Fundamenta Informaticae XX (2011) 1–26
IOS Press

The category of L-Chu correspondences and the structure of L-bonds*

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Abstract. An L-fuzzy generalization of the so-called Chu correspondences between formal contexts forms a category called L-ChuCors. In this work we show that this category naturally embeds ChuCors, and prove that it is *-autonomous. We also focus on the direct product of two L-fuzzy contexts, which is defined with the help of a binary operation, essentially a disjunction, on a lattice of truth-values L.

1. Introduction

Formal concept analysis (FCA) introduced by Ganter and Wille [11] has become an extremely useful theoretical and practical tool for formally describing structural and hierarchical properties of data with "object-attribute" character. Regarding applications, we can find papers ranging from ontology merging [24], to applications to the Semantic Web by using the notion of concept similarity [9], and from processing of medical records in the clinical domain [13] to the development of recommender systems [8].

Soon after the introduction of "classical" formal concept analysis, several approaches towards its generalization were introduced and, nowadays, there are recent works which extend the theory by using

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*This is an extended version of the paper "*L-bonds vs extents of direct products of two L-fuzzy contexts*" presented at the CLA'10 conference.
†Partially supported by grant VEGA 1/0131/09 and APVV-0035-10.
‡Partially supported by Spanish Ministry of Science project TIN09-14562-C05-01 and Junta de Andalucía project FQM-5233.

ideas from fuzzy set theory, or fuzzy logic reasoning, or from rough set theory, or some integrated approaches such as fuzzy and rough, or rough and domain theory [1, 19–22, 25, 26].

In this paper, we are concerned with fuzzy extensions of the classical concept lattice. Bělohlávek provided in [3, 5] an L-fuzzy extension of the main notions of FCA, such as context and concept, by extending its underlying interpretation on classical logic to the more general framework of L-fuzzy logic [12]. Later, Krajči introduced a further level of generalization [14].

In this work, we aim at formally describing some structural properties of intercontextual relationships [10] of L-fuzzy formal contexts. The categorical treatment of morphisms as fundamental structural properties has been advocated by [18] as a means for the modelling of data translation, communication, and distributed computing, among other applications. Our approach, broadly continues the research line which links the theory of Chu spaces with concept lattices [27] but, particularly, is based on the notion of Chu correspondences between formal contexts developed by Mori in [23]. Previous work in this categorical approach has been developed by the authors in [15, 16]. The category L-ChuCors is formed by considering the class of L-fuzzy formal contexts as objects and the L-fuzzy Chu correspondences as arrows between objects.

The main results here are the definition of the category L-ChuCors, which is proved to contain category ChuCorsas a subcategory, as well as being *-autonomous. Then, we focus on the extension of the relationship between bonds and extents of direct products of contexts to the realm of L-fuzzy FCA.

In order to obtain a mostly self-contained document, Section 2 introduces the basic definitions concerning the Lordered sets, the L-fuzzy extension of formal concept analysis, as well as those concerning L-Chu correspondences and L-bonds, the main results on these topics are stated too. The core of the paper starts at Section 3 with the introduction of the category of L-Chu correspondences, and the proof that the category of classical Chu correspondences is a (not full) subcategory of L-ChuCors. Then, in Section 4, we focus further on the structure of Lbonds, as an alternative way of connecting different contexts. Later, in Section 5, we introduce the dual of a context, and an internal Hom functor $C_1 \multimap C_2$ and a tensor product $C_1 \otimes C_2$ between L-fuzzy contexts, in order to prove that L-ChuCors is a *-autonomous category. Finally, the last section contains some conclusions and prospects for future work.

2. Preliminary definitions

In order to make this contribution as self-contained as possible, we proceed now with the preliminary definitions of complete residuated lattice, L-fuzzy context, L-fuzzy concept, and L-Chu correspondence.

Definition 2.1. An algebra $\langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ is said to be a **complete residuated lattice** if

1. $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete bounded lattice with least element 0 and greatest element 1,
2. $\langle L, \otimes, 1\rangle$ is a commutative monoid,
3. $\otimes$ and $\rightarrow$ are adjoint, i.e. $a \otimes b \leq c$ if and only if $a \leq b \rightarrow c$, for all $a, b, c \in L$, where $\leq$ is the ordering in the lattice generated from $\wedge$ and $\vee$.

We now introduce the notions of *L-fuzzy context*, *extended derivation operations*, and *L-fuzzy concept* provided by Bělohlávek [3,4]. Notice that we will use the notation Y^X to refer to the set of mappings from X to Y.

Definition 2.2. Let L be a complete residuated lattice, an L**-fuzzy context** is a triple $\langle B, A, r\rangle$ consisting of a set of objects B, a set of attributes A and an L-fuzzy binary relation r, i.e. a mapping $r\colon B \times A \to L$, which can be alternatively understood as an L-fuzzy subset of $B \times A$.

Given an L-fuzzy context $\langle B, A, r\rangle$, a pair of mappings $\uparrow\colon L^B \to L^A$ and $\downarrow\colon L^A \to L^B$ can be defined for every $f \in L^B$ and $g \in L^A$ as follows:

$$\uparrow f(a) = \bigwedge_{o\in B} (f(o) \to r(o,a)) \qquad\qquad \downarrow g(o) = \bigwedge_{a\in A} \big(g(a) \to r(o,a)\big) \tag{1}$$

Lemma 2.1. Let L be a complete residuated lattice, and let $r \in L^{B\times A}$ be an L-fuzzy relation between B and A. Then

1. The pair of operators $\uparrow$ and $\downarrow$, defined in (1) above, form a Galois connection between $\langle L^B; \subseteq\rangle$ and $\langle L^A; \subseteq\rangle$, that is, $\uparrow\colon L^B \to L^A$ and $\downarrow\colon L^A \to L^B$ are antitonic and, furthermore, for all $f \in L^B$ and $g \in L^A$ we have $f \subseteq \downarrow\uparrow f$ and $g \subseteq \uparrow\downarrow g$.

2. Furthermore, the following equalities hold for arbitrary $f \in L^B$ and $g \in L^A$, $\uparrow f = \uparrow\downarrow\uparrow f$ and $\downarrow g = \downarrow\uparrow\downarrow g$.

The second item in the previous lemma suggests to introduce a term to denote elements invariant under the compositions $\uparrow\downarrow$ and $\downarrow\uparrow$.

Definition 2.3. Consider an L-fuzzy context $C = \langle B, A, r\rangle$. An L-fuzzy set of objects $f \in L^B$ (resp. an L-fuzzy set of attributes $g \in L^A$) is said to be **closed in C** iff $f = \downarrow\uparrow f$ (resp. $g = \uparrow\downarrow g$).

Now, the second item in Lemma 2.1 can be rephrased as: both $\downarrow\uparrow f$ and $\uparrow\downarrow g$ are closed in C.

Definition 2.4. An L**-fuzzy concept** is a pair $\langle f, g\rangle$ such that $\uparrow f = g, \downarrow g = f$. The first component f is said to be the **extent** of the concept, whereas the second component g is the **intent** of the concept. The set of all L-fuzzy concepts associated to a fuzzy context (B, A, r) will be denoted as $L\text{-}FCL(B, A, r)$.

An ordering between L-fuzzy concepts is defined as follows: $\langle f_1, g_1\rangle \le \langle f_2, g_2\rangle$ if and only if $f_1 \subseteq f_2$ if and only if $g_1 \supseteq g_2$.

Proposition 2.1. The poset $(L\text{-}FCL(B, A, r), \le)$ is a complete lattice where

$$\bigwedge_{j\in J} \langle f_j, g_j\rangle = \Big\langle \bigwedge_{j\in J} f_j, \uparrow \big(\bigwedge_{j\in J} f_j\big)\Big\rangle \quad\text{and}\quad \bigvee_{j\in J} \langle f_j, g_j\rangle = \Big\langle \downarrow \big(\bigwedge_{j\in J} g_j\big), \bigwedge_{j\in J} g_j\Big\rangle$$

Finally, we proceed with the definition of L-Chu correspondences [16], for which we firstly to introduce a suitable extension of the notion of multifunction (also called, many-valued function, or correspondence) to that of L-multifunction.

Definition 2.5. An L-**multifunction** from X to Y is a mapping $\varphi\colon X \to L^Y$.

The **transposed** of an L-multifunction $\varphi\colon X \to L^Y$ is an L-multifunction $\varphi^t\colon Y \to L^X$ defined by $\varphi^t(y)(x) = \varphi(x)(y)$.

The set $L\text{-}Mfn(X, Y)$ of all the L-multifunctions from X to Y can be endowed with a poset structure by defining the ordering $\varphi_1 \le \varphi_2$ as $\varphi_1(x)(y) \le \varphi_2(x)(y)$ for all $x \in X$ and $y \in Y$.

Definition 2.6. Consider two L-fuzzy contexts $C_i = \langle B_i, A_i, r_i\rangle$, $(i = 1, 2)$, then the pair $\varphi = (\varphi_L, \varphi_R)$ is called a **correspondence** from C_1 to C_2 if φ_L and φ_R are L-multifunctions, respectively, from B_1 to B_2 and from A_2 to A_1 (that is, $\varphi_L : B_1 \to L^{B_2}$ and $\varphi_R : A_2 \to L^{A_1}$).

The L-correspondence φ is said to be a **weak L-Chu correspondence** if the following equality $\hat{r}_1(\chi_{o_1}, \varphi_R(a_2)) = \hat{r}_2(\varphi_L(o_1), \chi_{a_2})$ holds for all $o_1 \in B_1$ and $a_2 \in A_2$. By unfolding the definition of $\hat{r}_i$ this means that

$$\bigwedge_{a_1\in A_1} (\varphi_R(a_2)(a_1) \to r_1(o_1, a_1)) = \bigwedge_{o_2\in B_2} (\varphi_L(o_1)(o_2) \to r_2(o_2, a_2)) \tag{2}$$

A weak Chu correspondence φ is an **L-Chu correspondence** if $\varphi_L(o_1)$ is *closed* in C_2 and $\varphi_R(a_2)$ is *closed* in C_1 for all $o_1 \in B_1$ and $a_2 \in A_2$. We will denote the set of all Chu correspondences from C_1 to C_2 by L-ChuCors(C_1, C_2).

L-ordered sets of L-concepts and L-Chu correspondences

We assume that the reader knows the notions of L-equality, and completely L-ordered sets of L-concepts, see [7] for the definitions.

Given a formal context C, we will consider a completely L-ordered set based on the on the set of formal concepts $L\text{-}FCL(C)$.

Definition 2.7. We define an L-equality $\approx_1$ and L-ordering $\preceq_1$ on the set of formal concepts $L\text{-}FCL(C)$ of context C as follows:

- $\langle f_1, g_1\rangle \preceq_1 \langle f_2, g_2\rangle = \bigwedge_{o\in B}(f_1(o) \to f_2(o))$
- $\langle f_1, g_1\rangle \approx_1 \langle f_2, g_2\rangle = \bigwedge_{o\in B}(f_1(o) \leftrightarrow f_2(o))$

Definition 2.8. Let $C = \langle B, A, r\rangle$ be an L-fuzzy formal context and γ be an L-set from $L^{L\text{-}FCL(C)}$. We define L-sets of objects and attributes $\bigcup_B \gamma$ and $\bigcup_A \gamma$, respectively, as follows:

- $(\bigcup_B \gamma)(o) = \bigvee_{\langle f,g\rangle\in L\text{-}FCL(C)} (\gamma(\langle f, g\rangle) \otimes f(o))$, for $o \in B$
- $(\bigcup_A \gamma)(a) = \bigvee_{\langle f,g\rangle\in L\text{-}FCL(C)} (\gamma(\langle f, g\rangle) \otimes g(a))$, for $a \in A$

Theorem 2.1. (Bělohlávek)
Let $C = \langle B, A, r\rangle$ be an L-context. $\langle\langle L\text{-}FCL(C), \approx\rangle, \preceq\rangle$ is a completely L-ordered set in which infima and suprema can be described as follows: for an L-set $\gamma \in L^{L-FCL(C)}$ we have:

$$^1\inf(\gamma) = \{\langle\downarrow (\bigcup_A \gamma), \uparrow\downarrow (\bigcup_A \gamma)\rangle\} \qquad ^1\sup(\gamma) = \{\langle\downarrow\uparrow (\bigcup_B \gamma), \uparrow (\bigcup_B \gamma)\rangle\}.$$

Finally, given two formal context C_1, C_2, we will consider a completely L-ordered set based on the on the set of L-Chu correspondences between both contexts.

Definition 2.9. Given two L-fuzzy contexts $\langle B_i, A_i, r_i\rangle$ for $i \in \{1,2\}$ we define completely L-ordered set $\big\langle\langle L\text{-ChuCors}, \approx_2\rangle, \preceq_2\big\rangle$, where

$$\begin{aligned}\varphi_1 \approx_2 \varphi_2 &= \bigwedge_{o_1\in B_1}\bigwedge_{a_2\in A_2}(\uparrow_2(\varphi_{2L}(o_1))(a_2) \leftrightarrow \uparrow_2(\varphi_{1L}(o_1))(a_2))\\ &= \bigwedge_{o_1\in B_1}\bigwedge_{a_2\in A_2}(\downarrow_1(\varphi_{2R}(a_2))(o_1) \leftrightarrow \downarrow_1(\varphi_{1R}(a_2))(o_1))\\ &= \bigwedge_{o_1\in B_1}\bigwedge_{a_2\in A_2}(\beta_{\varphi_2}(o_1)(a_2) \leftrightarrow \beta_{\varphi_1}(o_1)(a_2))\end{aligned}$$

$$\begin{aligned}\varphi_1 \preceq_2 \varphi_2 &= \bigwedge_{o_1\in B_1}\bigwedge_{a_2\in A_2}(\uparrow_2(\varphi_{2L}(o_1))(a_2) \rightarrow \uparrow_2(\varphi_{1L}(o_1))(a_2))\\ &= \bigwedge_{o_1\in B_1}\bigwedge_{a_2\in A_2}(\downarrow_1(\varphi_{2R}(a_2))(o_1) \rightarrow \downarrow_1(\varphi_{1R}(a_2))(o_1))\\ &= \bigwedge_{o_1\in B_1}\bigwedge_{a_2\in A_2}(\beta_{\varphi_2}(o_1)(a_2) \rightarrow \beta_{\varphi_1}(o_1)(a_2))\end{aligned}$$

Other operations on an L-context

The corresponding notions of negation, disjunction and complement on an L-context, which will be used later, are introduced now.

Definition 2.10. Let us consider a unary operator *negation* and a binary *disjunction* operator on the underlying structure of truth values L as follows:

1. Negation $\neg\colon L \to L$ is defined by $\neg(l) = \neg l = l \to 0$
2. Disjunction $\ltimes\colon L\times L \to L$ is defined by $l_1 \ltimes l_2 = \neg l_1 \to l_2$

Some of the properties of negation appear in the following lemma.

Lemma 2.2. (Bělohlávek [6])
For any $a, b, c \in L$ the following holds.

1. $a \le \neg b \iff a \otimes b = 0$
2. $a \otimes \neg a = 0$
3. $a \le \neg\neg a$
4. $\neg 0 = 1$
5. $\neg a = \neg\neg\neg a$
6. $a \to b \le \neg b \to \neg a$
7. $a \le b \Longrightarrow \neg b \le \neg a$
8. $\neg(a \vee b) = \neg a \wedge \neg b$

From Property 6 above and the definition of disjunction, we can see that disjunction needs not be, in general, commutative. However, this property will be very important for the definition and properties of direct product of two L-contexts. Notice that commutativity will hold if the *law of double negation* ($\neg\neg a = a$) holds. The following result states some properties of residuated lattices satisfying double negation.

Proposition 2.2. (Bělohlávek [6])
If a residuated lattice satisfies the law of double negation then it also satisfies the following conditions:

1. $l \to k = \neg(k \otimes \neg l)$
2. $\neg(\bigwedge_{i\in I} l_i) = \bigvee_{i\in I} \neg l_i$
3. $l \to k = \neg k \to \neg l$

It is convenient here to recall that adding conditions of our underlying residuated lattice may change the class of structures we are working with. In particular, a residuated lattice satisfying the double negation law and divisibility (that is, $x \le y$ implies the existence of z such that $x = y \otimes z$), we are working with an MV-algebra. If divisibility is replaced by the fact that the product $\otimes$ coincides with the infimum of the lattice, then we are have just a Boolean algebra.

We finish this part with a specific notion of *complement* of a given L-fuzzy formal context.

Definition 2.11. The complement of an L-fuzzy formal context is a formal context with the binary relation $\neg r$ defined by $\neg r(o,a) = r(o,a) \to 0$ for all $o \in B$ and $a \in A$. The uparrow and downarrow mappings on the complement are denoted by $\uparrow_\neg$ and $\downarrow_\neg$.

Lemma 2.3. Let $C = \langle B, A, r\rangle$ be an L-fuzzy formal context. For all objects $o, b \in B$ the inequality $\downarrow\uparrow(\chi_o)(b) \le \downarrow_\neg\uparrow_\neg(\chi_b)(o)$ holds. If, moreover, the law of double negation holds we have the equality $\downarrow\uparrow(\chi_o)(b) = \downarrow_\neg\uparrow_\neg(\chi_b)(o)$.

Proof:
Follows from the chain of equalities below:

$$\begin{aligned}
\downarrow\uparrow(\chi_o)(b) &= \bigwedge_{a\in A}(\uparrow(\chi_o)(a) \to r(b,a)) \\
&= \bigwedge_{a\in A}(\bigwedge_{c\in B}(\chi_o(c) \to r(c,a)) \to r(b,a)) \\
&= \bigwedge_{a\in A}((\bigwedge_{c\in B, c\neq o}(\chi_o(c) \to r(c,a)) \wedge (\chi_o(o) \to r(o,a))) \to r(b,a)) \\
&= \bigwedge_{a\in A}((\bigwedge_{c\in B, c\neq o}(0 \to r(c,a)) \wedge (1 \to r(o,a))) \to r(b,a)) \\
&= \bigwedge_{a\in A}((1 \wedge (1 \to r(o,a))) \to r(b,a)) \\
&= \bigwedge_{a\in A}((1 \to r(o,a)) \to r(b,a)) \\
&= \bigwedge_{a\in A}(r(o,a) \to r(b,a)) \\
&\stackrel{*}{=} \bigwedge_{a\in A}(\neg r(b,a) \to \neg r(o,a)) = \cdots = \downarrow_\neg\uparrow_\neg(\chi_b)(o)
\end{aligned}$$

Equality $(*)$ follows from the law of double negation, otherwise we would obtain just the inequality $\downarrow\uparrow(\chi_o)(b) \le \downarrow_\neg\uparrow_\neg(\chi_b)(o)$. □

3. The category of L-Chu correspondences

In the following definition and lemma, we introduce some connections between the right and the left sides of L-Chu correspondences.

Definition 3.1. Given a mapping $\varphi : X \to L^Y$ we consider the associated mappings $\varphi_+ : L^X \to L^Y$ and $\varphi^+ : L^Y \to L^X$ defined as follows, for all $f \in L^X$ and $g \in L^Y$,

1. $\varphi_+(f)(y) = \bigvee_{x\in X}(f(x) \otimes \varphi(x)(y))$
2. $\varphi^+(g)(x) = \bigwedge_{y\in Y} \varphi(x)(y) \to g(y)$

Lemma 3.1. Given two L-fuzzy contexts $C_i = \langle B_i, A_i, r_i\rangle$ for $i = 1, 2$, consider $\varphi = (\varphi_L, \varphi_R) \in L$-ChuCors$(C_1, C_2)$. Then, the following equalities hold for all $f \in L^{B_1}$ and $g \in L^{A_2}$ and all $o_1 \in B_1$ and $a_2 \in A_2$:

- $\uparrow_2 (\varphi_{L+}(f)) = \varphi_R^+(\uparrow_1 (f))$ and $\downarrow_1 (\varphi_{R+}(g)) = \varphi_L^+(\downarrow_2 (g))$
- $\varphi_L(o_1) = \downarrow_2 (\varphi_R^+(\uparrow_1 (\chi_{o_1})))$ and $\varphi_R(a_2) = \uparrow_1 (\varphi_L^+(\downarrow_2 (\chi_{a_2})))$

Proof:
Let $a_2 \in A_2$

$$\begin{aligned}
\uparrow_2 (\varphi_{L+}(f))(a_2) &= \bigwedge_{o_2\in B_2} \Big(\bigvee_{o_1\in B_1} \big(f(o_1) \otimes \varphi_L(o_1)(o_2)\big) \to r_2(o_2, a_2)\Big) \\
&= \bigwedge_{o_2\in B_2} \bigwedge_{o_1\in B_1} \Big(\big(f(o_1) \otimes \varphi_L(o_1)(o_2)\big) \to r_2(o_2, a_2)\Big) \\
&= \bigwedge_{o_2\in B_2} \bigwedge_{o_1\in B_1} \Big(f(o_1) \to \big(\varphi_L(o_1)(o_2) \to r_2(o_2, a_2)\big)\Big) \\
&= \bigwedge_{o_1\in B_1} \Big(f(o_1) \to \bigwedge_{o_2\in B_2} \big(\varphi_L(o_1)(o_2) \to r_2(o_2, a_2)\big)\Big) \\
&= \bigwedge_{o_1\in B_1} \big(f(o_1) \to \widehat{r_2}(\varphi_L(o_1), \chi_{a_2})\big) \\
&= \bigwedge_{o_1\in B_1} \big(f(o_1) \to \widehat{r_1}(\chi_{o_1}, \varphi_R(a_2))\big) \\
&\;\;\vdots \\
&= \bigwedge_{o_1\in B_1} \bigwedge_{a_1\in A_1} \Big(\big(\varphi_R(a_2)(a_1) \otimes f(o_1)\big) \to r_1(o_1, a_1)\Big) \\
&= \bigwedge_{o_1\in B_1} \bigwedge_{a_1\in A_1} \Big(\varphi_R(a_2)(a_1) \to \big(f(o_1) \to r_1(o_1, a_1)\big)\Big) \\
&= \bigwedge_{a_1\in A_1} \Big(\varphi_R(a_2)(a_1) \to \bigwedge_{o_1\in B_1} \big(f(o_1) \to r_1(o_1, a_1)\big)\Big) \\
&= \bigwedge_{a_1\in A_1} \big(\varphi_R(a_2)(a_1) \to \uparrow_1 (f)(a_1)\big) = \varphi_R^+(\uparrow_1 (f)))(a_2)
\end{aligned}$$

The other equation can be proved similarly.

For the second part of the statement we will use that

$$\uparrow_2 (\varphi_L(o_1)) = \uparrow_2 (\varphi_{L^+}(\chi_{o_1})) = \varphi_R^+(\uparrow_1 (\chi_{o_1}))$$

and the definition of Chu correspondences which directly leads to $\varphi_L(o_1) = \downarrow_2 (\uparrow_2 (\varphi_L(o_1))) = \downarrow_2 (\varphi_R^+(\uparrow_1 (\chi_{o_1})))$. Again, the second equation can be proved similarly. □

3.1. The category L-ChuCors

We introduce now the category of L-Chu correspondences between L-fuzzy formal contexts as follows:

- **objects** L-fuzzy formal contexts
- **arrows** L-Chu correspondences
- **identity arrow** $\iota\colon C \to C$ of L-context $C = \langle B, A, r\rangle$
  - $\iota_L(o) = \downarrow\uparrow (\chi_o)$, for all $o \in B$
  - $\iota_R(a) = \uparrow\downarrow (\chi_a)$, for all $a \in A$
- **composition** $\varphi_2 \circ \varphi_1 : C_1 \to C_3$ **of arrows** $\varphi_1 : C_1 \to C_2$, $\varphi_2 : C_2 \to C_3$ ($C_i = \langle B_i, A_i, r_i\rangle$, $i \in \{1, 2\}$)
  - $(\varphi_2 \circ \varphi_1)_L : B_1 \to L^{B_3}$ and $(\varphi_2 \circ \varphi_1)_R : A_3 \to L^{A_1}$
  - $(\varphi_2 \circ \varphi_1)_L(o_1) = \downarrow_3\uparrow_3 (\varphi_{2L^+}(\varphi_{1L}(o_1)))$, where
  $$\varphi_{2L^+}(\varphi_{1L}(o_1))(o_3) = \bigvee_{o_2\in B_2} \varphi_{1L}(o_1)(o_2) \otimes \varphi_{2L}(o_2)(o_3)$$
  - $(\varphi_2 \circ \varphi_1)_R(a_3) = \uparrow_1\downarrow_1 (\varphi_{1R^+}(\varphi_{2R}(a_3)))$, where
  $$\varphi_{1R^+}(\varphi_{2R}(a_3))(a_1) = \bigvee_{a_2\in A_2} \varphi_{2R}(a_3)(a_2) \otimes \varphi_{1R}(a_2)(a_1)$$

Lemma 3.2. Let $C = \langle B, A, r\rangle$ be the L-fuzzy formal context. The identity arrow $\iota\colon C \to C$ defined above is a Chu correspondence.

Proof:
Consider the following chain of equalities:

$$\begin{aligned}
\widehat{r}(\iota_L(\chi_o), \chi_a) &= \bigwedge_{b\in B} (\downarrow\uparrow (\chi_o)(b) \to r(b, a)) = \uparrow\downarrow\uparrow (\chi_o)(a) = \uparrow (\chi_o)(a)\\
&= \bigwedge_{b\in B} (\chi_o(b) \to r(b, a)) = r(o, a) = \bigwedge_{d\in A} (\chi_a(d) \to r(o, d))\\
&= \downarrow (\chi_a)(o) = \downarrow\uparrow\downarrow (\chi_a)(o) = \bigwedge_{d\in A} (\uparrow\downarrow (\chi_a)(d) \to r(o, d))\\
&= \widehat{r}(\chi_o, \iota_R(\chi_a))
\end{aligned}$$

Hence ι is a weak Chu correspondence, but $\iota_L(o)$ and $\iota_R(a)$ are closed in C for all $o \in B$ and all $a \in A$, so $\iota \in L$-ChuCors(C, C). □

Lemma 3.3. Let $C_i = \langle B, A, r\rangle, i \in \{1,2\}$ be the L-fuzzy formal contexts. For identity arrows of L-ChuCors $\iota_i\colon C_i \to C_i, i \in \{1,2\}$ and $\varphi \in L\text{-ChuCors}(C_1, C_2)$ the following equalities hold:

$$(\iota_2 \circ \varphi) = \varphi \text{ and } (\varphi \circ \iota_1) = \varphi$$

Proof:
$(\iota_2 \circ \varphi)_L(o_2) = \downarrow_2\uparrow_2 (\iota_{2L+}(\varphi_L(o_1)))(o_2)$

$$\begin{aligned}
\uparrow_2 (\iota_{2L+}(\varphi_L(o_1)))(a_2) &= \bigwedge_{o_2\in B_2} (\iota_{2L+}(\varphi_L(o_1))(o_2) \to r_2(o_2, a_2)) \\
&= \bigwedge_{o_2\in B_2} (\bigvee_{b_2\in B_2} (\iota_{2L}(b_2)(o_2) \otimes \varphi_L(o_1)(b_2)) \to r_2(o_2, a_2)) \\
&= \bigwedge_{o_2\in B_2} \bigwedge_{b_2\in B_2} (\varphi_L(o_1)(b_2) \to (\iota_{2L}(b_2)(o_2) \to r_2(o_2, a_2))) \\
&= \bigwedge_{b_2\in B_2} (\varphi_L(o_1)(b_2) \to \bigwedge_{o_2\in B_2} (\iota_{2L}(b_2)(o_2) \to r_2(o_2, a_2))) \\
&= \bigwedge_{b_2\in B_2} (\varphi_L(o_1)(b_2) \to \uparrow_2 (\iota_{2L}(b_2))(a_2)) \\
&= \bigwedge_{b_2\in B_2} (\varphi_L(o_1)(b_2) \to \uparrow_2\downarrow_2\uparrow_2 (\chi_{b_2})(a_2)) \\
&= \bigwedge_{b_2\in B_2} (\varphi_L(o_1)(b_2) \to \uparrow_2 (\chi_{b_2})(a_2)) \\
&= \bigwedge_{b_2\in B_2} (\varphi_L(o_1)(b_2) \to r_2(b_2, a_2)) \\
&= \uparrow_2 (\varphi_L(o_1))(a_2)
\end{aligned}$$

So $(\iota_2 \circ \varphi)_L(o_2) = \downarrow_2\uparrow_2 (\varphi_L(o_1)) = \varphi_L(o_1)$. The second equation can be proved similarly. □

Lemma 3.4. Let $C_i = \langle B_i, A_i, r_i\rangle$ be an L-fuzzy formal context for $i \in \{1,2,3\}$, and let $\varphi_j\colon C_j \to C_{j+1}$ be an L-Chu correspondence for $j \in \{1,2\}$, then composition $\varphi_2 \circ \varphi_1 : C_1 \to C_3$ defined above is an L-Chu correspondence.

Proof:
Let $o_1 \in B_1$ be an arbitrary object of C_1 and $a_3 \in A_3$ be an arbitrary attribute of C_3

$$\begin{aligned}
\widehat{r}_1(\chi_{o_1}, (\varphi_2 \circ \varphi_1)_R(a_3)) &= \widehat{r}_1(\chi_{o_1}, \uparrow_1\downarrow_1 (\varphi_{1R+}(\varphi_{2R}(a_3)))) \\
&= \bigwedge_{a_1\in A_1} (\uparrow_1\downarrow_1 (\varphi_{1R+}(\varphi_{2R}(a_3)))(a_1) \to r_1(o_1, a_1)) \\
&= \downarrow_1\uparrow_1\downarrow_1 (\varphi_{1R+}(\varphi_{2R}(a_3)))(o_1) \\
&= \downarrow_1 (\varphi_{1R+}(\varphi_{2R}(a_3)))(o_1)
\end{aligned}$$

$$
\begin{aligned}
&= \bigwedge_{a_1 \in A_1} (\varphi_{1R+}(\varphi_{2R}(a_3))(a_1) \to r_1(o_1, a_1)) \\
&= \bigwedge_{a_1 \in A_1} (\bigvee_{a_2 \in A_2} (\varphi_{2R}(a_3)(a_2) \otimes \varphi_{1R}(a_2)(a_1)) \to r_1(o_1, a_1)) \\
&= \bigwedge_{a_1 \in A_1} \bigwedge_{a_2 \in A_2} ((\varphi_{2R}(a_3)(a_2) \otimes \varphi_{1R}(a_2)(a_1)) \to r_1(o_1, a_1)) \\
&= \bigwedge_{a_1 \in A_1} \bigwedge_{a_2 \in A_2} (\varphi_{2R}(a_3)(a_2) \to (\varphi_{1R}(a_2)(a_1) \to r_1(o_1, a_1))) \\
&= \bigwedge_{a_2 \in A_2} (\varphi_{2R}(a_3)(a_2) \to \bigwedge_{a_1 \in A_1} (\varphi_{1R}(a_2)(a_1) \to r_1(o_1, a_1))) \\
&= \bigwedge_{a_2 \in A_2} (\varphi_{2R}(a_3)(a_2) \to \widehat{r}_1(\chi_{o_1}, \varphi_{1R}(a_2))) \\
&= \bigwedge_{a_2 \in A_2} (\varphi_{2R}(a_3)(a_2) \to \widehat{r}_2(\varphi_{1L}(o_1), \chi_{a_2})) \\
&\quad \vdots \\
&= \bigwedge_{o_2 \in B_2} (\varphi_{1L}(o_1)(o_2) \to \widehat{r}_2(\chi_{o_2}, \varphi_{2R}(a_3))) \\
&= \bigwedge_{o_2 \in B_2} (\varphi_{1L}(o_1)(o_2) \to \widehat{r}_3(\varphi_{2L}(o_2), \chi_{a_3})) \\
&\quad \vdots \\
&= \uparrow_3 (\varphi_{2L+}(\varphi_{1L}(o_1)))(a_3) \\
&= \uparrow_3\downarrow_3\uparrow_3 (\varphi_{2L+}(\varphi_{1L}(o_1)))(a_3) \\
&= \bigwedge_{o_3 \in B_3} (\downarrow_3\uparrow_3 (\varphi_{2L+}(\varphi_{1L}(o_1)))(o_3) \to r_3(o_3, a_3)) \\
&= \widehat{r}_3(\downarrow_3\uparrow_3 (\varphi_{2L+}(\varphi_{1L}(o_1))), \chi_{a_3}) \\
&= \widehat{r}_3((\varphi_2 \circ \varphi_1)_L(o_1), \chi_{a_3})
\end{aligned}
$$

Hence the composition is a weak L-Chu correspondence. Moreover, either the left or the right side is a closed set of objects in C_3 or a closed set of attributes in C_1, respectively. Thus, the composition is an L-Chu correspondence. □

Lemma 3.5. Let $\varphi\colon C_1 \to C_2$ be the Chu correspondence between L-fuzzy formal contexts C_1 and C_2. Then for any arbitrary L-fuzzy set of objects $f \in L^{B_1}$ or attributes $g \in L^{A_1}$ the following equalities hold:

$$\downarrow_2\uparrow_2 (\varphi_{L+}(\downarrow_1\uparrow_1 (f))) = \downarrow_2\uparrow_2 (\varphi_{L+}(f)) \quad \text{and} \quad \uparrow_1\downarrow_1 (\varphi_{R+}(\uparrow_2\downarrow_2 (g))) = \uparrow_1\downarrow_1 (\varphi_{R+}(g)).$$

Proof:
By Lemma 3.1 we have $\uparrow_2 (\varphi_{L+}(\downarrow_1\uparrow_1 (f))) = \varphi_R^+(\uparrow_1\downarrow_1\uparrow_1 (f)) = \varphi_R^+(\uparrow_1 (f)) = \uparrow_2 (\varphi_{L+}(f))$. Hence $\downarrow_2\uparrow_2 (\varphi_{L+}(\downarrow_1\uparrow_1 (f))) = \downarrow_2\uparrow_2 (\varphi_{L+}(f))$.

The second equality can be proved similarly. □

Lemma 3.6. Let $C_i = \langle B_i, A_i, r_i \rangle$ for $i \in \{1,2,3,4\}$ be the L-fuzzy contexts. $\varphi_i : C_i \to C_{i+1}$ for $i \in \{1,2,3\}$ be the Chu correspondences. Then the associativity of composition holds $\varphi_3 \circ (\varphi_2 \circ \varphi_1) = (\varphi_3 \circ \varphi_2) \circ \varphi_1$.

Proof:
Consider the following chain of equalities:

$$((\varphi_3 \circ \varphi_2) \circ \varphi_1)_L(o_1) = \downarrow_4\uparrow_4 \left((\varphi_3 \circ \varphi_2)_{L^+}(\varphi_{1L}(o_1))\right)$$
$$= \downarrow_4\uparrow_4 \left(\downarrow_4\uparrow_4 \left(\varphi_{3L^+}(\varphi_{2L^+}(\varphi_{1L}(o_1)))\right)\right)$$

by the property of closure operator

$$= \downarrow_4\uparrow_4 \left(\varphi_{3L^+}(\varphi_{2L^+}(\varphi_{1L}(o_1)))\right)$$

by lemma 3.5

$$= \downarrow_4\uparrow_4 \left(\varphi_{3L^+}(\downarrow_3\uparrow_3 \left(\varphi_{2L^+}(\varphi_{1L}(o_1))\right))\right)$$
$$= \downarrow_4\uparrow_4 \left(\varphi_{3L^+}((\varphi_2 \circ \varphi_1)_L(o_1))\right)$$
$$= (\varphi_3 \circ (\varphi_2 \circ \varphi_1))_L(o_1)$$

□

As a consequence of the previous lemmas, we obtain the main result of this section.

Theorem 3.1. L-fuzzy Chu correspondences between L-fuzzy formal contexts form a category.

3.2. L-ChuCors embeds ChuCors

In this section we will show that the category of L-Chu correspondences embeds the category of classical Chu correspondences.

To begin with, recall that a *subcategory* S of a category C is a collection of some of the objects and some of the arrows of C which includes with each arrow f, both its domain and codomain, with each object its identity arrow, and with each pair of composable arrows, their composite. We will consider two complete residuated lattices $\langle L_1, \wedge^1, \vee^1, \otimes^1, \to^1, 0, 1\rangle$ and $\langle L_2, \wedge^2, \vee^2, \otimes^2, \to^2, 0, 1\rangle$ satisfying

- $\{0,1\} \subseteq L_1 \subseteq L_2$
- $\langle L_1, \wedge^1, \vee^1, 0, 1\rangle$ is a complete sublattice of $\langle L_2, \wedge^2, \vee^2, 0, 1\rangle$
- $k \otimes^1 l = k \otimes^2 l$ and $k \to^1 l = k \to^2 l$ for all $k, l \in L_1$

From the fact that $L_1 \subseteq L_2$ is easy to see that every L_1-context is an L_2-context as well. So every object of L_1-ChuCors is an object of L_2-ChuCors.

Lemma 3.7. For any L_1-context $C = \langle B, A, r\rangle$ we have that L_1-$FCL(C) \subseteq L_2$-$FCL(C)$.

Proof:
We will write $\uparrow^i$ and $\downarrow^i$ to denote the up- and down-arrow mappings defined on the L_i-context.

$$\uparrow^1 (f)(a) = \bigwedge_{o\in B}^{1} (f(o) \to^1 r(o,a)) = \bigwedge_{o\in B}^{2} (f(o) \to^1 r(o,a)) = \bigwedge_{o\in B}^{2} (f(o) \to^2 r(o,a)) = \uparrow^2 (f)(a)$$

And similarly for $\downarrow^1$ and $\downarrow^2$. As a result $\langle f, g\rangle \in L_1\text{-}FCL(C)$ implies $g = \uparrow^1 (f) = \uparrow^2 (f)$ and $f = \downarrow^1 (g) = \downarrow^2 (g)$. Hence $\langle f, g\rangle \in L_2\text{-}FCL(C)$. □

Now, consider an arbitrary $\varphi \in L_1$-ChuCors(C_1, C_2) for certain $C_i = \langle B_i, A_i, r_i\rangle$. Then

- $\varphi_L \colon B_1 \to L_1^{B_2} \subseteq L_2^{B_2}$, $\varphi_L(o_1)$ is closed in C_2 for any $o_1 \in B_1$
- $\varphi_R \colon A_2 \to L_1^{A_1} \subseteq L_2^{A_1}$, $\varphi_R(a_2)$ is closed in C_1 for any $a_2 \in A_2$

Note that "closed" in the items above is meant as L_1-closed, but Lemma 3.7 guarantees that are L_2-closed too, moreover

$$\uparrow_2^2 (\varphi_L(o_1))(a_2) = \uparrow_2^1 (\varphi_L(o_1))(a_2) = \downarrow_1^1 (\varphi_R(a_2))(o_1) = \downarrow_1^2 (\varphi_R(a_2))(o_1)$$

From the previous facts is clear that every $\varphi \in L_1$-ChuCors(C_1, C_2) is in L_2-ChuCors(C_1, C_2). So every arrow of L_1-ChuCors is an arrow in L_2-ChuCors.

Theorem 3.2. L_1-ChuCors is a subcategory of L_2-ChuCors.

Proof:
We have just to check the following items:

1. From the previous facts, it follows that $\iota_C \in L_1$-ChuCors$(C, C) \subseteq L_2$-ChuCors(C, C).
2. For all L_1-Chu correspondence φ viewed as an L_2-Chu correspondence, its source and target are, obviously, L_1-contexts.
3. For any two arrows $\varphi \colon C_1 \to C_2, \psi \colon C_2 \to C_3$, where C_i is L_1-context for all $i \in \{1, 2, 3\}$, the composition $\psi \circ \varphi$ is in L_1-ChuCors$(C_1, C_3) \subseteq L_2$-ChuCors(C_1, C_3).

□

Corollary 3.1. ChuCors is a subcategory of L-ChuCors.

As a consequence, we have that the inclusion functor is faithful. However, the following example shows that it is not full.

Example 3.1. Let us consider the complete residuated lattice structure defined on Belnap's diamond $L = \langle\{1, a, b, 0\}, \vee, \wedge, \otimes, \to, 0, 1\rangle$ where the conjunction $k \otimes l$ coincides with the meet $k \wedge l$ and $k \to l$ is its residuated implication $\bigwedge\{m \in L \mid m \otimes k \leq l\}$. Moreover, consider the two "classical" L-contexts $C_1 = \langle\{o_1\}, \{a_1\}, r_1(o_1, a_1) = 0\rangle$ and $C_2 = \langle\{o_2\}, \{a_2\}, r_2(o_2, a_2) = 0\rangle$.

It is not difficult to see that there exist L-Chu correspondences between C_1 and C_2 which are not classical. Specifically, we have L-ChuCors$(C_1, C_2) = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ where each L-Chu correspondence φ_i is defined as follows:

φ_{1L}	o_2	φ_{1R}	a_1
o_1	0	a_2	0

φ_{2L}	o_2	φ_{2R}	a_1
o_1	1	a_2	1

φ_{3L}	o_2	φ_{3R}	a_1
o_1	b	a_2	b

φ_{4L}	o_2	φ_{4R}	a_1
o_1	a	a_2	a

Hence, we have that the subcategory if classical Chu correspondences is not a full subcategory of L-ChuCors.

4. On the structure of L-bonds

The definition of L-bonds is based on the notion of multifunction, introduced in Definition 2.5. The usual definition of curry and uncurry operations can be adapted to the framework of L-multifunctions as follows:

Definition 4.1. Let us define for an arbitrary L-multifunction $\varphi \in L\text{-}Mfn(X,Y)$ an L-fuzzy relation $\varphi^{\mathrm{r}} \in L^{X\times Y}$ defined by $\varphi^{\mathrm{r}}(x,y) = \varphi(x)(y)$ for all $(x,y) \in X \times Y$. For arbitrary L-fuzzy relation $r \in L^{X\times Y}$ lets define an L-multifunction from $r^{\mathrm{mfn}} : X \to L^Y$ defined by $r^{\mathrm{mfn}}(x)(y) = r(x,y)$.

Finally, the notion of L-bond is given in the following definition:

Definition 4.2. An **L-bond** between two formal contexts $C_1 = \langle B_1, A_1, r_1\rangle$ and $C_2 = \langle B_2, A_2, r_2\rangle$ is a multifunction $\beta\colon B_1 \to L^{A_2}$ satisfying the condition that for all $o_1 \in B_1$ and $a_2 \in A_2$ both $\beta(o_1)$ and $\beta^{\mathrm{t}}(a_2)$ are closed L-fuzzy sets of, respectively, attributes in C_2 and objects in C_1. The set of all bonds from C_1 to C_2 is denoted as $L\text{-}Bonds(C_1, C_2)$.

Lemma 4.1. Let $\langle B_i, A_i, r_i\rangle$ be two L-fuzzy formal contexts for $i \in \{1,2\}$, where L satisfies the double negation law. For all L-bonds $\beta \in L\text{-}Bonds(C_1, C_2)$ and for all objects $o_1 \in B_1$ the equation $\beta(o_1) = \beta_+(\downarrow_{\neg_1}\uparrow_{\neg_1}(\chi_{o_1}))$ holds.

Proof:
We will prove the two inequalities separately.

$$\begin{aligned}
\beta(o_1)(a_2) &= \bigvee_{b_1\in B_1} (\beta(b_1)(a_2) \otimes \chi_{o_1}(b_1)) \\
&\leq \bigvee_{b_1\in B_1} (\beta(b_1)(a_2) \otimes \downarrow_{\neg_1}\uparrow_{\neg_1}(\chi_{o_1})(b_1)) = \beta_+(\downarrow_{\neg_1}\uparrow_{\neg_1}(\chi_{o_1}))(a_2)
\end{aligned}$$

For the other inequality, consider the following chain

$$\begin{aligned}
\beta_+(\downarrow_{\neg_1}\uparrow_{\neg_1}(\chi_{o_1}))(a_2) &= \bigvee_{b_1\in B_1}(\beta(b_1)(a_2)\otimes\downarrow_{\neg_1}\uparrow_{\neg_1}(\chi_{o_1})(b_1)) \\
&\overset{*}{=} \bigvee_{b_1\in B_1}(\beta(b_1)(a_2)\otimes\downarrow_1\uparrow_1(\chi_{b_1})(o_1)) \\
&= \bigvee_{b_1\in B_1}(\beta^{\mathrm{t}}(a_2)(b_1)\otimes\bigwedge_{a_1\in A_1}(\uparrow_1(\chi_{b_1})(a_1)\to r_1(o_1,a_1)))
\end{aligned}$$

$\beta^{\mathrm{t}}(a_2)$ is a closed L-set of objects of B_1, then $\beta^{\mathrm{t}}(a_2)(b_1)=\downarrow_1(g)(b_1)$ for some $g\in L^{A_1}$

$$\begin{aligned}
&= \bigvee_{b_1\in B_1}(\downarrow_1(g)(b_1)\otimes\bigwedge_{a_1\in A_1}((1\to r_1(b_1,a_1))\to r_1(o_1,a_1))) \\
&= \bigvee_{b_1\in B_1}(\bigwedge_{a_1\in A_1}g(a_1)\to r_1(b_1,a_1))\otimes\bigwedge_{a_1\in A_1}(r_1(b_1,a_1)\to r_1(o_1,a_1))) \\
&\overset{\star}{=} \bigvee_{b_1\in B_1}\bigwedge_{a_1\in A_1}((g(a_1)\to r_1(b_1,a_1))\otimes(r_1(b_1,a_1)\to r_1(o_1,a_1))) \\
&\leq \bigvee_{b_1\in B_1}\bigwedge_{a_1\in A_1}(g(a_1)\to r_1(o_1,a_1)) = \\
&= \bigvee_{b_1\in B_1}\downarrow_1(g)(o_1) = \bigvee_{b_1\in B_1}\beta^{\mathrm{t}}(a_2)(o_1) = \bigvee_{b_1\in B_1}\beta(o_1)(a_2) \\
&= \beta(o_1)(a_2)
\end{aligned}$$

where $(\star)$ follows from the inequality $(k\to l)\otimes(l\to m)\leq k\to l$ which holds for all $k,l,m\in L$. □

4.1. Direct product of two L-fuzzy contexts

Here we introduce the corresponding extension of the notion of direct product of two L-fuzzy contexts.

Definition 4.3. The *direct product* of two L-fuzzy contexts $C_1=\langle B_1,A_1,r_1\rangle$ and $C_2=\langle B_2,A_2,r_2\rangle$ is an L-fuzzy context $C_1\Delta C_2=\langle B_1\times A_2,A_1\times B_2,\Delta\rangle$, where Δ is defined as $\Delta((o_1,a_2),(a_1,o_2))=\neg r_1(o_1,a_1)\to r_2(o_2,a_2)$.

The following result states properties of the just defined direct product of L-fuzzy contexts.

Lemma 4.2. Let $C_1=\langle B_1,A_1,r_1\rangle$ and $C_2=\langle B_2,A_2,r_2\rangle$ be two L-fuzzy contexts, where L satisfies the double negation law. Given two arbitrary L-multifunctions $\varphi\colon B_1\to L^{A_2}$ and $\psi\colon A_2\to L^{B_1}$, for all $o_1\in B_1$ and $a_2\in A_2$ the following equalities hold

$$\uparrow_\Delta(\varphi^{\mathrm{r}})(o_2,a_1)=\downarrow_2(\varphi_+(\downarrow_{\neg_1}(\chi_{a_1})))(o_2)=\uparrow_1(\varphi^{\mathrm{t}}_+(\uparrow_{\neg_2}(\chi_{o_2})))(a_1)$$
$$\downarrow_\Delta(\psi^{\mathrm{r}})(o_1,a_2)=\uparrow_2(\psi_+(\uparrow_{\neg_1}(\chi_{o_1})))(a_2)=\downarrow_1(\psi^{\mathrm{t}}_+(\downarrow_{\neg_2}(\chi_{a_2})))(o_1)$$

Proof:
Consider the following chain of equalities:

$$\begin{aligned}
\uparrow_\Delta (\varphi^{\mathrm{r}})(o_2, a_1) &= \bigwedge_{(o_1,a_2)\in B_1\times A_2} (\varphi^{\mathrm{r}}(o_1, a_2) \to \Delta((o_1, a_2), (o_2, a_1))) \\
&= \bigwedge_{(o_1,a_2)\in B_1\times A_2} (\varphi^{\mathrm{r}}(o_1, a_2) \to (\neg r_1(o_1, a_1) \to r_2(o_2, a_2))) \\
&= \bigwedge_{(o_1,a_2)\in B_1\times A_2} ((\varphi^{\mathrm{r}}(o_1, a_2) \otimes \neg r_1(o_1, a_1)) \to r_2(o_2, a_2)) \\
&= \bigwedge_{(o_1,a_2)\in B_1\times A_2} ((\varphi^{\mathrm{r}}(o_1, a_2) \otimes (1 \to \neg r_1(o_1, a_1))) \to r_2(o_2, a_2)) \\
&= \bigwedge_{(o_1,a_2)\in B_1\times A_2} ((\varphi^{\mathrm{r}}(o_1, a_2) \otimes \bigwedge_{t_1\in A_1} (\chi_{a_1}(t_1) \to \neg r_1(o_1, t_1))) \to r_2(o_2, a_2)) \\
&= \bigwedge_{(o_1,a_2)\in B_1\times A_2} ((\varphi^{\mathrm{r}}(o_1, a_2) \otimes \downarrow_{\neg_1} (\chi_{a_1})(o_1)) \to r_2(o_2, a_2)) \\
&= \bigwedge_{o_1\in B_1} \bigwedge_{a_2\in A_2} ((\varphi^{\mathrm{r}}(o_1, a_2) \otimes \downarrow_{\neg_1} (\chi_{a_1})(o_1)) \to r_2(o_2, a_2)) \\
&= \bigwedge_{o_1\in B_1} \bigwedge_{a_2\in A_2} (\downarrow_{\neg_1} (\chi_{a_1})(o_1) \otimes (\varphi(o_1)(a_2)) \to r_2(o_2, a_2)) \\
&= \bigwedge_{o_1\in B_1} \downarrow_2 (\downarrow_{\neg_1} (\chi_{a_1})(o_1) \otimes \varphi(o_1))(o_2) \\
&= \downarrow_2 (\bigvee_{o_1\in B_1} (\downarrow_{\neg_1} (\chi_{a_1})(o_1) \otimes \varphi(o_1)))(o_2) \\
&= \downarrow_2 (\varphi(\downarrow_{\neg_1} (\chi_{a_1})))(o_2)
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\uparrow_\Delta (\varphi^{\mathrm{r}})(o_2, a_1) &= \bigwedge_{(o_1,a_2)\in B_1\times A_2} (\varphi^{\mathrm{r}}(o_1, a_2) \to (\neg r_1(o_1, a_1) \to r_2(o_2, a_2))) \\
&= \bigwedge_{(o_1,a_2)\in B_1\times A_2} (\varphi^{\mathrm{t}}(a_2)(o_1) \to (\neg r_2(o_2, a_2) \to r_1(o_1, a_1))) \\
&\ \vdots \\
&= \uparrow_1 (\varphi^{\mathrm{t}}_+(\uparrow_{\neg_2} (\chi_{o_2})))(a_1)
\end{aligned}$$

□

4.2. L-bonds vs direct products of L-fuzzy contexts

The main contribution of this section is presented here, in which a relationship between L-bonds and extents of direct products of L-fuzzy contexts is drawn by the following theorem.

Theorem 4.1. Let $C_i = \langle B_i, A_i, r_i \rangle$ be L-fuzzy contexts for $i \in \{1,2\}$, where L satisfies the double negation law. Let $\beta \in L\text{-}Mfn(B_1, A_2)$. Then:

1. If β^{r} is an extent of $C_1 \Delta C_2$, then $\beta \in L\text{-}Bond(C_1, C_2)$.
2. If $\beta \in L\text{-}Bond(C_1, C_2)$ and

$$\beta_+(\downarrow_{\neg_1}\uparrow_{\neg_1}(\chi_{o_1}))(a_2) = \bigwedge_{a_1 \in A_1} (\uparrow_{\neg_1}(\chi_{o_1})(a_1) \to \uparrow_2\downarrow_2(\beta_+(\downarrow_{\neg_1}(\chi_{a_1})))(a_2))$$

then ${}^{\mathrm{r}}\beta$ is an extent of $C_1 \Delta C_2$.

Proof:
1. For the first item, let β be an extent of $C_1 \Delta C_2$, then we know that $\beta(o_1)(a_2) = \downarrow_\Delta\uparrow_\Delta(\beta^{\mathrm{r}})(o_1, a_2)$
Let us write $\uparrow_\Delta(\beta^{\mathrm{r}})^{\mathrm{mfn}} = \psi$, then

$$\beta(o_1)(a_2) = \downarrow_\Delta(\psi)(o_1, a_2) = \uparrow_2(\psi^+(\uparrow_{\neg_1}(\chi_{o_1})))(a_2)$$

As a result, $\beta(o_1)$ is a closed L-set from L^{A_2}.

Similarly, we have that $\beta^{\mathrm{t}}(a_2)(o_1) = \downarrow_1(\psi^{+\mathrm{t}}(\downarrow_{\neg_2}(\chi_{a_2})))(o_1)$. Hence $\beta^{\mathrm{t}}(a_2)$ is a closed L-set of objects from L^{B_1}.

2. The proof for the second item is as follows:

$$\begin{aligned}
&\bigwedge_{a_1 \in A_1} (\uparrow_{\neg_1}(\chi_{o_1})(a_1) \to \uparrow_2\downarrow_2(\beta_+(\downarrow_(\neg\chi_{a_1})))(a_2)) = \\
&= \bigwedge_{a_1 \in A_1} (\uparrow_{\neg_1}(\chi_{o_1})(a_1) \to \bigwedge_{o_2 \in B_2} (\downarrow_2(\beta_+(\downarrow_{\neg_1}(\chi_{a_1})))(o_2) \to r_2(o_2, a_2))) \\
&= \bigwedge_{a_1 \in A_1} \bigwedge_{o_2 \in B_2} (\uparrow_{\neg_1}(\chi_{o_1})(a_1) \to (\downarrow_2(\beta_+(\downarrow_{\neg_1}(\chi_{a_1})))(o_2) \to r_2(o_2, a_2))) \\
&= \bigwedge_{a_1 \in A_1} \bigwedge_{o_2 \in B_2} ((\uparrow_{\neg_1}(\chi_{o_1})(a_1) \otimes \downarrow_2(\beta_+(\downarrow_{\neg_1}(\chi_{a_1})))(o_2)) \to r_2(o_2, a_2)) \\
&= \bigwedge_{o_2 \in B_2} (\bigvee_{a_1 \in A_1} (\uparrow_{\neg_1}(\chi_{o_1})(a_1) \otimes \downarrow_2(\beta_+(\downarrow_{\neg_1}(\chi_{a_1})))(o_2)) \to r_2(o_2, a_2)) \\
&= \uparrow_2(\bigvee_{a_1 \in A_1} (\uparrow_{\neg_1}(\chi_{o_1})(a_1) \otimes \downarrow_2(\beta_+(\downarrow_{\neg_1}(\chi_{a_1}))))(a_2) \\
&= \uparrow_2(\bigvee_{a_1 \in A_1} (\uparrow_{\neg_1}(\chi_{o_1})(a_1) \otimes (\uparrow_\Delta(\beta^{\mathrm{r}}))^{\mathrm{mfn}}(a_1))(a_2) \\
&= \uparrow_2((\uparrow_\Delta(\beta^{\mathrm{r}}))^{\mathrm{mfn}+}(\uparrow_{\neg_1}(\chi_{o_1})))(a_2) \\
&= \downarrow_\Delta\uparrow_\Delta(\beta^{\mathrm{r}})(o_1, a_2) \\
&\stackrel{\star}{=} \beta_+(\downarrow_{\neg_1}\uparrow_{\neg_1}(\chi_{o_1}))(a_2) = \beta(o_1)(a_2)
\end{aligned}$$

where $(\star)$ follows, firstly, from the hypothesis, which states that it equals to $\beta_+(\downarrow_{\neg_1}\uparrow_{\neg_1}(\chi_{o_1}))(a_2)$ and, as $\beta \in L\text{-}Bond(C_1, C_2)$, by Lemma 4.1. □

The following theorem generalizes the previous one, presented in [17], by providing a characterization instead of just an implication.

Theorem 4.2. Let $C_i = \langle B_i, A_i, r_i \rangle$ for $i \in \{1, 2\}$ be two L-fuzzy contexts, where L satisfies the double negation law. $\langle \beta, \gamma \rangle \in L\text{-}FCL(C_1 \Delta C_2)$ if and only if $\beta \in L$-(Bonds)$(\mathrm{C}_1, \mathrm{C}_2)$, $\gamma \in L$-Bonds$(\mathrm{C}_2, \mathrm{C}_1)$ and for L-Chu correspondences asigned to β and γ following equalities hold:

$$\varphi_{\beta R}(a_2)(a_1) = \uparrow_1\downarrow_1 (\neg\varphi^t_{\gamma R}(a_2))(a_1) \quad \text{and} \quad \varphi_{\gamma L}(o_2)(o_1) = \uparrow_1\downarrow_1 (\neg\varphi^t_{\beta L}(o_2))(o_1).$$

Proof:
Let us assume that $\langle \beta^r, \gamma^r \rangle \in L\text{-}FCL(C_1 \Delta C_2)$. Then

$$\begin{aligned}
\beta(o_1)(a_2) &= \downarrow_\Delta (\gamma)(o_1, a_2) = \bigwedge_{o_2 \in B_2} \bigwedge_{a_1 \in A_1} (\gamma(o_2)(a_1) \to (\neg r_1(o_1, a_1) \to r_2(o_2, a_2))) \\
&= \bigwedge_{o_2 \in B_2} \bigwedge_{a_1 \in A_1} ((\gamma(o_2)(a_1) \otimes \neg r_1(o_1, a_1)) \to r_2(o_2, a_2))) \\
&= \bigwedge_{o_2 \in B_2} \bigwedge_{a_1 \in A_1} (\neg r_1(o_1, a_1) \to (\gamma(o_2)(a_1) \to r_2(o_2, a_2))) \\
&= \bigwedge_{a_1 \in A_1} (\neg r_1(o_1, a_1) \to \uparrow_2 (\gamma^t(a_1))(a_2)) = \bigwedge_{a_1 \in A_1} (\neg r_1(o_1, a_1) \to \varphi_{\gamma R}(a_1)(a_2)) \\
&= \bigwedge_{a_1 \in A_1} (\neg \varphi_{\gamma R}(a_1)(a_2) \to r_1(o_1, a_1)) = \downarrow_1 (\neg \varphi^t_\gamma(a_2))(o_1)
\end{aligned}$$

Hence $\varphi_{\beta R}(a_2)(a_1) = \uparrow_1 (\beta^t(a_2))(a_1) = \uparrow_1\downarrow_1 (\neg\varphi^t_\gamma(a_2))(o_1)$. The second equivalence can be proved similarly.

Now let us assume that equality $\varphi_{\gamma L}(o_2)(o_1) = \uparrow_1\downarrow_1 (\neg\varphi^t_{\beta L}(o_2))(o_1)$ holds. Then

$$\begin{aligned}
\uparrow_\Delta (\beta^r)(o_1, a_2) &= \bigwedge_{o_1 \in B_1} \bigwedge_{a_2 \in A_2} (\beta(o_1)(a_2) \to (\neg r_1(o_1, a_1) \to r_2(o_2, a_2))) \\
&= \bigwedge_{o_1 \in B_1} \bigwedge_{a_2 \in A_2} (\neg r_1(o_1, a_1) \to (\beta(o_1)(a_2) \to r_2(o_2, a_2))) \\
&= \bigwedge_{o_1 \in B_1} (\neg r_1(o_1, a_1) \to \bigwedge_{a_2 \in A_2} (\beta(o_1)(a_2) \to r_2(o_2, a_2))) \\
&= \bigwedge_{o_1 \in B_1} (\neg r_1(o_1, a_1) \to \downarrow_2 (\beta(o_1))(o_2)) \\
&= \bigwedge_{o_1 \in B_1} (\neg r_1(o_1, a_1) \to \varphi_{\beta L}(o_1)(o_2)) \\
&= \bigwedge_{o_1 \in B_1} (\neg \varphi_{\beta L}(o_1)(o_2) \to r_1(o_1, a_1)) \\
&= \uparrow_1 (\neg\varphi^t_{\beta L}(o_2))(a_1) = \uparrow_1\downarrow_1\uparrow_1 (\neg\varphi^t_{\beta L}(o_2))(a_1) \\
&= \uparrow_1 (\varphi_{\gamma L}(o_2))(a_1) = \gamma(o_2)(a_1)
\end{aligned}$$

□

5. L-ChuCors is a *-autonomous category

This final part of the paper is devoted to proving that L-ChuCors is a *-autonomous category [2]. Among the various equivalent formulations to define a *-autonomous category, we will consider that of a symmetric monoidal closed category with a duality functor. In the next sections, we will be introducing each of the corresponding constructions.

5.1. The internal Hom functor

Definition 5.1. Given two L-fuzzy contexts $C_i = \langle B_i, A_i, r_i \rangle$ for $i \in \{1, 2\}$ the formal L-fuzzy context $C_1 \multimap C_2 = \langle L\text{-ChuCors}(C_1, C_2), B_1 \times A_2, r_{C_1 \multimap C_2} \rangle$, the object part of the internal Hom functor, is defined where the mapping $r_{C_1 \multimap C_2} \colon L\text{-ChuCors}(C_1, C_2) \times B_1 \times A_2 \to L$ is given by

$$r_{C_1 \multimap C_2}(\varphi, (o_1, a_2)) = \uparrow_2 (\varphi_L(o_1))(a_2) = \downarrow_1 (\varphi_R(a_2))(o_1)$$

The rest of this section focuses on properties of $\multimap$ concerning L-fuzzy orderings and L-fuzzy equalities, together with the introduction of a *canonical form* of a context, which will be used later.

Note that in several proofs in this section, specifically Theorem 5.1 and Lemma 5.3, we will use the result proved in [16] which states the existence of an anti-isomorphism between the complete lattices of L-bonds and L-ChuCors.

Theorem 5.1. Let $C_i = \langle B_i, A_i, r_i \rangle$ be L-fuzzy contexts for $i \in \{1, 2\}$, then there is an isomorphism

$$\big\langle \langle L\text{-}FCL(C_1 \multimap C_2), \approx_1 \rangle, \preceq_1 \big\rangle \quad \cong \quad \big\langle \langle L\text{-ChuCors}(C_1, C_2), \approx_2 \rangle, \preceq_2 \big\rangle.$$

Proof:
Consider an arbitrary concept $\langle \Phi, \beta \rangle$, where $\Phi \in L^{L\text{-ChuCors}(\mathrm{C}_1, \mathrm{C}_2)}$ and $\beta \in L^{B_1 \times A_2}$, then

$$\begin{aligned}
\beta(o_1)(a_2) &= \uparrow_{C_1 \multimap C_2} (\Phi)(o_1, a_2) \\
&= \bigwedge_{\varphi \in L\text{-ChuCors}(\mathrm{C}_1, \mathrm{C}_2)} (\Phi(\varphi) \to r_{C_1 \multimap C_2}(\varphi, (o_1, a_2))) \\
&= \bigwedge_{\varphi} (\Phi(\varphi) \to \uparrow_2 (\varphi_L(o_1))(a_2)) \\
&= \bigwedge_{\varphi} (\Phi(\varphi) \to \bigwedge_{o_2 \in B_2} (\varphi_L(o_1) \to r_2(o_2, a_2))) \\
&= \bigwedge_{o_2 \in B_2} \bigwedge_{\varphi} (\Phi(\varphi) \to (\varphi_L(o_1) \to r_2(o_2, a_2))) \\
&= \bigwedge_{o_2 \in B_2} \bigwedge_{\varphi} ((\Phi(\varphi) \otimes \varphi_L(o_1)) \to r_2(o_2, a_2)) \\
&= \bigwedge_{o_2 \in B_2} (\bigvee_{\varphi} (\Phi(\varphi) \otimes \varphi_L(o_1)) \to r_2(o_2, a_2)) \\
&= \bigwedge_{o_2 \in B_2} ((\bigcup \Phi)_L(o_1)(o_2) \to r_2(o_2, a_2)) \\
&= \uparrow_2 ((\bigcup \Phi)_L(o_1))(a_2)
\end{aligned}$$

Similarly we obtain:

$$
\begin{aligned}
\beta^t(a_2)(o_1) &= \uparrow_{C_1 \multimap C_2} (\Phi)(o_1, a_2) \\
&= \cdots = \bigwedge_{\varphi} (\Phi(\varphi) \to \downarrow_1 (\varphi_R(a_2))(o_1)) \\
&= \cdots = \bigwedge_{a_1 \in A_1} (\bigvee_{\varphi} (\Phi(\varphi) \otimes \varphi_R(a_2)(a_1)) \to r_1(o_1, a_1)) \\
&= \downarrow_1 ((\bigcup \Phi)_R(a_2))(o_1)
\end{aligned}
$$

Now, as we have seen that $\beta \in L^{B_1 \times A_2}$ is closed in $C_1 \multimap C_2$, then β is in L-Bonds(C_1, C_2).

Every bond $\beta \in L$-Bonds(C_1, C_2) is closed in $C_1 \multimap C_2$, because of the following chain of equalities:

$$
\begin{aligned}
\beta(o_1)(a_2) &= \uparrow_2 (\varphi_\beta(o_1))(a_2) = r_{C_1 \multimap C_2}(\varphi_\beta, (o_1, a_2)) \\
&= 1 \to r_{C_1 \multimap C_2}(\varphi_\beta, (o_1, a_2)) \\
&= \bigwedge_{\varphi} (\chi_{\varphi_\beta}(\varphi) \to r_{C_1 \multimap C_2}(\varphi_\beta, (o_1, a_2))) \\
&= \uparrow_{C_1 \multimap C_2} (\chi_{\varphi_\beta})(o_1, a_2)
\end{aligned}
$$

As a result we obtain that there is a bijection between L-ChuCors(C_1, C_2) and L-$FCL(C_1 \multimap C_2)$.

Let $\langle \Phi_i, \beta_i \rangle$ for $i \in \{1, 2\}$ be two concepts of $C_1 \multimap C_2$, then

$$
\begin{aligned}
\langle \Phi_1, \beta_1 \rangle \preceq_1 \langle \Phi_2, \beta_2 \rangle &= \bigwedge_{o_1 \in B_1} \bigwedge_{a_2 \in A_2} (\beta_2(o_1)(a_2) \to \beta_1(o_1)(a_2)) \\
&= \varphi_{\beta_1} \preceq_2 \varphi_{\beta_2}
\end{aligned}
$$

Similarly for the L-equalities $\approx_i$. □

Now, a canonical form for any context will be introduced. The first result here, however, states an isomorphism between the set of concepts of an L-context and a set of L-Chu correspondences from a 'constant' context.

Theorem 5.2. Let $C = \langle B, A, r \rangle$ be an arbitrary L-context. Then there is an isomorphism between L-ordered sets

$$
\Big\langle \langle L\text{-}FCL(C), \approx_1 \rangle, \preceq_1 \Big\rangle \quad \cong \quad \Big\langle \langle L\text{-ChuCors}(\top, C), \approx_2 \rangle, \preceq_2 \Big\rangle,
$$

where $\top = \langle \{\diamond\}, L, \lambda \rangle$, where $\lambda(\diamond, l) = l$, for any $l \in L$.

Proof:
Let $\varphi \in L$-ChuCors$(\top, C)$ be an arbitrary L-Chu correspondence. Then we have $\varphi_L : \{\diamond\} \to L^B$ and $\varphi_R : A \to L^L$ where $\varphi_L(\diamond)$ is closed in C and $\varphi_R(a)$ is closed in $\top$ for any $a \in A$. It means that every left side of any Chu correspondence from $\top$ to C is an object part of some concept of C.

Now let $\langle f, g\rangle$ be an arbitrary concept of C. Then we can construct the L-Chu correspondence from $\top$ to C, $\varphi_L(\diamond) = f$. From Lemma 3.1 we know that

$$\varphi_R(a) = \uparrow_\lambda (\varphi_L^+(\downarrow (\chi_a))) = \uparrow_\lambda (\bigwedge_{o\in B} (\varphi_L(\diamond)(o) \to r(o,a)))$$
$$= \uparrow_\lambda (\bigwedge_{o\in B} (f(o) \to r(o,a))) = \uparrow_\lambda (\uparrow (f)(a)) = \uparrow_\lambda (g(a))$$

Hence φ_R will assign a closed L-set in $\top$ to every $a \in A$. And with any closed $g \in L^A$ there will be a new L-set from L^L such that $\varphi_R(a)(l) = \uparrow_\lambda (g(a))(l) = (l \to g(a))$.

Consider two new L-concepts $\langle f_1, g_1\rangle$, $\langle f_2, g_2\rangle$ of context C and two L-Chu correspondences φ_{f_1} and φ_{f_2} assigned to the concepts. Then

$$\langle f_1, g_1\rangle \preceq_1 \langle f_2, g_2\rangle = \bigwedge_{a\in A} (g_2(a) \to g_1(a)) = \bigwedge_{a\in A} (\uparrow (f_2)(a) \to\uparrow (f_1)(a))$$
$$= \bigwedge_{a\in A} (\uparrow (\varphi_{f_2})(a) \to\uparrow (\varphi_{f_1})(a)) = \varphi_{f_1} \preceq_2 \varphi_{f_2}.$$

The equality $\langle f_1, g_1\rangle \approx_1 \langle f_2, g_2\rangle = \varphi_{f_1} \approx_2 \varphi_{f_2}$ can be proved similarly. □

Definition 5.2. Let $C = \langle B, A, r\rangle$ be an L-fuzzy formal context. The *canonical form* of context C is an L-fuzzy formal context $\mathrm{cf}(C) = \langle L\text{-FCL}(C), A, r_C\rangle$, such that $r_C(\langle f, g\rangle, a) = g(a)$.

Corollary 5.1. For any L-concept $C = \langle B, A, r\rangle$ there is an isomorphism between L-ordered sets

$$\Big\langle\langle L\text{-}FCL(C), \approx_1\rangle, \preceq_1\Big\rangle \quad\cong\quad \Big\langle\langle L\text{-}FCL(\mathrm{cf}(C)), \approx_1\rangle, \preceq_1\Big\rangle.$$

Proof:
Consider an arbitrary $\gamma \in L^{L\text{-}FCL(C)}$.

$$\uparrow_C (\gamma)(a) = \bigwedge_{\langle f',g'\rangle\in L\text{-}FCL(C)} (\gamma(\langle f', g'\rangle) \to r_C(\langle f', g'\rangle, a))$$
$$= \bigwedge_{\langle f',g'\rangle} (\gamma(\langle f', g'\rangle) \to g'(a)) = \bigwedge_{\langle f',g'\rangle} (\gamma(\langle f', g'\rangle) \to\uparrow (f')(a))$$
$$= \bigwedge_{\langle f',g'\rangle} (\gamma(\langle f', g'\rangle) \to \bigwedge_{o\in B} (f'(o) \to r(o,a)))$$
$$= \bigwedge_{o\in B} \bigwedge_{\langle f',g'\rangle} (\gamma(\langle f', g'\rangle) \to (f'(o) \to r(o,a)))$$
$$= \bigwedge_{o\in B} \bigwedge_{\langle f',g'\rangle} ((\gamma(\langle f', g'\rangle) \otimes f'(o)) \to r(o,a)))$$
$$= \bigwedge_{o\in B} (\bigvee_{\langle f',g'\rangle} (\gamma(\langle f', g'\rangle) \otimes f'(o)) \to r(o,a))$$
$$= \bigwedge_{o\in B} (\bigcup_B \gamma) \to r(o,a)) = \uparrow (\bigcup_B \gamma)(a)$$

Then $\langle \downarrow (\uparrow_C (\gamma)), \uparrow_C (\gamma)\rangle = \langle \downarrow\uparrow (\bigcup_B \gamma), \uparrow (\bigcup_B \gamma)\rangle$ the concept of C is the only element of ${}^1\sup(\gamma)$ from Theorem 2.1. □

By considering the transposed, we can define the dual form of a context as follows:

Definition 5.3. Dual form of formal L-fuzzy context $C = \langle B, A, r\rangle$ is a context $C^* = \langle A, B, r^{\mathrm{t}}\rangle$.

The following result rephrases Corollary 5.1 in terms of the $\top$ context and its dual $\bot$ as follows:

Corollary 5.2. Let C be an arbitrary L-context and let us write $\bot = \langle L, \{\diamond\}, \lambda^{\mathrm{t}}\rangle$, then the following isomorphisms hold

$$\begin{aligned}\big\langle\langle L\text{-}FCL(\mathrm{cf}(C)), \approx_1\rangle, \preceq_1\big\rangle &\cong \big\langle\langle L\text{-}FCL(\top \multimap C), \approx_1\rangle, \preceq_1\big\rangle\\ \big\langle\langle L\text{-}FCL(\mathrm{cf}(C^*)), \approx_1\rangle, \preceq_1\big\rangle &\cong \big\langle\langle L\text{-}FCL(C \multimap \bot), \approx_1\rangle, \preceq_1\big\rangle\end{aligned}$$

Lemma 5.1. For any two arbitrary L-contexts C_1 and C_2 there is an isomorphism

$$\big\langle\langle L\text{-ChuCors}(C_1, C_2), \approx_2\rangle, \preceq_2\big\rangle \cong \big\langle\langle L\text{-ChuCors}(\mathrm{cf}(C_1), \mathrm{cf}(C_2)), \approx_2\rangle, \preceq_2\big\rangle$$

Proof:
Consider $\varphi \in L$-ChuCors(C_1, C_2). Now by Lemma 3.1 we can construct an L-Chu correspondence $\mathrm{cf}(\varphi) \in L$-ChuCors$(\mathrm{cf}(C_1), \mathrm{cf}(C_2))$ with right component $\mathrm{cf}(\varphi)_R \colon A_2 \to L^{A_1}$ and left component $\mathrm{cf}(\varphi)_L \colon L\text{-}FCL(C_1) \to L^{L\text{-}FCL(C_2)}$ in the following way:

- $\mathrm{cf}(\varphi)_R = \varphi_R$
- $\mathrm{cf}(\varphi)_L(\langle f_1, g_1\rangle) = \downarrow_{C_2} (\varphi_R^{+}(g_1))$

Thus, we have the following chain of equalities

$$\begin{aligned}\uparrow_{C_2} (\mathrm{cf}(\varphi)_L(\langle f_1, g_1\rangle))(a_2) &= \uparrow_{C_2}\downarrow_{C_2} (\varphi_R^{+}(g_1))(a_2) = \varphi_R^{+}(g_1)(a_2)\\ &\Big[\text{because of } \varphi_R^{+}(g_1) = \varphi_R^{+}(\uparrow_1 (f_1)) = \uparrow_2 (\varphi_{L^+}(f_1)), \text{ so } \varphi_R^{+}(g_1) \text{ is closed in } C_2 \text{ and in } \mathrm{cf}(C_2)\Big]\\ &= \bigwedge_{a_1 \in A_1} (\varphi_R(a_2)(a_1) \to g_1(a_1))\\ &= \downarrow_{C_1} (\varphi_R(a_2))(\langle f_1, g_1\rangle)\\ &= \downarrow_{C_1} (\mathrm{cf}(\varphi)_R(a_2))(\langle f_1, g_1\rangle)\end{aligned}$$

Conversely, given an L-Chu correspondence $\mathrm{cf}(\varphi) \in L$-ChuCors$(\mathrm{cf}(C_1), \mathrm{cf}(C_2))$ then we can construct $\varphi \in L$-ChuCors(C_1, C_2) as follows:

- $\varphi_R = \mathrm{cf}(\varphi)_R$
- $\varphi_L(o) = \downarrow_2 (\varphi_R^{+}(\uparrow_1 (\chi_o))) = \downarrow_2 (\varphi_R^{+}(\uparrow_1 (\chi_o))) = \downarrow_2 (\mathrm{cf}(\varphi)_R^{+}(\uparrow_1 (\chi_o)))$ for any object $o \in B_1$

For any pair $\varphi_1, \varphi_2 \in L$-ChuCors$(C_1, C_2)$ we have

$$\begin{aligned}\varphi_1 \preceq_2 \varphi_2 &= \bigwedge_{o_1\in B_1}\bigwedge_{a_2\in A_2}(\downarrow_1(\varphi_{2R}(a_2))(o_1) \to \downarrow_1(\varphi_{1R}(a_2))(o_1))\\ &= \bigwedge_{o_1\in B_1}\bigwedge_{a_2\in A_2}(\downarrow_1(\mathrm{cf}(\varphi)_{2R}(a_2))(o_1) \to \downarrow_1(\mathrm{cf}(\varphi)_{1R}(a_2))(o_1))\\ &= \mathrm{cf}(\varphi)_1 \preceq_2 \mathrm{cf}(\varphi)_2\end{aligned}$$

Similarly for $\approx_2$. □

It is not difficult to check that cf satisfies all the requirements of an endofunctor in L-ChuCors.

Proposition 5.1. cf is an endofunctor in L-ChuCors.

Note that we can define a correspondence $\kappa\colon C \to \mathrm{cf}(C)$ from any context $C = \langle B, A, r\rangle$ to its canonical form $\mathrm{cf}(C)$ in a very natural way.

- $\kappa_L\colon B \to L^{L-\mathrm{FCL(C)}}$ is given by $\kappa_L(o) = \downarrow_C(\uparrow(\chi_o))$
- $\kappa_R\colon A \to L^A$ is given by $\kappa_R(a) = \uparrow\downarrow(\chi_a)$

It is not difficult to check that $\kappa \in L$-ChuCors$(C, \mathrm{cf}(C))$, because the following equalities hold:

$$\begin{aligned}\uparrow_C(\kappa_L(o))(a) &= \uparrow_C\downarrow_C(\uparrow(\chi_o))(a) = \uparrow(\chi_o)(a)\\ &= r(o,a)\\ &= \downarrow(\chi_a)(o) = \downarrow\uparrow\downarrow(\chi_a)(o) = \downarrow(\kappa_R(a))(o)\end{aligned}$$

Theorem 5.3. The family of L-Chu correspondences κ between a context and its canonical form is a natural isomorphism from the identity functor of L-ChuCors to the functor cf.

Proof:
We can create an inverse correspondence $\kappa^{-1}\colon \mathrm{cf}(C) \to C$

- $\kappa_L^{-1}\colon L\text{-FCL}(C) \to L^B$ defined by $\kappa_L^{-1}(\langle f, g\rangle) = f$
- $\kappa_R^{-1}\colon A \to L^A$ defined by $\kappa_R^{-1}(a) = \kappa_R(a) = \uparrow\downarrow(\chi_a)$

and the following equalities hold:

$$\begin{aligned}\uparrow(\kappa_L^{-1}(\langle f,g\rangle))(a) &= \uparrow(f)(a) = \bigwedge_{o\in B}(f(o)\to r(o,a)) = \bigwedge_{o\in B}(f(o)\to\downarrow(\chi_a)(o))\\ &= \bigwedge_{a\in A}(\uparrow\downarrow(\chi_a)(o)\to g(a)) = \uparrow_C(\kappa_R^{-1}(a))(\langle f,g\rangle)\end{aligned}$$

□

5.2. The tensor product

We introduce here new operation between L-contexts in order to provide the structure of symmetric monoidal category. This notion is given in the definition below:

Definition 5.4. Let $C_i = \langle B_i, A_i, r_i \rangle$ be two L-contexts. The new context $C_1 \otimes C_2$ is defined as the triple $\langle B_1 \times B_2, L\text{-ChuCors}(C_1, C_2^*), r_\otimes \rangle$, where

$$r_\otimes(\varphi, (o_1, o_2)) = \downarrow_1 (\varphi_L(o_1))(o_2) = \downarrow_2 (\varphi_R(o_2))(o_1).$$

This construction satisfies commutativity and has a neutral element, which is exactly the $\top$ context defined previously.

Lemma 5.2. Let C, C_1, C_2 be arbitrary L-contexts. Then following isomorphisms hold

$$\begin{array}{ccccc} \top \otimes C & \cong & \text{cf}(C) & \cong & C \otimes \top \\ C_1 \otimes C_2 & & \cong & & C_2 \otimes C_1 \end{array}$$

Proof:
It is not difficult to check the following

$$\begin{array}{rcl} \top \otimes C = \langle \{\diamond\} \times B, L\text{-ChuCors}(\top, C^*), r_\otimes \rangle & \cong & \text{cf}(C) \\ C \otimes \top = \langle B \times \{\diamond\}, L\text{-ChuCors}(C, \bot), r_\otimes \rangle & \cong & \text{cf}(C) \end{array}$$

For the second part, it is enough to take into account that $r_\otimes((o_1, o_2), \varphi) = r_\otimes((o_2, o_1), \psi)$, where $\psi_L = \varphi_R$ and $\psi_R = \varphi_L$ □

Proposition 5.2. L-ChuCors is a symmetric monoidal category.

Proof:
The commutativity of the corresponding diagrams can be checked by diagram chasing. □

Now, we have to prove that the tensor product and the internal hom functor are adjoint, thus providing the monoidal closedness of the category.

Lemma 5.3. Let $C_i = \langle B_i, A_i, r_i \rangle$ for $i \in \{1, 2, 3\}$ be arbitrary L-contexts. The following isomorphism holds

$$L\text{-ChuCors}(C_1 \otimes C_2, C_3) \cong L\text{-ChuCors}(C_2, C_1 \multimap C_3)$$

As a result, the monoidal category L-ChuCors is closed.

Proof:
The structure of the proof is the following: firstly, given an L-Chu correspondence in the left-hand side we consider its L-bond associated by the anti-isomorphism stated in the previous paragraph. Then a new L-bond is built which is shown to correspond to the L-Chu correspondence in the right hand side.

Essentially, we will show that, given any L-bond from L-Bonds$(C_1 \otimes C_2, C_3)$, we can construct from a new L-bond from L-Bonds$(C_1, C_2 \multimap C_3)$.

Let us consider an arbitrary $\varphi \in L\text{-ChuCors}(C_1 \otimes C_2, C_3)$, this means means that there exist:

- $\varphi_L : B_1 \times B_2 \to L^{B_3}$
- $\varphi_R : A_3 \to L^{L\text{-ChuCors}(C_1, C_2^*)}$

satisfying that $\uparrow_3 (\varphi_L(o_1, o_2))(a_3) = \downarrow_{C_1 \otimes C_2} (\varphi_R(a_3))(o_1, o_2)$

We define now $\gamma(o_1)(o_2, a_3) = \beta_\varphi(o_1, o_2)(a_3)$, which satisfies the following facts:

1. $\gamma(o_1)(o_2, a_3) = \beta_\varphi(o_1, o_2)(a_3) = \uparrow_3 (\varphi_L(o_1, o_2))(a_3)$
2. $\gamma(o_1)(o_2, a_3) = \beta^t_\varphi(a_3)(o_1, o_2) = \downarrow_2 (\bigvee_{\omega \in L\text{-ChuCors}(C_1, C_2^*)} (\varphi_R(a_3)(\omega) \otimes \omega_L(o_1)))(o_2)$
3. $\gamma^t(o_2, a_3)(o_1) = \beta^t_\varphi(a_3)(o_1, o_2) = \downarrow_1 (\bigvee_\omega (\varphi_R(a_3)(\omega) \otimes \omega_R(o_2)))(o_1)$

The proof of the second item above is the following:

$$\begin{aligned}
\beta^t_\varphi(a_3)(o_1, o_2) &= \downarrow_{C_1 \otimes C_2} (\varphi_R(a_3))(o_1, o_2) \\
&= \bigwedge_\omega (\varphi_R(a_3)(\omega) \to r_{C_1 \otimes C_2}((o_1, o_2), \omega)) \\
&= \bigwedge_\omega (\varphi_R(a_3)(\omega) \to \downarrow_2 (\omega_L(o_1))(o_2)) \\
&= \bigwedge_\omega (\varphi_R(a_3)(\omega) \to \bigwedge_{a_2 \in A_2} (\omega_L(o_1)(a_2) \to r_2(o_2, a_2))) \\
&= \bigwedge_{a_2 \in A_2} \bigwedge_\omega (\varphi_R(a_3)(\omega) \to (\omega_L(o_1)(a_2) \to r_2(o_2, a_2))) \\
&= \bigwedge_{a_2 \in A_2} \bigwedge_\omega ((\varphi_R(a_3)(\omega) \otimes \omega_L(o_1)(a_2)) \to r_2(o_2, a_2)) \\
&= \downarrow_2 (\bigvee_\omega (\varphi_R(a_3)(\omega) \otimes \omega_L(o_1)))(o_2)
\end{aligned}$$

The third item can be proved similarly by using $r_{C_1 \otimes C_2}((o_1, o_2), \omega)) = \downarrow_1 (\omega_R(o_2))(o_1)$.

Now from first and second facts we have that $\gamma(o_1) \in L$-Bonds(C_2, C_3), hence $\gamma(o_1)$ is closed in $C_2 \multimap C_3$. Now, by the third fact, $\gamma^t(o_2, a_3)$ is closed in C_1, hence $\gamma \in L$-Bonds$(C_1, C_2 \multimap C_2)$. This enables the construction of $\psi \in L$-ChuCors$(C_1, C_2 \multimap C_3)$

- $\psi_L \colon B_1 \to L^{L\text{-ChuCors}(C_2, C_3)}$, $\psi_L(o_1) = \uparrow_{C_2 \multimap C_3} (\gamma(o_1))$
- $\psi_R \colon B_2 \times A_3 \to L^{A_1}$, $\psi_R(o_2, a_3) = \downarrow_1 (\gamma^t(o_2, a_3))$

□

The final part of the section shows that the construction of the dual context actually provides a duality functor.

Lemma 5.4. Let $C_i = \langle B_i, A_i, r_i \rangle$ for $i \in \{1, 2\}$ be arbitrary L-contexts. Then the following natural isomorphism holds

$$C_1 \multimap C_2 \quad \cong \quad C_2^* \multimap C_1^*$$

Proof:
Given $\psi \in L$-ChuCors(C_2^*, C_1^*), we have that $\psi_L : A_2 \to L^{A_1}$ and $\psi_R : B_1 \to L^{B_2}$.

The new correspondence φ is defined in the following way: $\varphi_L = \psi_R$ and $\varphi_R = \psi_L$. Its easy to see that $\varphi \in L$-ChuCors(C_1, C_2). □

As a consequence of the previous results we obtain

Theorem 5.4. L-ChuCors is a *-autonomous category.

6. Conclusions and future work

The categorical treatment of morphisms as fundamental structural properties has been advocated by several authors as a means for the modeling of data translation, communication, and distributed computing, among other applications.

The contributions presented in this paper seem to pave the way towards determining possible categories on which to model knowledge transfer and information sharing. We have shown that the classical crisp category ChuCors is a (not full) subcategory of the category L-ChuCors, introduced in [16]. This result supports the coherence of the proposed extension of the theory of Chu correspondences which, in addition, has been proved to be *-autonomous and, hence, some kind of generalized topology can be defined on L-ChuCors. Furthermore, we have introduced an adequate generalization of the study of L-bonds as morphisms among contexts, initiated in [18], by showing how the classical relationships between bonds and contexts can be lifted to a more general framework; an improved result, with respect to the content of [17], has been included here.

Similar categorial studies are being developed with the aim of describing some structural properties of intercontextual relationships of L-fuzzy formal contexts in terms of L-bonds which, interestingly enough, can be related to L-Chu correspondences. Current research on (extensions of) the theory of Chu spaces studies morphisms among contexts in order to obtain categories with certain specific properties.

A thorough study of the properties of the extended categorical framework of Chu correspondences and L-Chu correspondences still needs to be carried out, in order to identify their natural interpretation within the theory of knowledge representation.

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