From Theorem 1 we have just to check that, for all $a, b, c \in M$, $a \le b$ and $a \equiv_\cap b$ imply that $ac \equiv_\cap bc$.

From Proposition 2, if $z \in bc$ then there exists $w \in ac$ such that $w \le z$ and, by Proposition 1, for all $w \in ac$ with $w \le z$ and all $i \in \Lambda$, $w \equiv_i z$ (so $w \equiv_\cap z$).

Conversely, from Proposition 3 and Proposition 1, if $w \in ac$ then there exists $z \in bc$ such that $w \le z$ and, for all $z \in bc$ with $w \le z$ and all $i \in \Lambda$, $w \equiv_i z$ (so $w \equiv_\cap z$).

The proof for the transitive closure of union follows by a routine calculation. $\square$

4 Congruences on multilattices and homomorphisms

Similarly to the case of lattice theory, a multilattice is an nd-structure $(A, \vee, \wedge)$ in which $(A, \vee)$ and $(A, \wedge)$ are multisemilattices and both nd-operations satisfy the absorption properties. Moreover, if both operations are m-distributive, then the following holds for all $a, b \in A$ with $a \le b$ and $c \in A$:

1. $(a \wedge b) \vee c \subseteq (a \vee c) \wedge (b \vee c)$
2. $(a \vee b) \wedge c \subseteq (a \wedge c) \vee (b \wedge c)$

In fact, the two latter conditions are equivalent to the m-distributivity of $\wedge$ and $\vee$, as the following result shows.

Proposition 4 *Let $(A, \vee, \wedge)$ be a multilattice and $a, b, c \in A$. The following conditions are equivalent:*

1. *If $a \le b$ and $w \in a \vee c$, then $b \vee w \cap b \vee c \neq \varnothing$.*
2. *If $a \le b$, then $(a \wedge b) \vee c \subseteq (a \vee c) \wedge (b \vee c)$.*

Proof

– (1 ⇒ 2) If $w \in (a \wedge b) \vee c = a \vee c$ then, by hypothesis 1, there exists $u \in b \vee w \cap b \vee c$. Since $u \in b \vee w$, by property C_1, we have $w \leq u$ and, therefore, $w = w \wedge u \subseteq (a \vee c) \wedge (b \vee c)$.
– (2 ⇒ 1) Conversely, assume $w \in a \vee c$, since $a \leq b$, we have $a = a \wedge b$ and, by hypothesis 2 we have $w \in (a \wedge b) \vee c \subseteq (a \vee c) \wedge (b \vee c)$. Thus, there exists an element $u \in a \vee c$ and $v \in b \vee c$ such that $w \in u \wedge v$. Moreover,[1] by C_1, $w \leq v$ and $b \leq v$. By m-associativity we have now that $v = b \vee v = b \vee (w \vee v) \subseteq (b \vee w) \vee v$. This means that there exists $v' \in b \vee w$ such that $v = v' \vee v$ (that is, $v' \leq v$). Finally, we will prove that $v \leq v'$ and thus $v = v' \in b \vee w \cap b \vee c$ as desired.

  By C_1 we have that $b \leq v'$ and $c \leq w \leq v'$ which, by transitivity, implies $c \leq v'$. This means that v' is an upper bound of b and c and, by definition of multilattice, there should exist $v'' \in b \vee c$ such that $v'' \leq v'$. But we already know that $v' \leq v$, hence $v'' \leq v' \leq v$, but as v and v'' are multi-suprema, the only consistent possibility is that $v = v' = v''$. □

As one might expect, a congruence on a multilattice $(M, \vee, \wedge)$ is an equivalence relation that is compatible with both operations. Therefore, the set of congruences on the multilattice is the intersection of the set of congruences on both multisemilattices, $(M, \vee)$ and $(M, \wedge)$. So, as an immediate consequence of Theorem 2 we have the following result.

Theorem 3 *The set of congruence relations in an m-distributive multilattice M is a sublattice of $Eq(M)$. Moreover, it is a complete lattice wrt the inclusion ordering.*

[1] Recall that, as stated in the definition of multilattice, the order used considered for $\wedge$ is $\geq$, this explains the following use of property C_1.

The following characterization theorem follows the schema of that given in [7].

Theorem 4 *Let* $(M, \vee, \wedge)$ *be a multilattice and* $\mathcal{R}$ *be a binary relation. Then* $\mathcal{R}$ *is a congruence relation if and only if the following conditions hold:*

1. $\mathcal{R}$ *is reflexive*
2. $x\mathcal{R}y$ *if and only if there exist* $z \in x \wedge y$ *and* $w \in x \vee y$ *with* $z\mathcal{R}w$
3. *If* $x \leq y \leq z$ *with* $x\mathcal{R}y$ *and* $y\mathcal{R}z$*, then* $x\mathcal{R}z$
4. *If* $x \leq y$ *with* $x\mathcal{R}y$*, then* $x \wedge t \,\mathcal{R}\, y \wedge t$ *and* $x \vee t \,\mathcal{R}\, y \vee t$*.*

Proof If $\mathcal{R}$ is a congruence relation, then all the conditions are satisfied using the previous results. Conversely, let us suppose that all the conditions are satisfied. Firstly we will prove that $\mathcal{R}$ is an equivalence relation. Symmetry is a straightforward consequence of (2).

For the transitivity, let us suppose that $x\mathcal{R}y$ and $y\mathcal{R}z$. Then by (2) there exist $u \in x \wedge y$, $w \in x \vee y$, $u' \in y \wedge z$ and $w' \in y \vee z$ such that $u\mathcal{R}w$ and $u'\mathcal{R}w'$. Since $u \leq w$ and $u\mathcal{R}w$, by (4), $w' = u \vee w' \mathcal{R} w \vee w'$, thus,[2] there exists $q \in w \vee w'$ such that $w'\mathcal{R}q$. Analogously, $u \wedge u' \,\mathcal{R}\, w \wedge u' = u'$ so, there exists $p \in u' \wedge u$ such that $p\mathcal{R}u'$. Since $p \leq u' \leq y \leq w' \leq q$ and $p\mathcal{R}u'\mathcal{R}w'\mathcal{R}q$, by (3), $p\mathcal{R}q$.

Since $p \leq x, z \leq q$ in the multilattice $(M, \vee, \wedge)$, then there exist $u'' \in x \wedge z$ and $w'' \in x \vee z$ such that $p \leq u'' \leq w'' \leq q$. So $p \leq q$ and $p\mathcal{R}q$, by (4), $p = p \wedge w'' \mathcal{R} q \wedge w'' = w''$. Now, as $p \leq w''$ and $p\mathcal{R}w''$, by (4), $u'' = p \vee u'' \mathcal{R} w'' \vee u'' = w''$. Finally, by (2), we have that $x\mathcal{R}z$.

Now let us prove the compatibility with the operations. If $a\mathcal{R}b$, by (2) there exist $z \in a \wedge b$ and $w \in a \vee b$ such that $z\mathcal{R}w$ and so $a\mathcal{R}w$. Then using (4) we have that

[2] We are abusing the notation here, in that singletons are not written between braces.

since $a \leq w$ then $a \vee t\,\mathcal{R}\,w \vee t$ and since $b \leq w$ then $b \vee t\,\mathcal{R}\,w \vee t$. Then we have that $a \vee t\,\mathcal{R}\,b \vee t$. $\square$

The notion of homomorphism is naturally extended to the theory of multilattices as a map which preserves multi-valued operations $h\colon M \to M'$ is a homomorphism if $h(a \vee b) \subseteq h(a) \vee h(b)$ and $h(a \wedge b) \subseteq h(a) \wedge h(b)$.

Proposition 5 *Let $(M, \vee, \wedge)$ be a multilattice and $\equiv$ a congruence relation, then $M/_{\equiv}$ is a multilattice with*

$$[a] \vee [b] = \{[x] \mid x \in a \vee b\} \qquad \textit{and} \qquad [a] \wedge [b] = \{[x] \mid x \in a \wedge b\}$$

Moreover, the mapping $h : M \to M/_{\equiv}$ such that $h(x) = [x]$ is a surjective homomorphism.

Theorem 5 *Let $h : M \to M'$ be a homomorphism between multilattices. The relation in M given by $a \equiv b \Leftrightarrow h(a) = h(b)$ is a congruence iff:*

1. *$h(a) = h(b)$ implies $a \vee b \neq \varnothing$ and $a \wedge b \neq \varnothing$*
2. *$h(a \vee b) = h(a)$ implies $a \wedge b \neq \varnothing$*
3. *$h(a \wedge b) = h(a)$ implies $a \vee b \neq \varnothing$*

Proof We will prove just the converse implication, for which we will use the characterisation given in Theorem 4.

It is obvious that $\equiv$ is an equivalence relation. Now, let us consider $x, y \in M$ where $x \leq y$ and $h(x) = h(y)$, and $z \in x \wedge t$. As $z \leq y$ and $z \leq t$ there must be $w \in y \wedge t$ with $z \leq w$; consequently, $h(z) \leq h(w)$. On the other hand, by properties of homomorphism, we have $h(w) \in h(y \wedge t) \subseteq h(y) \wedge h(t) = h(x) \wedge h(t)$ and $h(z) \in h(x \wedge t) \subseteq h(x) \wedge h(t)$. As a result, $h(z) = h(w)$, that is, $z \equiv w$.

Let us consider $x, y, t \in M$ with $x \leq y$, $h(x) = h(y)$ and $w \in y \wedge t$. Firstly, $x \leq y$ and $w \leq y$ so there must exists $y' \in x \vee w$ with $x \leq y' \leq y$. As h is a homomorphism we have that $h(x) \leq h(y') \leq h(y) = h(x)$, that is, $h(y') = h(x)$. As $h(y') = h(x) \in h(x) \vee h(w)$ so $h(x) \vee h(w) = h(x)$. In a nutshell, $h(x \vee w) = h(x)$.

Furthermore, $h(x \wedge w) \subseteq h(x) \wedge h(w) = h(w)$, thus $h(x \wedge w) = h(w)$. By condition (2), $x \wedge w \neq \varnothing$, so we can take $x' \in x \wedge w$ and, by definition of multilattice, there exists $z \in x \wedge t$ such that $x' \leq z$. Notice that $h(x') \leq h(z)$ and $h(z), h(w) \in h(x) \wedge h(t)$; hence, we obtain that $h(w) = h(z)$. □

It is remarkable that in most of the applications of multilattices it is the case that $\operatorname{multisup}(\{a, b\}) \neq \varnothing \neq \operatorname{multinf}(\{a, b\})$ and, as a result, every homomorphism defines a congruence.

5 Conclusions and future work

We have started the investigation of congruences on several hyperstructures, and shown that the set of congruences of an m-distributive multisemilattice is a complete latice and a sublattice of the set of its equivalence relations. Moreover, the well-known relation between congruences and homomorphisms has been shown to be preserved when considered in the framework of multilattices.

As future work, we are planning to investigate the multilattice-based generalization of the concept of L-fuzzy congruence, following the line of the several papers published on the lattice of fuzzy congruences on different algebraic structures [1, 4, 5, 12, 13]. Then, continuing towards our main aim, studying the computational properties of multilattices as a suitable algebraic structure on which found an extended theory of fuzzy structures, we will try to lift from a crisp to a fuzzy setting concepts such as congruences, ideal, closure systems and homomorphisms over nd-structures.

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