On residuation in multilattices: filters, congruences, and homomorphisms

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Abstract

Continuing with our general study of algebraic hyperstructures, we focus on the residuated operation in the framework of multilattices. Firstly, we recall the existing relation between filters, homomorphisms and congruences in the framework of multilattices; then, introduce the notion of residuated multilattice and further study the notion of filter, which has to be suitably modified so that the results in the first section are conveniently preserved also in the residuated case.

Key words: hyperalgebra, pocrim, multilattices, residuation, logic.

1. Introduction

The algebraic study of logical systems has a prominent role in artificial intelligence, in that such systems are usually modeled as partially ordered sets together with some operations reflecting the properties of the connectives. More generally, algebraic methods are often used in formalism to handle uncertainty or related notions [30]. The algebraic notion of *hyperstructure* arises as well as an interesting theoretical tool when considering

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topics such as non-determinism, or reasoning handling of uncertain information [18, 14, 8].

On the other hand, residuation has been studied as a mathematical entity since 1930s; however, its interest as a research topic has been recently increased because of the fact that residuated lattices have been identified as the algebraic structures underlying substructural logics [21, 28, 25], but it has application in different fields, such as network calculus [19], or in algebraic structures used in soft constraint satisfaction problems [5], or when considering the fuzzy extensions of crisp formalisms, for instance, description logic [11], or formal concept analysis [3, 2, 17]. Current research conducted on residuated lattices shows them as suitable structures to represent roughness [31]; other authors have focused on residuated frames as a valuable tool for solving both algebraic and logical problems; in another approach, residuation in a broad sense has been studied in relation with quantum structures [20, 33] and, furthermore, operations failing associativity, commutativity are used as underlying carrier to generalized residuated structures [16]

In this work we focus on new theoretical developments and links between the residuation and the hyperstructure called multilattice [26, 9], in the line of recent work done of bilattices [24]. Multilattices, introduced by Benado [4] and contrariwise to bilattices (which include two different orderings), are defined on just one order relation, and it is the requirement of the existence of suprema and infima what is relaxed. Much more recently, Cordero *et al* [26] proposed an alternative algebraic definition of multilattice which is more closely related to that of lattice, allowing for natural definitions of related structures such that multisemilattices and, in addition, is better suited for applications; for instance, Medina *et al* [27] developed a general approach to fuzzy logic programming based on a multilattice as underlying set of truth-values for the logic.

The contributions in this paper are the following: firstly, we start with the notion of filter in a multilattice, and present its relation to homomorphisms and congruences; then we move to residuated multilattices and consider the most adequate notion of filter (as the appropriate generalization of deductive system in a residuated multilattice), prove that the set of filters of a residuated multilattice is a complete lattice, and study again the relation between the filters, homomorphisms and congruences; moreover, the main properties of residuated multilattices are stated and proved.

The organization of this paper in the following: in Section 2, after introducing the preliminary definitions of filter, congruence and homomorphism between multilattices, we prove that the set of congruences on a multilattice forms a complete lattice, together with the suitable extension of the well-known relations between filters, congruences, and homomorphisms which are not preserved by other definitions provided in the literature. Then, in Section 3 we introduce a residuation operation on a multilattice, leading to the class of the so-called residuated multilattices, which is somewhere in between the class of residuated lattices and the class of pocrims. We identify some properties of residuated multilattices which make them to collapse to residuated lattices, or even to Heyting algebras. Finally, in Section 4 we argument on the need of a new definition of filter which behaves adequately with residuation. The last section draws some conclusions and present prospects for future work.

2. Filters, homomorphisms and congruences on multilattices

Given $(M, \leq)$ a partially ordered set (henceforth poset) and $B \subseteq M$, a *multi-supremum of* B is a minimal element of the set of upper bounds of B and multisup(B) denote the set of multi-suprema of B. Dually, we define the *multi-infima* which will be denoted multinf(B).

Definition 1. A poset, $(M, \leq)$, is an **ordered multilattice** if and only if it satisfies that, for all $a, b, x \in M$ with $a \leq x$ and $b \leq x$, there exists[4] $z \in \text{multisup}\{a, b\}$ such that $z \leq x$, and its dual version for multinf$\{a, b\}$.

A multilattice is said to be **full** if multisup$\{a, b\} \neq \varnothing$ and multinf$\{a, b\} \neq \varnothing$ for all $a, b \in M$.

Remark 2 (Notation).

- We will write $a \sqcup b$ to denote multisup$\{a, b\}$ and $a \sqcap b$ to denote the set multinf$\{a, b\}$.
- In the rest of the paper we will frequently write singletons without braces if no confusion arises.
- Let $\mathcal{R}$ be a binary relation in M and $X, Y \subseteq M$, then $X \widehat{\mathcal{R}} Y$ denotes that, for all $x \in X$, there exists $y \in Y$ such that $x \mathcal{R} y$ and for all $y \in Y$ there exists $x \in X$ such that $x \mathcal{R} y$.

[4]Note that the definition is consistent with the existence of two incomparable elements *without* any multisupremum. In other words, multisup$\{a, b\}$ can be empty.

- If $\mathcal{R}$ is an equivalence relation in M, we will denote $[x]$ the equivalence class of an element $x \in M$.
- Let $(M, \leq)$ be a poset, the set of upper bounds of $x \in M$ will be denoted $x\!\uparrow = \{a \in M : a \geq x\}$ and dually $x\!\downarrow = \{a \in M : a \leq x\}$ denotes the set of lower bounds. Similarly, for $X \subseteq M$, the upper closure is $X\!\uparrow = \bigcup_{x \in X} x\!\uparrow$ and the lower closure is $X\!\downarrow = \bigcup_{x \in X} x\!\downarrow$.

Similarly to the case of lattices, one can consider a multilattice as a structure $(M, \sqcup, \sqcap)$ with certain properties [26]. It is worth to note that a multilattice is not an algebraic structure in the usual sense that this expression has in universal algebra, since multisup and multinf are not algebraic operations, i.e. they are not functions from M^n to M, but hyperoperations, functions from M^n to 2^M. Moreover, any finite poset is always a multilattice. These facts are especially important in adding value to the results in the paper.

The concept of homomorphism between hyperstructures [15, 22, 13] has been considered from different points of view in the literature. The most used definition is that introduced originally by Benado [4] in the framework of multilattices.

Definition 3. A map $h\colon M \to M'$ between multilattices is said to be a **homomorphism** if

$$h(a \sqcup b) \subseteq h(a) \sqcup h(b) \quad \text{and} \quad h(a \sqcap b) \subseteq h(a) \sqcap h(b) \quad \text{for all } a, b \in M$$

It would be nice to have conditions of homomorphism expressed in terms of equalities instead of set-inclusion. When the initial multilattice is full, the notion of homomorphism can be characterized in terms of equalities, as the following result shows.

Proposition 4. *Let $h\colon M \to M'$ be a map between multilattices where M is full. Then h is a homomorphism if and only if, for all $a, b \in M$,*

$$\begin{aligned} h(a \sqcup b) &= \big(h(a) \sqcup h(b)\big) \cap h(M) \\ h(a \sqcap b) &= \big(h(a) \sqcap h(b)\big) \cap h(M) \end{aligned} \qquad (1)$$

Proof. It is sufficient to prove that, for all $a, b, c \in M$, if $h(c) \in h(a) \sqcup h(b)$ then there exists $c' \in a \sqcup b$ such that $h(c') = h(c)$.

Firstly, $h(a \sqcup c) \subseteq h(a) \sqcup h(c) = h(c)$ and $h(b \sqcup c) \subseteq h(b) \sqcup h(c) = h(c)$. Since $a \sqcup c \neq \varnothing \neq b \sqcup c$, there exist $x \in a \sqcup c$ and $y \in b \sqcup c$ such that $h(x) = h(c) = h(y)$.

On the other hand, any element $z \in x \sqcup y$ satisfies $h(z) \in h(x \sqcup y) \subseteq h(x) \sqcup h(y) = h(c) \sqcup h(c) = h(c)$.

As $a \leq z$ and $b \leq z$, there exists $c' \in a \sqcup b$ such that $c' \leq z$. Therefore, $h(c) = h(z) = h(c' \sqcup z) \subseteq h(c') \sqcup h(z) = h(c') \sqcup h(c)$ which implies that $h(c') \leq h(c)$. Since also $h(c'), h(c) \in h(a) \sqcup h(b)$ we deduce $h(c') = h(c)$. □

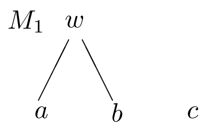
Notice that when the initial multilattice is not full there exist situations in which equations (1) hold (for instance consider the identity), and situations in which the equations do not hold (see Example 6 below).

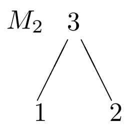
Definition 5. Let $h\colon A \to B$ be a mapping. The **kernel relation** of h is defined as follows

$$a \,\mathcal{K}_h\, b \text{ if and only if } h(a) = h(b)$$

Obviously, for every mapping h, the kernel relation is an equivalence relation. However, for a multilattice homomorphism, this relation is not necessarily compatible with the operations as the following example shows:

Example 6. Let M_1 and M_2 be the multilattices described by the following diagrams and the homomorphism $h : M_1 \to M_2$ given by the following table:





x	a	b	c	w
$h(x)$	1	2	2	3

Observe that $h(b) = h(c)$ but since $a \sqcup b = w$ and $a \sqcup c = \varnothing$, it does not satisfy that $h(a \sqcup b) = h(a \sqcup c)$. □

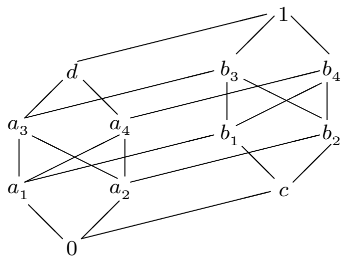
It is remarkable that the conditions given in (1) are the key to the compatibility between the kernel relation and the operations in a multilattice, as we will see later. Notice that the homomorphism in the example above does not verify (1), because

$$\varnothing = h(a \sqcup c) \neq (h(a) \sqcup h(c)) \cap \{1, 2, 3\} = \{3\}$$

We now move to the notion of congruences, whose study is important both from a theoretical standpoint and for their applications in the field of logic-based approaches to uncertainty. Regarding applications, the notion of congruence is intimately related to the foundations of fuzzy reasoning and their relationships with other logics of uncertainty. More focused on the theoretical aspects of Computer Science, some authors [1, 29] have pointed out the relation between congruences, fuzzy automata and determinism.

Definition 7. Let $(M, \sqcup, \sqcap)$ be a multilattice, a **congruence** on M is any equivalence relation $\equiv$ such that if $a \equiv b$, then $a \sqcup c \mathrel{\widehat{\equiv}} b \sqcup c$ and $a \sqcap c \mathrel{\widehat{\equiv}} b \sqcap c$, for all $a, b, c \in M$.

Example 8. Let $(M, \sqcup, \sqcap)$ be the multilattice depicted in the figure below.



The partition $\{\{0, a_1, a_2, a_3, a_4, d\}, \{c, b_1, b_2, b_3, b_4, 1\}\}$ defines a non-trivial congruence. However, the following partition

$$\{\{0, a_1, c, b_1\}, \{a_2, a_3, b_2, b_3\}, \{a_4, d, b_4, 1\}\}$$

defines an equivalence relation $\sim$ which is not a congruence because $0 \sim a_1$ but $a_4 \in a_1 \sqcup a_2$ and no $x \in 0 \sqcup a_2 = a_2$ exists such that $x \sim a_4$. □

Lemma 9. *Let $\equiv$ be a congruence relation in a multilattice M and $a, b \in M$. Then, the following conditions hold:*

1. *If $b \in [a]$ then $\varnothing \neq a \sqcup b \subseteq [a]$ and $\varnothing \neq a \sqcap b \subseteq [a]$*
2. *If there exist $z \in a \sqcap b$ and $w \in a \sqcup b$ such that $z \equiv w$, then $a \equiv b$*
3. *If $z, w \in a \sqcup b$, $z \equiv w$ and $a, b \in z \sqcap w$ then $a \sqcup b \subseteq [a] = [b]$.*
4. *If $z, w \in a \sqcap b$, $z \equiv w$ and $a, b \in z \sqcup w$ then $a \sqcap b \subseteq [a] = [b]$.*

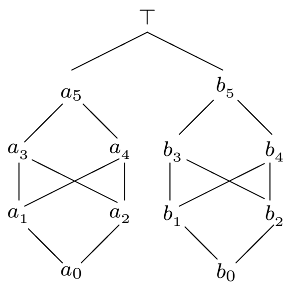
Proof.

1. As $b \equiv a$ then $a \sqcup b \mathrel{\widehat{\equiv}} a \sqcup a = a$ which implies $\varnothing \neq a \sqcup b \subseteq [a]$. The other result is proved similarly.

2. Assume that there are $z \in a \sqcap b$ and $w \in a \sqcup b$ such that $z \equiv w$. Then $a = a \sqcap w \mathrel{\widehat{\equiv}} a \sqcap z = z = b \sqcap z \mathrel{\widehat{\equiv}} b \sqcap w = b$ and, as a result, $a \equiv b$.
3. It is trivial in the case that $z = w$. Consider $z, w \in a \sqcup b$ with $z \neq w$ and $a, b \in z \sqcap w$. If $z \equiv w$, by item 1, $a, b \in z \sqcap w \subseteq [z]$ and, therefore, $a \equiv b$. Finally, applying again item 1, we get $a \sqcup b \subseteq [a] = [b]$.
4. Similar to 3. □

Notice that it would seem that if a multilattice has at least one congruence (different from the identity), then it is full. It is not true, as shown by the following counterexample:

Example 10. Let $(M, \leq)$ be the following multilattice.



This multilattice is not full, yet there exists at least one non-trivial congruence: for instance, that given by the partition

$$\{\{\top\}, \{a_i \mid 0 \leq i \leq 5\}, \{b_i \mid 0 \leq i \leq 5\}\}$$

□

Lemma 11. *Let $\equiv$ be a congruence relation in a multilattice M, and consider $a, b, t \in M$. If $a \leq b$ with $a \equiv b$, then*

1. *For all $z \in a \sqcap t$ we have that $\varnothing \neq (b \sqcap t) \cap z\!\uparrow \subseteq [z]$.*
2. *For all $z \in b \sqcup t$ we have that $\varnothing \neq (a \sqcup t) \cap z\!\downarrow \subseteq [z]$.*

Proof. If $z \in a \sqcap t$, then $z \leq a \leq b$ and $z \leq t$ and, since M is a multilattice, there exists $w \in b \sqcap t$ with $z \leq w$. Moreover, for any $w \in (b \sqcap t) \cap z\!\uparrow$, it is easy to prove that $z \in a \sqcap w$. As $a \equiv b$, we have that $z \in a \sqcap w \mathrel{\widehat{\equiv}} b \sqcap w = w$. The second item can be proved analogously. □

The following result can be viewed as a suitable generalisation to multilattices of a similar result about lattices given by Grätzer [23, page 26]. Its usefulness arises in that it reduces the set of requirements to be checked in order to prove that a given binary relation is indeed a congruence.

Theorem 12 (See [10]). *Let $(M, \sqcup, \sqcap)$ be a multilattice and $\mathcal{R}$ be a binary relation. Then $\mathcal{R}$ is a congruence relation if and only if the following hold:*

1. *$\mathcal{R}$ is reflexive*
2. *$x\mathcal{R}y$ if and only if there exist $z \in x \sqcap y$ and $w \in x \sqcup y$ with $z\mathcal{R}w$*
3. *If $x \leq y \leq z$ with $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$*
4. *If $x \leq y$ with $x\mathcal{R}y$, then $x \sqcap t\, \widehat{\mathcal{R}}\, y \sqcap t$ and $x \sqcup t\, \widehat{\mathcal{R}}\, y \sqcup t$.*

It is well-known that, for every set A, the set of equivalence relations on A, $Eq(A)$, with the inclusion ordering (in the powerset of $A \times A$) is a complete lattice in which the infimum is the meet and the supremum is the transitive closure of the join. In [10] the authors proved that the set of congruences on a multilattice is a complete lattice under the assumption of m-distributivity. This requirement can be avoided by means of a more involved proof which is given below:

Theorem 13. *The set of the congruences in a multilattice is a complete lattice with respect to the inclusion ordering.*

Proof. Let $\{\equiv_i\}_{i \in \Lambda}$ be a set of congruences in M, consider $\equiv_\cap$ to be its intersection and $\equiv_{tc}$ be the transitive closure of their union.

Since $\equiv_\cap$ and $\equiv_{tc}$ are equivalence relations, they satisfy the items 1 and 3 of Theorem 12. On the other hand, item 2 is a consequence of Lemma 9. Thus, in order to show that both relations $\equiv_\cap$ and $\equiv_{tc}$ are congruences, we have just to check item 4. We begin with the latter, $\equiv_{tc}$, and prove its compatibility with the operations.

Consider $x \equiv_{tc} y$, that is, there exists a sequence $x_1, \ldots, x_n$ such that $x_1 = x$, $x_n = y$ and $x_1 \equiv_{i_1} x_2 \equiv_{i_2} \cdots \equiv_{i_{n-1}} x_n$ with $i_1, i_2, \cdots, i_{n-1} \in \Lambda$. Then $x_1 \sqcup t \widehat{\equiv}_{i_1} x_2 \sqcup t \widehat{\equiv}_{i_2} \ldots \widehat{\equiv}_{i_{n-1}} x_n \sqcup t$ and $x_1 \sqcap t \widehat{\equiv}_{i_1} x_2 \sqcap t \widehat{\equiv}_{i_2} \ldots \widehat{\equiv}_{i_{n-1}} x_n \sqcap t$. Therefore, $x \sqcup t \;\widehat{\equiv}_{tc}\; y \sqcup t$ and $x \sqcap t \;\widehat{\equiv}_{tc}\; y \sqcap t$.

Now, for $\equiv_\cap$, let us consider $x \leq y$ with $x \equiv_\cap y$. Lemma 11 (item 1) ensures that, if $z \in x \sqcap t$, then there exists $w \in y \sqcap t$ with $z \equiv_\cap w$.

Conversely, given $w \in y \sqcap t$, we should prove that there exists $z \in x \sqcap t$ such that $z \equiv_\cap w$. For all $i \in \Lambda$, we have $x \sqcap w \;\widehat{\equiv}_i\; y \sqcap w = w$. That is, for all $u \in x \sqcap w$, the congruence $u \equiv_i w$ holds. Now, since $u \leq x, t$, there exists elements $z \in x \sqcap t$ with $u \leq z$. For **any** of these elements z, by Lemma 11 (item 1), for all $w' \in y \sqcap t$ with $z \leq w'$ (and there exist at least one such element) we have that $w' \equiv_i z$.

As $w, w' \in y \sqcap t$, then, in particular, y and t are upper bounds of $\{w, w'\}$. Thus, there exist $a, b \in w \sqcup w'$ with $a \leq y$ and $b \leq t$ and, moreover, $w, w' \in a \sqcap b$.

Now, recalling that $w \equiv_i u$ and $u \leq z \leq w'$, applying the properties of congruence we obtain $a, b \in w \sqcup w' \mathrel{\widehat{\equiv}_i} u \sqcup w' = w'$, (i.e. $a \equiv_i w' \equiv_i b$) $a, b \in w \sqcup w'$ and $w, w' \in a \sqcap b$. By Lemma 9 (item 3), $w \sqcup w' \subseteq [w]_i = [w']_i$. Therefore, as we already proved $z \equiv_i w'$, we have $z \equiv_i w$ as well.

Finally, as this argument is uniformly applicable to all $i \in \Lambda$, we conclude that, for any $u \in x \sqcap w$ and any $z \in x \sqcap t$ with $u \leq z$, $z \equiv_\cap w$.

For $\sqcup$ we proceed similarly. □

Theorem 14. *Let $h\colon M \to M'$ be a map between multilattices such that*

$$h(a \sqcup b) = \big(h(a) \sqcup h(b)\big) \cap h(M)$$
$$h(a \sqcap b) = \big(h(a) \sqcap h(b)\big) \cap h(M)$$

Then, the kernel relation of h is a congruence.

Proof. Let us denote the kernel relation as $\equiv$; trivially, it is an equivalence. Let us now consider $a \equiv b$ and $x \in a \sqcup c$, for $c \in M$. Then, $h(x) \in h(a \sqcup c) = \big(h(a) \sqcup h(c)\big) \cap h(M) = \big(h(b) \sqcup h(c)\big) \cap h(M) = h(b \sqcup c)$. Hence, there exists $y \in b \sqcup c$ such that $h(x) = h(y)$, that is $x \equiv y$. In the same way, given $y \in b \sqcup c$, there exists $x \in a \sqcup c$ with $y \equiv x$, thus $a \sqcup c \mathrel{\widehat{\equiv}} b \sqcup c$. Dually, $a \sqcap c \mathrel{\widehat{\equiv}} b \sqcap c$, for all $a, b, c \in M$. □

Let $\equiv$ be a congruence relation defined over a multilattice M and $M/{\equiv}$ be the set of equivalence classes. Our aim is to prove that $M/{\equiv}$ has a multilattice structure, therefore we start by defining an ordering relation.

Consider the following binary relation on $M/{\equiv}$:

$$\begin{aligned} [a] \leq [b] \quad &\overset{\text{def}}{\iff} \quad \varnothing \neq a \sqcup b \subseteq [b] \\ &\iff \quad a \sqcup b \mathrel{\widehat{\equiv}} b \end{aligned} \tag{2}$$

Among other things, we have to prove that $\leq$ is an ordering relation; thus, firstly we will establish an alternative characterization of the relation in terms of the so-called Hoare, Smyth, and Egli-Milner powerset preorders (which are recalled below):

Given P an arbitrary set and $\leq$ a preorder (reflexive and transitive relation) defined over P, it is possible to lift the preorder structure to the powerset $\mathcal{P}(P)$ by defining

$$\begin{array}{lcl} X \sqsubseteq_H Y & \Leftrightarrow & \text{for all } x \in X \text{ there exists } y \in Y \text{ such that } x \leq y \\ X \sqsubseteq_S Y & \Leftrightarrow & \text{for all } y \in Y \text{ there exists } x \in X \text{ such that } x \leq y \\ X \sqsubseteq_{EM} Y & \Leftrightarrow & \text{for all } x \in X \text{ there exists } y \in Y \text{ such that } x \leq y \text{ and} \\ & & \text{for all } y \in Y \text{ there exists } x \in X \text{ such that } x \leq y \end{array}$$

The following result shows that, in our particular framework, the three definitions above coincide and are equivalent to the relation defined in (2).

Proposition 15. *Let $\equiv$ be a congruence relation defined over a multilattice M. Then, for all $a, b \in M$, the following conditions are equivalent:*

1. $[a] \leq [b]$
2. $[a] \sqsubseteq_{EM} [b]$
3. $[a] \sqsubseteq_H [b]$
4. $[a] \sqsubseteq_S [b]$

Proof. $(1 \Rightarrow 2)$. Assume $[a] \leq [b]$, that is $\varnothing \neq a \sqcup b \subseteq [b]$, and consider $x \in [a]$. Then, $x \sqcup b \mathrel{\widehat{\equiv}} a \sqcup b \mathrel{\widehat{\equiv}} b$, so there exists $y \in x \sqcup b$ such that $y \equiv b$ (i.e. $x \leq y$ and also $y \in [b]$). Given $y \in [b]$, there exists $z \in a \sqcup b$ such that $z \equiv y$ and, therefore, $a = z \sqcap a \mathrel{\widehat{\equiv}} y \sqcap a$. Every element $x \in y \sqcap a$ satisfies $x \in [a]$ and $x \leq y$.

$(2 \Rightarrow 3)$ and $(2 \Rightarrow 4)$. Trivial.

$(3 \Rightarrow 1)$. Assuming $[a] \sqsubseteq_H [b]$, there exists $y \equiv b$ such that $a \leq y$. Observe that $y = a \sqcup y$ and then $b \equiv y \mathrel{\widehat{\equiv}} a \sqcup b$, that is $\varnothing \neq a \sqcup b \subseteq [b]$.

$(4 \Rightarrow 1)$. Similarly. □

Theorem 16. *Let $(M, \sqcup, \sqcap)$ be a multilattice and $\equiv$ a congruence relation, then $M/\!\equiv$ is a multilattice with the operations*

$$[a] \sqcup [b] = \{[x] : x \in a \sqcup b\} \qquad [a] \sqcap [b] = \{[x] : x \in a \sqcap b\}$$

As a consequence, the mapping $p \colon M \to M/\!\equiv$ defined as $p(x) = [x]$ is a surjective homomorphism that preserves multisuprema and multiinfima.

Proof. Firstly, by the previous proposition, the binary relation $\leq$ defined in (2) can be seen as any of the powerset preorders, hence it is reflexive and transitive; furthermore, the second line in the definition of $\leq$ shows that is antisymmetric as well. Hence, it is a partial order relation.

We will prove now that for every upper bound $[z]$ of $\{[a],[b]\}$, there exists $c \in a \sqcup b$ such that $[c] \leq [z]$: given $[z] \geq [a],[b]$, there exist $x, y \in M$ such that $x \equiv z \equiv y$ with $a \leq x$ and $b \leq y$. From $x \equiv y$ follows that $a \sqcup x \mathrel{\widehat{\equiv}} a \sqcup y$ and since $a \leq x$, we have $x \mathrel{\widehat{\equiv}} a \sqcup y$. As a consequence, $a \sqcup y$ is nonempty and any $t \in a \sqcup y$ satisfies $x \equiv t$. Observe that $b \leq y \leq t$ and $a \leq t$, so being M a multilattice, there exists $c \in a \sqcup b$ such that $c \leq t$. Since $x \equiv t \equiv z$, then $c \sqcup x \mathrel{\widehat{\equiv}} c \sqcup t = t$ and $c \sqcup x \mathrel{\widehat{\equiv}} z$; moreover, using again $x \equiv z$, we have $c \sqcup x \mathrel{\widehat{\equiv}} c \sqcup z$ which, together with the previous $\widehat{\equiv}$-chains, implies $c \sqcup z \mathrel{\widehat{\equiv}} z$ or, equivalently, $c \sqcup z \subseteq [z]$, that is $[c] \leq [z]$.

To show that $[a] \sqcup [b]$ consists of the minimal elements of the set of upper bounds of $\{[a],[b]\}$, we firstly note that two different elements $[c],[d] \in [a] \sqcup [b]$ should be incomparable: suppose that $c, d \in a \sqcup b$ such that $[c] \leq [d]$. As $c, d \in a \sqcup b$, then, in particular, a, b are lower bounds of $\{c, d\}$. Thus, there exist $a', b' \in c \sqcap d$ with $a \leq a'$ and $b \leq b'$. Moreover, $c, d \in a' \sqcup b'$.

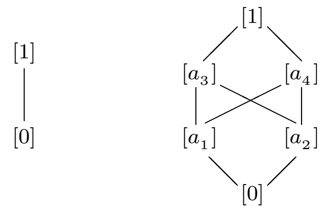
Since $[c] \leq [d]$, for all $x \in c \sqcup d$ we have that $x \equiv d$, then $c = c \sqcap x \mathrel{\widehat{\equiv}} c \sqcap d$ therefore, $a' \equiv c \equiv b'$. By Lemma 9 (item 4) we obtain $c \equiv d$.

Finally, to prove that $[a] \sqcup [b]$ consists of minimal elements, consider $x \in a \sqcup b$ and assume that $[z]$ is an upper bound of $\{[a],[b]\}$ such that $[z] \leq [x]$. As we have proved above, there exists $y \in a \sqcup b$ such that $[y] \leq [z]$. Then, we have $x, y \in a \sqcup b$ such that $[y] \leq [x]$ and then, $[y] = [x] = [z]$. □

Example 17. In the multilattice described in Example 8, we have the following congruence relations:

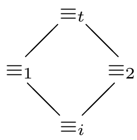
$\equiv$	$M/_{\equiv}$
$\equiv_t$	$\{\{0, a_1, a_2, a_3, a_4, d, c, b_1, b_2, b_3, b_4, 1\}\}$
$\equiv_1$	$\{\{0, a_1, a_2, a_3, a_4, d\},$ $\{c, b_1, b_2, b_3, b_4, 1\}\}$
$\equiv_2$	$\{\{0, c\}, \{a_1, b_1\}, \{a_2, b_2\},$ $\{a_3, b_3\}, \{a_4, b_4\}, \{d, 1\}\}$
$\equiv_i$	$\{\{0\}, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}, \{d\},$ $\{c\}, \{b_1\}, \{b_2\}, \{b_3\}, \{b_4\}, \{1\}\}$

The non-trivial quotient multilattices are the following:



$(M/{\equiv_1}, \leq)$ $(M/{\equiv_2}, \leq)$

and the lattice of congruences $(Con(M), \subseteq)$ is:



$\square$

We move now to the study of the notion of filter in a multilattice. There exist several ways to give a definition for this notion; in the rest of the section, we introduce the one which is more suitable for extending the classical results about congruences and homomorphisms.

Definition 18. Let $(M, \sqcup, \sqcap)$ be a multilattice. A non-empty set $F \subseteq M$ is said to be a **filter** if the following conditions hold:

1. $i, j \in F$ implies $\varnothing \neq i \sqcap j \subseteq F$.
2. $i \in F$ implies $i \sqcup a \subseteq F$ for all $a \in M$.
3. For all $a, b \in M$, if $(a \sqcup b) \cap F \neq \varnothing$ then $a \sqcup b \subseteq F$.

Theorem 19. *The set of filters in a full multilattice is a complete lattice with the inclusion ordering.*

Proof. If M is a full multilattice then, the arbitrary intersection of filters is non-empty and trivially satisfies the three conditions of the definition of filter. As M itself is a filter, therefore the set of filters $(\mathcal{F}(M), \subseteq)$ is a complete inf-semilattice with top element, hence, a complete lattice. $\square$

Theorem 20. *Let $(M, \sqcup, \sqcap)$ be a multilattice with top element $\top$, and let $\equiv$ be a congruence relation. Then, the equivalence class $[\top]$ is a filter of M.*

Proof. We prove the three conditions in the definition of filter:

1. Consider $i, j \in [\top]$. In particular, since both are congruent with $\top$, we have that $i \equiv j$ and, as a result, $i \sqcap j \mathrel{\widehat{\equiv}} j \sqcap j = j$, which implies $\varnothing \neq i \sqcap j \subseteq [\top]$.
2. Now, given $i \in [\top]$, for all $a \in M$ we have $i \sqcup a \mathrel{\widehat{\equiv}} \top \sqcup a = \top$ which implies $i \sqcup a \subseteq [\top]$.
3. Consider $a, b \in M$ and $x \in (a \sqcup b) \cap [\top]$. Then, $x \in a \sqcup b$, and $x \in [\top]$ and so $x \equiv \top$. Now, given $y \in a \sqcup b$ with $x \neq y$, we will prove that $y \equiv \top$.
   Since $a \leq x$ and $a \leq y$ there exists $a' \in x \sqcap y$ such that $a \leq a'$. Analogously, $b \leq x$ and $b \leq y$ there exists $b' \in x \sqcap y$ such that $b \leq b'$. We have $a', b' \in x \sqcap y$, $x, y \in a' \sqcup b'$ and, since $x \sqcap y \mathrel{\widehat{\equiv}} \top \sqcap y = y$, thus $a' \equiv b'$. By Lemma 9 (item 4), we have that $x \sqcap y \subseteq [y] = [x] = [\top]$.
   □

Finally, the relation between filters and homomorphisms is stated in the following result which, follows from the particular choice for the definition of filter that we have introduced.

Theorem 21. *Let $h \colon M \to M'$ be a multilattice homomorphism and assume that M is full and M' has a top element $\top$ such that $\top \in h(M)$. Then $h^{-1}(\top)$ is a filter of M, called the* ***kernel filter***.

Proof.

1. Given $i, j \in h^{-1}(\top)$ and $x \in i \sqcap j$ (that there exists because M is full), we have that $h(x) \in h(i \sqcap j) \subseteq h(i) \sqcap h(j) = \top$.
2. Consider $i \in h^{-1}(\top)$ and $a \in M$. For all $x \in i \sqcup a$, it holds that $h(x) \in h(i) \sqcup h(a) = \top \sqcup h(a) = \top$.
3. Finally, consider $a, b \in M$ and $x_0 \in (a \sqcup b) \cap h^{-1}(\top)$. Firstly, observe that $\top = h(x_0) \in h(a \sqcup b) \subseteq h(a) \sqcup h(b)$. Moreover, for all $x \in a \sqcup b$, we have $h(x) \in h(a) \sqcup h(b)$, as $\top$ is an element of $h(a) \sqcup h(b)$, therefore $\top = h(x)$, which proves that $a \sqcup b \subseteq h^{-1}(\top)$. □

Remark 22. It is worth to note that the previous proof can be easily adapted to prove that the inverse image of a filter is a filter.

From the previous results, the following theorem holds straightforwardly.

Theorem 23. *Let $h\colon M \to M'$ be a map between multilattices such that*

$$h(a \sqcup b) = \big(h(a) \sqcup h(b)\big) \cap h(M)$$
$$h(a \sqcap b) = \big(h(a) \sqcap h(b)\big) \cap h(M)$$

Let $\equiv$ be the kernel relation of h. Then, h can be canonically decomposed as $h = i \circ \overline{h} \circ p$ where the mapping $\overline{h}\colon M/{\equiv} \to h(M)$ is the isomorphism defined as $\overline{h}([x]) = h(x)$ and $i\colon h(M) \to M'$ is the inclusion monomorphism.

3. Residuated multilattices: algebraic structures

Residuation has a prominent role in the algebraic study of logical systems, which usually are modeled as partially ordered sets together with some operations reflecting the properties of the connectives. This section is related to the use of residuated implication in the framework of multilattices as a useful tool for fuzzy logic reasoning.

Although the most used structure in this context is that of residuated lattice, there are reasons for weakening some of its properties, thus leading to a more general class of algebraic structures for computation. A commonly considered algebraic structure is that of partially ordered commutative residuated integral monoid [6].

Definition 24. A tuple $\mathcal{A} = (A, \leq, \odot, \to, \top)$ is said to be a *partially ordered commutative residuated integral monoid*, briefly a **pocrim**, if the following properties hold:

1. $(A, \odot, \top)$ is a commutative monoid with neutral element $\top$.
2. $(A, \leq)$ is a poset with maximum $\top$.
3. the operations $\odot$ and $\to$ satisfy the adjointness condition, that is $a \odot c \leq b$ if and only if $c \leq a \to b$, for every $a, b, c \in A$.

A pocrim $\mathcal{A}$ is said to be a **residuated lattice** if $(A, \leq)$ is a lattice.

We now recall the following useful conditions that hold in a pocrim A, for all $x, y, z \in A$, which will be used hereinafter:

P1 $\quad x \odot y \leq x, y$

P2 $\quad x \odot (x \to y) \leq x \leq y \to (x \odot y)$ and $x \odot (x \to y) \leq y \leq x \to (x \odot y)$

P3 $\quad$ If $x \leq y$, then $x \odot z \leq y \odot z$, $z \to x \leq z \to y$, and $y \to z \leq x \to z$

P4 $x \to (y \to z) = y \to (x \to z)$

P5 $(x \to y) \odot (y \to z) \leq x \to z$

P6 $x \to y \leq (x \odot z) \to (y \odot z)$

P7 $x \to y \leq (z \to x) \to (z \to y)$ and $x \to y \leq (y \to z) \to (x \to z)$

In this section we focus on the translation of the main properties of residuated lattices, as those introduced in [12], by using residuated multilattices. To begin with, we should define what such a term means:

Definition 25. A **residuated multilattice** is a pocrim whose underlying poset is a multilattice. If, in addition, there exists a bottom element, the residuated multilattice is said to be **bounded**.

Remark 26.

1. Observe that, as a consequence of the definition of pocrim, $\top$ is the top element in the multilattice. To ease notation, from now on, every residuated multilattice will be denoted just by M, when no confusion arises.
2. Notice that every residuated multilattice is full: for all $a, b \in M$ we have that $a, b \leq \top$ and, therefore, $a \sqcup b \neq \varnothing$. Furthermore, $a \odot b \leq a$, and $a \odot b \leq b$, hence $a \sqcap b \neq \varnothing$.
3. Recall that any finite poset is actually a multilattice, in fact the only proper examples of pocrims not multilattices have to be infinite.

The following example, taken from [6] where it was included with a different purpose, shows a *proper* residuated multilattice, in that its carrier is not a lattice.

Example 27. Let $\mathbb{Z}$, $\mathbb{Z}^-$ and $\mathbb{Z}^+$ denote, respectively, the sets of all integers, of all non-positive integers, and of all non-negative integers. Let $\bot, \top \notin \mathbb{Z}$, the set $A = \big(\{\bot\} \times \mathbb{Z}^+\big) \cup \big(\mathbb{Z}^+ \times \mathbb{Z}\big) \cup \big(\{\top\} \times \mathbb{Z}^-\big)$ and $\leq$ be the partial ordering on A depicted in Figure 1, which can be expressed as follows

$$\langle \alpha, i \rangle \leq \langle \beta, j \rangle \quad \text{iff} \quad i + |\alpha - \beta| \leq j$$

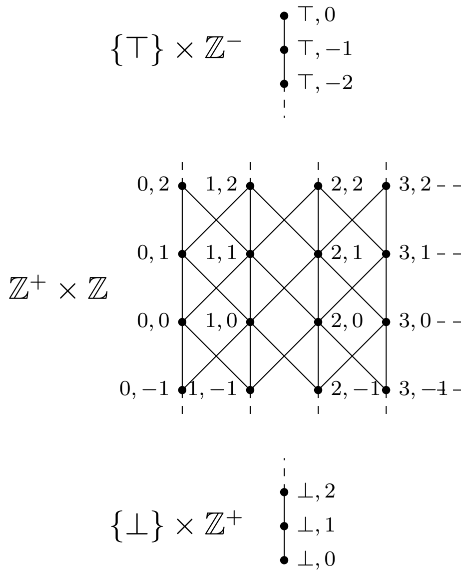


Figure 1: Hasse Diagram of $(A, \leq)$

The operation $\odot$ on A is defined as

$$
\begin{aligned}
& x \odot y = y \odot x \\
& \langle \top, i\rangle \odot \langle \top, j\rangle = \langle \top, i+j\rangle && (i, j \leq 0) \\
& \langle \top, i\rangle \odot \langle \alpha, j\rangle = \langle \alpha, i+j\rangle && (i \leq 0) \\
& \langle \top, i\rangle \odot \langle \bot, j\rangle = \langle \bot, \max\{0, i+j\}\rangle && (i \leq 0 \leq j) \\
& \langle \alpha, i\rangle \odot \langle \beta, j\rangle = \langle \bot, \max\{0, i+j+|\alpha-\beta|\}\rangle \\
& \langle \alpha, i\rangle \odot \langle \bot, j\rangle = \langle \bot, k\rangle \odot \langle \bot, j\rangle = \langle \bot, 0\rangle && (0 \leq j, k)
\end{aligned}
$$

Finally, $(A, \leq, \odot, \rightarrow, \langle \top, 0\rangle)$ is a residuated multilattice when considering the following residuated implication:

$$x \to y = \langle \top, 0 \rangle \quad \text{iff} \quad x \le y$$
$$\langle \top, i \rangle \to \langle \top, j \rangle = \langle \top, \min\{0, j - i\} \rangle \qquad (i, j \le 0)$$
$$\langle \top, i \rangle \to \langle \alpha, j \rangle = \langle \alpha, j - i \rangle \qquad (i \le 0)$$
$$\langle \top, i \rangle \to \langle \bot, j \rangle = \langle \bot, j - i \rangle \qquad (i \le 0 \le j)$$
$$\langle \alpha, i \rangle \to \langle \beta, j \rangle = \langle \top, \min\{0, j - i - |\alpha - \beta|\} \rangle$$
$$\langle \alpha, i \rangle \to \langle \bot, j \rangle = \langle \alpha, j - i \rangle \qquad (0 \le j)$$
$$\langle \bot, i \rangle \to \langle \bot, j \rangle = \langle \top, \min\{0, j - i\} \rangle \qquad (0 \le i, j)$$

□

Before starting the systematic study of residuated multilattices, let us introduce a lemma with a general property of multilattices which will be useful later.

Lemma 28. *Let M be a multilattice, $X \subseteq M$ and $a, b \in M$.*

1. $a \sqcup b \subseteq X \subseteq (a \sqcup b)\!\uparrow$ *implies* $a \sqcup b = \text{minimals}(X)$.
2. $a \sqcap b \subseteq X \subseteq (a \sqcap b)\!\downarrow$ *implies* $a \sqcap b = \text{maximals}(X)$.

Proof. We only prove the first item, as the second follows similarly.

For all $x \in a \sqcup b \subseteq X$, if $y \in X$ such that $y \le x$, then there exists $z \in a \sqcup b$ such that $z \le y \le x$ and, as the elements in $a \sqcup b$ are pairwise incomparable, $z = y = x$, and x is a minimal element of X.

Conversely, if $x \in \text{minimals}(X)$, using now $X \subseteq (a \sqcup b)\!\uparrow$, there exists $y \in a \sqcup b$ with $y \le x$ and, since $y \in a \sqcup b \subseteq X$ and x is minimal, then $y = x$. Thus $x \in a \sqcup b$. □

Proposition 29. *In a residuated multilattice M, the following conditions hold, for all $x, y, z \in M$:*

1. $x \odot y, x \odot (x \to y) \in (x \sqcap y)\!\downarrow$
2. $x \odot (y \sqcap z) \subseteq [(x \odot y) \sqcap (x \odot z)]\!\downarrow$
3. $x \odot (y \sqcup z) \subseteq [(x \odot y) \sqcup (x \odot z)]\!\uparrow$
4. $(x \odot y) \sqcup (x \odot z) \subseteq x \odot (y \sqcup z)$

PROOF.

1. It follows from **P1** and **P2**, and the definition of multilattice.
2. For $m \in y \sqcap z$ and using **P3**, we have that $x \odot m \leq x \odot y, x \odot z$, thus, by definition of multilattice, there exists $c \in (x \odot y) \sqcap (x \odot z)$ such that $x \odot m \leq c$.
3. Given $m' \in y \sqcup z$, as $y, z \leq m'$, then $x \odot y, x \odot z \leq x \odot m'$. Then there exists $m'' \in (x \odot y) \sqcup (x \odot z)$ such that $m'' \leq x \odot m'$.
4. If $a \in (x \odot y) \sqcup (x \odot z)$ then $x \odot y, x \odot z \leq a$, by adjointness condition, $y, z \leq x \to a$ and then, there exists $b \in y \sqcup z$ such that $b \leq x \to a$ which implies $x \odot b \leq a$. But $x \odot y, x \odot z \leq x \odot b$, therefore, by minimality, $a = x \odot b \in x \odot (y \sqcup z)$. □

The above proposition and Lemma 28 lead to the following result.

Corollary 30. *In a residuated multilattice M, for all $x, y, z \in M$,*

$$(x \odot y) \sqcup (x \odot z) = \text{minimals}\{x \odot (y \sqcup z)\}$$

The equality in the previous corollary need not hold if the minimals are not considered in the right-hand side.

Example 31. The multilattice in Example 27 illustrates the fact that, in general, one has $(a \odot b) \sqcup (a \odot c) \neq a \odot (b \sqcup c)$. Specifically,

$$\langle 0,0\rangle \sqcup \langle 1,0\rangle = \{\langle 0,1\rangle, \langle 1,1\rangle\}$$

$$\langle 2,0\rangle \odot (\langle 0,0\rangle \sqcup \langle 1,0\rangle) = \{\langle 2,0\rangle \odot \langle 0,1\rangle, \langle 2,0\rangle \odot \langle 1,1\rangle\} = \{\langle \bot,3\rangle, \langle \bot,2\rangle\}$$

$$\langle 2,0\rangle \odot \langle 0,0\rangle \sqcup \langle 2,0\rangle \odot \langle 1,0\rangle = \langle \bot,2\rangle \sqcup \langle \bot,1\rangle = \langle \bot,2\rangle$$

□

Proposition 32. *In a residuated multilattice M, the following inclusions hold for all $x, y, z \in M$:*

1. $(x \sqcap y) \to z \subseteq [(x \to z) \sqcup (y \to z)]\!\uparrow$
2. $z \to (x \sqcup y) \subseteq [(z \to x) \sqcup (z \to y)]\!\uparrow$
3. $(x \sqcup y) \to z \subseteq [(x \to z) \sqcap (y \to z)]\!\downarrow$
4. $z \to (x \sqcap y) \subseteq [(z \to x) \sqcap (z \to y)]\!\downarrow$
5. $[(x \to z) \sqcap (y \to z)] \subseteq (x \sqcup y) \to z$

6. $[(z \to x) \sqcap (z \to y)] \subseteq z \to (x \sqcap y)$

Proof. Inclusions 1-4 follow from **P3**, adjointness, and the definition of multilattices.

For item 5, let $m \in (x \to z) \sqcap (y \to z)$. Then, as $m \leq x \to z, y \to z$, using adjointness, we have that $x \odot m, y \odot m \leq z$ and so there exists $m' \in (x \odot m) \sqcup (y \odot m)$ such that $m' \leq z$. Using Proposition 29 (item 4), there exists $m'' \in x \sqcup y$ such that $m' = m \odot m'' \leq z$ which implies $m \leq m'' \to z$. On the other hand, as $m'' \in x \sqcup y$, we have that $x, y \leq m''$ and using **P3**, $m'' \to z \leq x \to z, y \to z$ and so there exists $m''' \in (x \to z) \sqcap (y \to z)$ such that $m'' \to z \leq m'''$. So, we have that $m \leq m'' \to z \leq m'''$ and, as m and m''' should be either equal or incomparable, we obtain $m = m''' = m'' \to z$.

Inclusion 6 follows the same pattern as the previous one. $\square$

Again in conjunction with Lemma 28, the previous proposition leads to the following result.

Corollary 33. *In a residuated multilattice M, the following conditions hold, for all $x, y, z \in M$:*

1. $(x \to z) \sqcap (y \to z) = \text{maximals}\{(x \sqcup y) \to z\}$
2. $(z \to x) \sqcap (z \to y) = \text{maximals}\{z \to (x \sqcap y)\}$
3. $x \to y = \max\{(x \sqcup y) \to y\} = \max\{x \to (x \sqcap y)\}$

The next proposition introduces two other results on the relation between product and multilattice operations:

Proposition 34. *In a residuated multilattice M, the following conditions hold for all $x, x', y, y' \in M$:*

1. $(x \to y) \odot (x' \to y') \in [(x \sqcup x') \to (y \sqcup y')]\downarrow$
2. $(x \to y) \odot (x' \to y') \in [(x \sqcap x') \to (y \sqcap y')]\downarrow$

Proof.

1. For all $m \in y \sqcup y'$, the element $(x \to y) \odot (x' \to y')$ is a lower bound of $\{x \to m, x' \to m\}$ because $(x \to y) \odot (x' \to y') \leq x \to y \leq x \to m$ and the same for x'. By using Proposition 32 (item 5), observe that

$$(x \to m) \sqcap (x' \to m) \subseteq (x \sqcup x') \to m \subseteq (x \sqcup x') \to (y \sqcup y')$$

Therefore, we obtain that

$$(x \to y) \odot (x' \to y') \in [(x \to m) \sqcap (x' \to m)]\downarrow \subseteq [(x \sqcup x') \to (y \sqcup y')]\downarrow$$

2. It follows the same scheme as previous item. □

In the rest of the section, we will focus on the relation between residuated multilattices and Heyting algebras.

Definition 35. A residuated lattice in which $\odot$ coincides with the meet operation is said to be a **Heyting algebra**.

To begin with, the following proposition shows a first condition which collapses a residuated multilattice into a Heyting algebra.

Proposition 36. *Let M be a residuated multilattice such that $a \odot b \in a \sqcap b$ for all $a, b \in M$, then M is a Heyting algebra.*

Proof. Consider $x \in a \sqcap b$, then $x \leq a$ and $x = a \sqcap x = a \odot x$ (the latter equality holds by the hypothesis), and the same for b. Thus $a \odot b \odot x = a \odot x = x$ which implies that $x \leq a \odot b$. Now, as both $x, a \odot b$ belong to $a \sqcap b$, then $x = a \odot b$. We have obtained that, for all $a, b \in M$, $a \odot b = a \sqcap b$, that is, there exists the infimum for all a and b. Since M is full, by results in [26, Lemma 5.1 and Theorem 5.14 (part of the proof)], there also exists the supremum of a and b and M is a lattice in which $\odot$ coincides with the infimum operation. □

The following lemma elaborates on the previous proposition in order to obtain the same result on the basis of alternative hypotheses.

Lemma 37. *Let M be a residuated multilattice with idempotent product, then, for all $a, b \in M$,*

1. *If $x \in a \sqcup b$, then $a \odot x = a$.*
2. *$a \leq b$ if and only if $a \odot b = a$.*
3. *$a \odot b \in a \sqcap b$.*

Proof. The key property here is the idempotence of the product $\odot$:

1. Observe that $a = a \sqcup (a \odot b) = (a \odot a) \sqcup (a \odot b)$ which, by Corollary 30, equals to minimals$\{a \odot (a \sqcup b)\}$. Therefore, if $x \in a \sqcup b$, then $a \leq a \odot x$. As, by monotonicity, $a \odot x \leq a$, we have $a \odot x = a$.
2. If $a \leq b$, then $a \odot b \leq a \odot \top = a$ and $a = a \odot a \leq a \odot b$ and, hence, $a \odot b = a$. Conversely, if $a \odot b = a$, then $\top = a \to a = a \to a \odot b \leq a \to b$ which implies $a \leq b$.

3. By Proposition 29 (item 1), there exists $c \in a \sqcap b$ such that $a \odot b \leq c$ On the other hand, by item 2 above, since $c \leq b$ and $c \leq a$, we have that $a \odot b \odot c = a \odot c = c$ and, again by item 2, $c \leq a \odot b$. Therefore, $a \odot b = c \in a \sqcap b$. □

Theorem 38. *Any idempotent residuated multilattice is a Heyting algebra.*

Proof. Direct consequence of the previous lemma and proposition. □

Usually, the so-called *natural ordering* relation is considered in connection to an algebraic structure with a binary operation $\odot$:

$$a \sqsubseteq b \quad \text{if and only if} \quad a \odot b = a$$

In the framework of residuated multilattices, the operation $\odot$ is assumed to be both associative and commutative, and this implies anti-symmetry and transitivity of $\sqsubseteq$. Moreover, this relation is included in $\leq$. That is, $a \sqsubseteq b$ implies $a \leq b$ (it is due to **P1**). Note, finally, that $\sqsubseteq$ is reflexive if and only if the product is idempotent. As a result, we obtain that the natural ordering $\sqsubseteq$ is a partial ordering relation (in a residuated multilattice) exactly in the subclass of Heyting algebras.

We close the section by showing that what is really important is the combination between the structure of residuated multilattice together with idempotency since, in general, the presence of idempotency in a pocrim is not a sufficient condition to guarantee the structure of Heyting algebra, as the following example shows:

Example 39. Let us consider the meet-semilattice $(A, \leq)$ depicted in Figure 2, the product being the meet operator and the residuated implication $\rightarrow$ defined by

$$\begin{array}{ll} x \rightarrow y = \top & \text{iff } x \leq y \\ c_i \rightarrow x = x & \text{for all } x \leq c_i \\ a \rightarrow \bot = a \rightarrow b = b & \\ b \rightarrow \bot = b \rightarrow a = a & \end{array}$$

then $(A, \leq, \odot, \rightarrow, \top)$ is an idempotent pocrim, but it is not a lattice (elements a and b do not have a supremum) and, hence, is not a Heyting algebra.

□

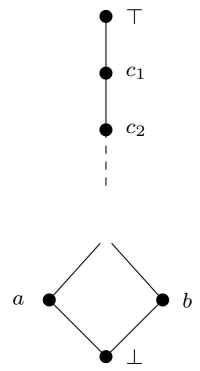


Figure 2: Hasse Diagram of $(A, \leq)$

4. On filters, homomorphisms, and congruences in a residuated multilattice

Concerning applications in logic, the notions of filter and deductive system, closely related to *modus ponens*, deserve to be studied in depth. Thus, we proceed to study the necessary adaptation of this notion to fit the structure of residuated multilattice; to begin with, let us recall the definition of filter in a pocrim. In [7] several kinds of filters and congruence relations in pocrims were introduced.

Definition 40. Given $\mathcal{A} = (A, \leq, \odot, \rightarrow, \top)$ a pocrim, a non-empty subset $F \subseteq A$ is said to be a **filter** if the following conditions hold:

i) if $a, b \in F$, then $a \odot b \in F$

ii) if $a \leq b$ and $a \in F$, then $b \in F$.

On the other hand, F is said to be a **deductive system** if

i) $\top \in F$ and

ii) $a \rightarrow b \in F$ and $a \in F$ imply $b \in F$.

It is not difficult to see that both definitions are equivalent.

Due to the fact that any residuated multilattice combines the structures of multilattice and pocrim, it is possible to use the notion of filter on the multilattice, filter on the pocrim, or give a new definition that combines both. These three approaches are not equivalent, as we will show below. So, in order to distinguish them, we will write **p-filter** to denote a filter of the pocrim (Definition 40), and **m-filter**, a filter of the multilattice (Definition 18). The notion that we are interested in throughout this work, will be called just **filter**.

Definition 41. Let M be a residuated multilattice. A non-empty subset $F \subseteq M$ is said to be a **filter** if it is a deductive system and the following condition holds: $a \to b \in F$ implies $a \sqcup b \to b \subseteq F$ and $a \to a \sqcap b \subseteq F$.

Example 42. Let $(M, \leq)$ be the multilattice described in Figure 3.

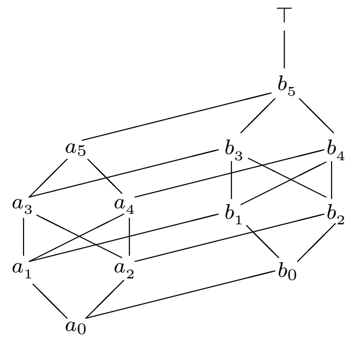


Figure 3:

Let the subsets $A = \{a_i \mid 0 \leq i \leq 5\}$, $B = \{b_i \mid 0 \leq i \leq 5\}$ and $C = \{b_i \mid 2 \leq i \leq 5\}$ and the operations $\odot$ and $\to$ be defined as follows

$$x \odot y = \begin{cases} x & \text{if } y = \top \\ y & \text{if } x = \top \\ b_2 & \text{if } x, y \in C \\ b_0 & \text{if } x \in B \smallsetminus C, y \in B \\ b_0 & \text{if } x \in B, y \in B \smallsetminus C \\ a_0 & \text{otherwise.} \end{cases} \qquad x \to y = \begin{cases} \top & \text{if } x \leq y \\ y & \text{if } x = \top \\ a_5 & \text{if } x \in B, y \in A \\ b_1 & \text{if } x \in C \\ & \text{and } y \in B \smallsetminus C \\ b_5 & \text{otherwise.} \end{cases}$$

It is routine calculation that $(M, \leq, \odot, \to, \top)$ is a residuated multilattice.

We have that $B \cup \{\top\}$ is a filter, p-filter and an m-filter. In the following we show that the alternative notions of filter are, actually, different.

The subset $C \cup \{\top\}$ is a p-filter but it is not a filter because $b_3 \sqcap b_4 = \{b_1, b_2\} \not\subseteq C$. Likewise, $D = \{b_5, \top\}$ is an m-filter that is not a filter because $b_5 = b_5 \to b_2 \in D$, but $b_2 \notin D$. □

Example 43. Consider now the multilattice $(M, \leq)$ described in Fig. 4, in which b_1 is no longer smaller than b_4.

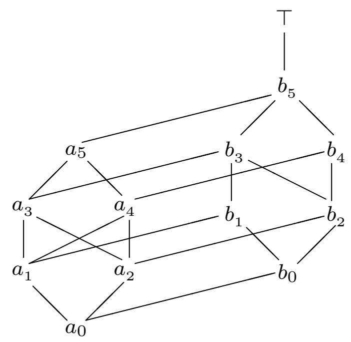


Figure 4:

Let $A_0 = \{a_0, a_1\}, A_1 = \{a_i \mid 2 \leq i \leq 5\}, B_0 = \{b_0, b_1\}, B_1 = \{b_i \mid 2 \leq i \leq 5\}$; moreover, denote $A = A_0 \cup A_1$ and $B = B_0 \cup B_1$ and define the operations $\odot$ and $\rightarrow$ as follows:

$$x \odot y = \begin{cases} x & \text{if } y = \top \\ y & \text{if } x = \top \\ a_0 & \text{if } (x \in A \cup B \text{ and } y \in A_0) \text{ or } (x \in B_0 \text{ and } y \in A \cup B) \\ a_2 & \text{if } (x \in A_1 \cup B_1 \text{ and } y \in A_1) \text{ or } (x \in A_2 \text{ and } y \in B_1) \\ b_0 & \text{if } (x \in B \text{ and } y \in B_0) \text{ or } (x \in B_0 \text{ and } y \in B_1) \\ b_2 & \text{if } x, y \in B_1 \end{cases}$$

$$x \rightarrow y = \begin{cases} \top & \text{if } x \leq y \\ y & \text{if } x = \top \\ a_1 & \text{if } x \in B_1 \text{ and } y \in A_0 \\ a_5 & \text{if } (x \in B_0 \text{ and } y \in A) \text{ or } (x \in B_1 \text{ and } y \in A_1) \\ b_1 & \text{if } (x \in A_1 \text{ and } y \in A_0 \cup B_0) \text{ or } (x \in B_1 \text{ and } y \in B_0) \\ b_5 & \text{otherwise.} \end{cases}$$

Observe that $(M, \leq, \odot, \rightarrow, \top)$ is a residuated multilattice, in which $F = B_1 \cup \top$ is a p-filter and an m-filter, but it is not a filter because $a_3 \rightarrow a_4 = b_5 \in F$ whereas $a_1 \in a_3 \sqcap a_4$ but $a_3 \rightarrow a_1 = b_1 \notin F$. □

The following result characterizes the equivalence between the notion of filter and m-filter.

Theorem 44. *Let M be a residuated multilattice and F a deductive system, then F is a filter* ***if and only if***

1. *F is an m-filter*
2. *for all $x, y \in a \sqcup b$, if $x \to y \in F$ then $y \to x \in F$.*
3. *for all $x, y \in a \sqcap b$, if $x \to y \in F$ then $y \to x \in F$.*

Proof. Assume that F is a filter and, firstly, let us prove that it is an m-filter:

- Consider $a, b \in F$. As $a \leq b \to a$, then $b \to a \in F$. Therefore, $b \to a \sqcap b \subseteq F$. So, given $x \in a \sqcap b$, as $b \to x \in F$ and $b \in F$, then $x \in F$.

- Given $a \in F$ and $b \in M$, consider $x \in a \sqcup b$. Then, as $a \leq x$ we have $a \to x = \top \in F$, hence, by the properties of deductive system, $x \in F$. So, we have $a \sqcup b \subseteq F$.

- On the other hand, suppose that there exists $x \in (a \sqcup b) \cap F$. If $a \sqcup b$ is a singleton, then, trivially, $a \sqcup b \subseteq F$. Otherwise, let $y \in a \sqcup b$ such that $x \neq y$. As $a, b \leq x, y$, there exist two different elements $a', b' \in x \sqcap y$ such that $a \leq a'$ and $b \leq b'$, and also $x, y \in a' \sqcup b'$. Observe that $\top = a' \to x = a' \to y \in F$. As $x \in F$ and $x \leq y \to x$, then $y \to x \in F$. Thus, $y \to x \sqcup y \subseteq F$ which implies that $y \to a', y \to b' \in F$. From $y \geq a'$, we obtain $y \to b' \leq a' \to b'$ and so, $a' \to b' \in F$. Therefore, $a' \sqcup b' \to b' \subseteq F$, which leads to $x \to b' \in F$. As also $\top = b' \to y \in F$, then $x \to y \in F$. Finally, as $x \in F$, so $y \in F$.

We will now prove item 1 in the statement: given $x, y \in a \sqcup b$, in the same manner as in the previous paragraph, there exist $a', b' \in x \sqcap y$, with $a \leq a'$ and $b \leq b'$. Assuming that $x \to y \in F$, by conditions of filter we have $x \to x \sqcap y \subseteq F$, so we have $x \to a' \in F$ and, by **P3**, $b' \to a' \in F$. Once again by properties of filter, we obtain $a' \sqcup b' \to a' \subseteq F$ and, particularly, $y \to a' \in F$. Finally, **P3** implies $y \to x$. The proof for item 2 is similar.

Conversely, suppose now that F is an m-filter satisfying conditions 1 and 2. As we are assuming that F is a deductive system, we have just to prove the two conditions in Definition 41; so let $a, b \in M$ such that $a \to b \in F$. By Proposition 32 (item 5)

$$(a \to b) \sqcap (b \to b) = (a \to b) \sqcap \top = (a \to b) \subseteq (a \sqcup b) \to b$$

Thus, there exists $x_1 \in a \sqcup b$ such that $a \to b = x_1 \to b$. If $a \sqcup b$ is a singleton, there is nothing to prove. Otherwise, given $x_2 \in a \sqcup b$, since $\top = b \to x_2 \in F$ and $x_1 \to b \in F$, we have that $x_1 \to x_2 \in F$. Using the hypothesis, it implies that also $x_2 \to x_1 \in F$ and again with $x_1 \to b \in F$, we obtain that $x_2 \to b \in F$.

The other condition follows similarly. □

Definition 45. Let $(M, \leq, \odot, \to, \top)$ be a residuated multilattice, a **congruence** on M is any equivalence relation $\equiv$ such that if $a \equiv b$, then $a \sqcup c \mathrel{\widehat{\equiv}} b \sqcup c$, $a \sqcap c \mathrel{\widehat{\equiv}} b \sqcap c$, $a \odot c \equiv b \odot c$, $a \to c \equiv b \to c$ and $c \to a \equiv c \to b$, for all $a, b, c \in M$.

Example 46. In the residuated multilattice given in Example 42 the partition $\{A, B \cup \{\top\}\}$ defines a non-trivial congruence relation. □

Given a multilattice M, Theorem 13 ensures that the set of congruences, $Con_m(M)$, is a sublattice of the complete lattice $(Eq(M), \subseteq)$. It is not difficult to see that if M is a residuated multilattice, the set of the congruences $Con_r(M)$, is a sublattice of $Con_m(M)$ as well.

Definition 47. Let $h\colon M \to M'$ be a map between residuated multilattices, h is said to be an **homomorphism** if h is a multilattice homomorphism and also $h(a \odot b) = h(a) \odot h(b)$ and $h(a \to b) = h(a) \to h(b)$ for all $a, b \in M$.

Observe that $h(\top) = \top$, for all homomorphism h between residuated multilattices. It is also remarkable that $h(a \sqcup b) = \big(h(a) \sqcup h(b)\big) \cap h(M)$ and $h(a \sqcap b) = \big(h(a) \sqcap h(b)\big) \cap h(M)$, since M is a full multilattice, Proposition 4, and Remark 26 (item 2).

The relation among filters, congruences and homomorphisms is stated below and follows from the convenient definition of filter just introduced.

Theorem 48. *Let $h\colon M \to M'$ be an homomorphism between residuated multilattices.*

1. *The **kernel relation**, defined as $a \equiv b$ if and only if $h(a) = h(b)$, is a congruence.*
2. *$h^{-1}(\top) = \{x \in M \mid h(x) = \top\}$ is a filter of M, the **kernel filter**.*

Proof.

1. Firstly, from Theorem 14, the kernel relation is compatible with $\sqcup$ and $\sqcap$. The compatibility with $\odot$ and $\rightarrow$ follows trivially from the definition of homomorphism.
2. As previously stated, $\top \in h^{-1}(\top)$.
   If $a, a \rightarrow b \in h^{-1}(\top)$ then $\top = h(a \rightarrow b) = h(a) \rightarrow h(b) = \top \rightarrow h(b)$ which implies $\top \leq h(b)$, that is, $\top = h(b)$.
   Consider now $a \rightarrow b \in h^{-1}(\top)$, and notice that $\top = h(a \rightarrow b) = h(a) \rightarrow h(b)$, and hence, $h(a) \leq h(b)$. As a result, we obtain that $h(a \sqcup b \rightarrow b) = h(a \sqcup b) \rightarrow h(b) \subseteq h(a) \sqcup h(b) \rightarrow h(b) = h(b) \rightarrow h(b) = \top$. Similarly, one can prove $a \rightarrow a \sqcap b \subseteq h^{-1}(\top)$. □

Again, the inverse image of any filter, not only $\{\top\}$, is a filter. In this case, the adaptation of the proof is not that straightforward, so we include it below:

Theorem 49. *Let $h\colon M \rightarrow M'$ be an homomorphism between residuated multilattices, and F a filter of M', then $h^{-1}(F)$ is a filter of M.*

Proof. We already proved that $\top \in h^{-1}(\top)$.

If $a, a \rightarrow b \in h^{-1}(F)$ then $h(a) \rightarrow h(b) = h(a \rightarrow b) \in F$. Now, as $h(a) \in F$, and F is a filter $h(b) \in F$, that is, $b \in h^{-1}(F)$.

Finally, consider $a \rightarrow b \in h^{-1}(F)$, as above, we obtain $h(a) \rightarrow h(b) \in F$, and hence, $h(a) \sqcup h(b) \rightarrow h(b) \subseteq F$. Now

$$h(a \sqcup b \rightarrow b) = h(a \sqcup b) \rightarrow h(b) \subseteq h(a) \sqcup h(b) \rightarrow h(b) \subseteq F$$

As a result, $a \sqcup b \rightarrow b \subseteq h^{-1}(F)$.

Similarly, one can prove $a \rightarrow a \sqcap b \subseteq h^{-1}(F)$. □

Theorem 50. *Let $(M, \leq, \odot, \rightarrow, \top)$ be a residuated multilattice and $\equiv$ a congruence relation on M. The mapping $p\colon M \rightarrow M/{\equiv}$ such that $p(x) = [x]$ is a surjective homomorphism of residuated multilattices and, as a consequence, the equivalence class $[\top]$ is a filter of M.*

Proof. Given a congruence relation $\equiv$ on M, it is clear that the quotient set $M/{\equiv}$ is a pocrim and the multilattice structure follows from Theorem 16. Moreover, it is trivial that p preserves the operations $\odot$ and $\rightarrow$. □

Now, we are interested in whether there exists a biunivocal relation between filters and congruences as in the case of pocrims, stated in the proposition below.

Proposition 51 (See [32]). *Given a pocrim $\mathcal{A} = (A, \leq, \odot, \to, \top)$ and F a deductive system, the following relation*

$$a \equiv_F b \quad \textit{if and only if} \quad a \to b, b \to a \in F$$

is an equivalence relation compatible with $\odot$ and $\to$, that is, it is a congruence of the pocrim.

It is remarkable that when underlying pocrim is a residuated multilattice, the previous congruence of pocrims behaves well with the multilattice structure, as the following theorem shows:

Theorem 52. *Let $(M, \leq, \odot, \to, \top)$ be a residuated multilattice and F be a filter. Then, the relation $a \equiv_F b$ if and only if $a \to b$, $b \to a \in F$ defines a congruence of residuated multilattices.*

Proof. By Proposition 51 we already know that $\equiv_F$ is a congruence of pocrims. We will use Theorem 12 to prove that it is a congruence of multilattices as well; we only have to prove conditions 2 and 4 in the statement of that theorem, as the other ones are straightforward.

2. We have to prove that, for all $a, b \in M$, we have that $a \equiv_F b$ if and only if there exist $z \in a \sqcap b$ and $w \in a \sqcup b$ such that $z \equiv_F w$:

   Assume $a \equiv_F b$, then for all $z \in a \sqcap b$ and $w \in a \sqcup b$, we have $z \leq w$, so $z \to w = \top \in F$. Moreover, since $a \equiv_F b$ implies $b \to a \in F$, using that F is a filter, we obtain $a \sqcup b \to a \subseteq F$; in particular, for $w \in a \sqcup b$, it holds that $w \to a \in F$. Analogously, $a \to b \in F$, implies $a \to a \sqcap b \subseteq F$; in particular, for $z \in a \sqcap b$, we have $a \to z \in F$, which together with $w \to a \in F$ implies $w \to z \in F$ and then $z \equiv_F w$.

   Conversely, assume $z \in a \sqcap b$ and $w \in a \sqcup b$ such that $z \equiv_F w$, in particular, $w \to z \in F$. Then, from $z \leq b$, by **P3**, we obtain $w \to z \leq w \to b$, then $w \to b \in F$. Likewise, from $a \leq w$, it is deduced that $w \to b \leq a \to b$ and thus, $a \to b \in F$. Analogously, by replacing b by a above, we have $b \to a \in F$ and, therefore, $a \equiv_F b$.

4. We have to prove that for elements $a, b \in M$ such that $a \leq b$ and $a \equiv_F b$, then $a \sqcap c \mathrel{\widehat{\equiv}_F} b \sqcap c$ and $a \sqcup c \mathrel{\widehat{\equiv}_F} b \sqcup c$, for all $c \in M$.

   For $x \in a \sqcap c$, since $x \leq a \leq b$ and $x \leq c$, there exists $y \in b \sqcap c$ such that $x \leq y$. On one hand, $x \to y = \top \in F$. On the other hand, as $b \leq y$, then $b \to a \leq y \to a$, which implies $y \to a \in F$ since $b \to a \in F$, because of $a \equiv_F b$ and properties of filter. As a consequence, $y \to (a \sqcap y) \subseteq F$. It is not difficult to check that $x \in a \sqcap y$ and, hence, $y \to x \in F$. Now, Theorem 44 implies that $x \to y \in F$ as well and, as a result, $x \equiv_F y$.

Conversely, for $y \in b \sqcap c$ and $z \in a \sqcap y$, there exists $x \in a \sqcap c$ with $z \leq x$ and also $y' \in b \sqcap c$ with $x \leq y'$, by definition of ordered multilattice. By **P3**, as $y \leq b$ then $b \to a \leq y \to a$ and thus $y \to a \sqcap y \subseteq F$, using the definition of filter. So $y \to z \in F$ which yields to $y \to x \in F$. Besides, $\top = x \to y' \in F$, then $y \to y' \in F$. By Theorem 44, as we have also $y' \to y \in F$ and $x \to y' \in F$, we obtain $x \to y \in F$.

Following the same pattern as above, one proves $a \sqcup c \,\widehat{\equiv}\, b \sqcup c$. □

The rest of the section focuses on a particular class of filters and deductive systems in a residuated multilattice in which the implication behaves consistently with respect to multiinfima and multisuprema. Formally,

Definition 53. Let $(M, \leq, \odot, \to, \top)$ be a residuated multilattice. A deductive system F is said to be **consistent** if for all $a, b, c \in M$ the following conditions hold:

1. If $a \to c, b \to c \in F$, then $(a \sqcup b) \to c \subseteq F$
2. If $c \to a, c \to b \in F$, then $c \to (a \sqcap b) \subseteq F$

Proposition 54. *Every consistent deductive system is a filter.*

PROOF. We have just to prove the specific condition in Definition 41.

If $a \to b \in F$, as also $\top = b \to b = a \to a \in F$ then $(a \sqcup b) \to b \subseteq F$ and $a \to (a \sqcap b) \subseteq F$. □

Example 55. In the residuated multilattice described in the Example 42, the filter $B \cup \{\top\}$ is consistent.

On the other hand, the subset $\{\top\}$ is a filter that it is not consistent because $b_1 \to b_3 = b_1 \to b_4 = \top$ but $b_1 \to (b_3 \sqcap b_4) = \{b_5, \top\}$. □

Theorem 56. *Let $h\colon M \to M'$ be a map between residuated multilattices. Then, $h(M)$ is a lattice if and only if $h^{-1}(\top)$ is a consistent filter.*

PROOF. Assume that $h(M)$ is a lattice, then according to Theorem 49 it suffices to prove that $h^{-1}(\top)$ is consistent. Let $a \to c, b \to c \in h^{-1}(\top)$. Then, $\top = h(a \to c) = h(a) \to h(c)$ and $\top = h(b \to c) = h(b) \to h(c)$, so, by Proposition 32 (item 5), we have $\top = (h(a) \to h(c)) \sqcap (h(b) \to h(c)) \subseteq (h(a) \sqcup h(b)) \to h(c)$. Since $h(M)$ is a lattice, $(h(a) \sqcup h(b)) \to h(c) = \{\top\}$. On the other hand, $h((a \sqcup b) \to c) \subseteq (h(a) \sqcup h(b)) \to h(c) = \{\top\}$, thus, $(a \sqcup b) \to c \subseteq h^{-1}(\top)$. Analogously, from $c \to a, c \to b \in h^{-1}(\top)$, we obtain $c \to (a \sqcap b) \subseteq h^{-1}(\top)$.

Conversely, consider $a, b, x, y \in M$ such that $h(x), h(y) \in h(a) \sqcup h(b)$, and let us prove that $h(x) = h(y)$. Observe that $\top = h(a) \to h(x) = h(a \to x)$ and the same for b. Since $h^{-1}(\top)$ is consistent, $h((a \sqcup b) \to x) = \top$. Since $h(y) \in (h(a) \sqcup h(b)) \cap h(M) = h(a \sqcup b)$, there exists $y' \in a \sqcup b$ such that $h(y) = h(y')$. Then $h(y) \to h(x) = h(y') \to h(x) = h(y' \to x) \in h((a \sqcup b) \to x) = \top$ which leads to $h(y) \leq h(x)$ and then, $h(x) = h(y)$. Similarly, it can be obtained that $h(a) \sqcap h(b)$ is a singleton, for all $a, b \in M$. □

Corollary 57. *Let F be a filter of a residuated multilattice $(M, \leq, \odot, \to, \top)$ and $\equiv_F$ the equivalence relation defined as $a \equiv_F b$ if and only if $a \to b, b \to a \in F$. Then, F is consistent if and only if $M/{\equiv_F}$ is a residuated lattice.*

PROOF. It suffices to consider the canonical projection $p \colon M \to M/{\equiv_F}$ which is a surjective homomorphism. □

5. Conclusions and future work

We have proceeded gradually towards the study of residuated multilattices and their main algebraic properties, specifically, those related to filters, congruences, and homomorphisms.

Then, residuated multilattices are defined as an intermediate structure between the class of residuated lattices and the class of pocrims. In such a general framework, apparently innocuous properties may well collapse the whole structure, and some of these properties are identified: for instance, idempotence of $\odot$ makes a residuated multilattice collapse to a Heyting algebra. It is worth studying other common properties which generate similar situations.

There exist a lot of other interesting topics for future research: on the one hand, we will study the influence of considering the residuated operations $\odot$ and $\to$ as hyperoperations, thus leading to a complete embedding of the structure into a hyperalgebraic framework. In this context, it would be convenient to obtain a satisfactory notion of distributivity on residuated multilattices and deepening in the relationship between boolean multilattices and hyperrings. Last but not least, we will try to identify further occurrences of this structure, and devise new applications on those fields.

On the other hand, it is worth to note that all the inequations involving implications in the framework of residuated lattices have a generalized counterpart, in a residuated multilattice, which greatly resembles approximation by a rough set, again suggesting a potential relation to results in [31].

Concerning possible relations between these lattice-theoretical structures and formal logic, we have already studied (non-residuated) multilattices by using a coalgebraic approach [8, 9], which can be interpreted in some sense within the realm of modal logic. Now, when residuation is considered as well, it is reasonable to consider the existence of logics, in the sense of syntactic calculus, of which multilattices would be the semantical counterpart. Certainly, given the non-deterministic character of the multilattice operators multiinf and multisup, the target logics will be necessarily non-classical.

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