# A logic framework for reasoning with movement based on fuzzy qualitative representation[☆]

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**Abstract**

We present a logic approach to reason with moving objects under fuzzy qualitative representation. This way, we can deal both with qualitative and quantitative information, and consequently, to obtain more accurate results. The proposed logic system is introduced as an extension of Propositional Dynamic Logic: this choice, on the one hand, simplifies the theoretical study concerning soundness, completeness and decidability; on the other hand, provides the possibility of constructing complex relations from simpler ones and the use of a language very close to programming languages.

*Key words:* Qualitative reasoning with fuzzy data, Moving objects, Order-of-magnitude reasoning, Propositional dynamic logic. Reasoning under uncertainty.

## 1. Introduction

Qualitative Reasoning is an interesting tool in order to deal with incomplete information which sometimes happens when we are dealing with moving objects. Some papers have been published which study and develop qualitative kinematics models [8, 21, 19, 37], following the ideas presented in [12, 26, 11].

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Different approaches have been used in order to face the problem of the relative movement of one physical object with respect to another [10, 36, 9]. However, to the best of our knowledge, the only paper which introduces a logic framework to manage qualitative velocity is [5].

Sometimes, using just qualitative reasoning is not precise enough, especially when we have to take into account precise absolute locations that may be known in advance in some applications [22]. In particular, this can be a problem in specific tasks related to moving objects, such as collision avoidance, catching an object, etc. As a consequence, a combination of qualitative and quantitative data would be required, and it seems that Fuzzy Qualitative Reasoning (FQR) can be a good choice for that purpose [34]. FQR uses fuzzy numbers in order to represent qualitative classes and can be applied to robot kinematics [19] by using fuzzy qualitative trigonometry [20]. Several recently published papers develop different applications of FQR to human motion [6], dynamic systems [7], geographical systems [16], Fuzzy Spatial Reasoning [31, 30, 32].

On the other hand, fuzzy logic controllers have been designed and used to improve navigation in mobile robots, presenting a set of IF-THEN rules to formulate the attributes of human reasoning and decision-making [27, 28]. Furthermore, a fuzzy control system for reactive navigation of mobile robots has been presented recently in [25].

In this paper, we continue the line of [5] by presenting a logic approach to deal with moving objects with fuzzy qualitative representation. We exploit the advantages of using fuzzy numbers in qualitative reasoning for simplifying the tables of compositions of movements. In some sense, this choice allows for using both qualitative and quantitative information, and consequently, obtain more accurate results. The use of logic improves also the capability of formal representation of problems and provides insights into their most suitable solving methods. As examples of logics for qualitative reasoning see, for example [29, 23]. Our logic approach is based on Propositional Dynamic Logic (PDL) because it provides the possibility of constructing complex relations from simpler ones and the use of a language very close to programming languages. We will exploit these advantages of PDL by giving specific axioms for collision avoidance. We choose PDL which is a decidable logic and, as a consequence, we have the advantage that reasoning can be performed by theorem proving. Some applications of PDL in AI can be seen in [35, 4, 3].

The present approach focuses on the movement of objects with respect to others with or without obstacles. Our aim is to develop a formalism capable of indicating any relative position of an object with respect to another one, in such a way that allows us to calculate different movements and represent certain actions in the chosen scenarios. We are specially interested in representing specific actions such as collision avoidance and intercepting an object. In order to establish a sufficiently detailed and accurate calculus, this formalism is integrated into the logic PDL. This choice is highly pertinent because PDL is an excellent tool for managing operations on these vectors and represent the dynamism of actions.

We represent the movement of an object with respect to another by a tuple whose components include information about objects, velocity, orientation, relative movement, allowed movements, qualitative latitude and longitude. Some of these components were inspired by previous works in the literature whereas others have been included in order to increase the expressive power of our approach. For instance, [10] uses two components, for velocity and orientation, which are considered as relative magnitudes; our approach considers velocity and orientation as absolute magnitudes, because so are the values obtained from devices such as velocimeters, GPS, etc. On the other hand, [9] considers two components as well, but their interpretation is different: the relative movement and the relative velocity of one object with respect to another; the former is included in our approach.

The advantages of our approach are two-fold: on the one hand, it subsumes in some sense several previous approaches, and the formalization provided by the logic allows reasoning without using too many case-based tables; on the other hand, our approach is flexible enough so that the number and/or the specifications of the components of a movement can be modified without altering much the general framework: for instance, the components of relative position and cardinal direction could be enriched in the line of [33, 24].

The paper is organized as follows: Section 2 introduces the preliminary definitions and notations to be used in the rest of the paper; Section 3 is devoted to present our approach in different scenarios used in the literature; in Section 4, we introduce our logic approach to reasoning with moving objects with fuzzy qualitative data; then, in Section 5, we present our logic approach and provide a set of specific axioms for collision avoidance; the soundness, completeness and decidability of the proposed logic are studied in Section 6; finally, we draw some conclusions and prospects of future work.

## 2. Preliminary definitions

We represent the movement of an object with respect to another with different labels such as velocity, orientation, relative movement, possible directions and relative position. The values of these labels are given by different qualitative classes, and the granularity can be changed depending on the problem in question. To be more precise, we represent the movement of an object $\mathsf{A_i}$ with respect to $\mathsf{A_j}$ by $(x_1; \ldots; x_7) \in L$, being $L = L_1 \times \ldots \times L_7$ defined as follows. As some of the sets $L_i$ are defined also by a cartesian product, for an easy reading we will eliminate some parentheses by using "; " to indicate the seven components of our label, while we will use "," for the components of each $L_i$. In this case, we will use superscripts to each element of this cartesian product. For example $x_7 = (x_7^1, x_7^2)$.

The set $L_1$ defined on $\mathcal{A} = \{\mathsf{A}_1, \dots, \mathsf{A}_\mathsf{k}\}$ with $\mathsf{k} \in \mathbb{N}$, consisting of all pairs $(\mathsf{A}_\mathsf{i}, \mathsf{A}_\mathsf{j})$ with $\mathsf{i} \neq \mathsf{j}$, representing the movement of object $\mathsf{A}_\mathsf{i}$ *with respect to* object $\mathsf{A}_\mathsf{j}$.

The set of qualitative velocities $L_2 = 2^{\{\mathsf{v}_0,\mathsf{v}_1,\mathsf{v}_2,\mathsf{v}_3\}} \smallsetminus \varnothing$, where $\mathsf{v}_0, \mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3$ represent *zero*, *slow*, *normal* and *quick* velocity, respectively.[1]

The set of qualitative orientations is $L_3 = 2^{\{\mathsf{o}_0,\mathsf{o}_1,\mathsf{o}_2,\mathsf{o}_3,\mathsf{o}_4\}} \smallsetminus \varnothing$, where the labels $\mathsf{o}_0, \mathsf{o}_1, \mathsf{o}_2, \mathsf{o}_3, \mathsf{o}_4$ represent, respectively, *none*, *North*, *South*, *East* and *West* orientations.[2]

We consider the set $L_4 = (2^{\{0,-,+\}} \smallsetminus \varnothing) \times (2^{\{0,-,+\}} \smallsetminus \varnothing)$ in order to represent relative movements , where $0, -, +$ mean, respectively, *stable*, *moving towards*, *moving away from* (following [9]). For short, we will denote the subset $\{-, +\}$ by $\pm$, to represent the object is *moving* (but it is not determined either *towards* or *away from*).

The set $L_5 = 2^{\{\mathsf{o}_0,\mathsf{o}_1,\mathsf{o}_2,\mathsf{o}_3,\mathsf{o}_4\}}$ represents the possible directions that object $\mathsf{A}_\mathsf{i}$ can follow, this is suited for movements in a network, as presented in [9]. In this case, we do not eliminate the empty list $\varnothing$, because the complete list $\mathsf{o}_0\mathsf{o}_1\mathsf{o}_2\mathsf{o}_3\mathsf{o}_4$ means that the object can follow *every* possible direction, while the empty set means that there is *lack of information* about the possible directions.

The sets $L_6$ and $L_7$ are used for representing the qualitative latitude and longitude of $\mathsf{A}_\mathsf{i}$ with respect to $\mathsf{A}_\mathsf{j}$. Namely:

$L_6 = (2^{\{\mathsf{o}_1,\mathsf{o}_2\}} \smallsetminus \varnothing) \times (2^{\{\mathsf{d}_0,\mathsf{d}_1,\mathsf{d}_2,\mathsf{d}_3\}} \smallsetminus \varnothing)$ means the *North-South* position and the distance (this is the *qualitative latitude*), where $\mathsf{d}_0, \mathsf{d}_1, \mathsf{d}_2, \mathsf{d}_3$ mean *zero, close, normal*, *distant*.

Finally, $L_7 = (2^{\{\mathsf{o}_3,\mathsf{o}_4\}} \smallsetminus \varnothing) \times (2^{\{\mathsf{d}_0,\mathsf{d}_1,\mathsf{d}_2,\mathsf{d}_3\}} \smallsetminus \varnothing)$ means the *qualitative longitude*, that is, a pair *East-West* position, together with the distance.

Notice that we assume an underlying external reference system for some of the attributes, such as velocity $L_2$, orientation $L_3$, and allowed orientations for the movements $L_5$. On the other hand, we use an object as the reference in the representation of relative movement $L_4$ and qualitative latitude $L_6$ and longitude $L_7$. This choice fits both the examples in the literature and our purposes; however it could be changed depending on the problem in question since it does not substantially change our logic-based approach.

The following table summarizes the definitions of every component presented

[1] As usual, $2^X$ denotes the set of subsets of $X$, for any set $X$. The use of the power set $2^X$ allows us consider lists as components of the qualitative label. We exclude $\varnothing$ because it means that the object can move at any velocity $\mathsf{v}_0\mathsf{v}_1\mathsf{v}_2\mathsf{v}_3$.

[2] We use subscripts instead of the usual abbreviations N, S, E, W, in order to give a more general and modular approach. We do the same to represent velocities, distances, etc

above.

| $L_i$ | **Description** | **Example** |
|---|---|---|
| $L_1$ | objects ( $\mathsf{A_i}$, $\mathsf{A_j}, \ldots$) | (a) |
| $L_2$ | velocity (zero $\mathsf{v_o}$, slow $\mathsf{v_1}$, normal $\mathsf{v_2}$, quick $\mathsf{v_3}$) | (b) |
| $L_3$ | orientation (none $\mathsf{o_0}$, North $\mathsf{o_1}$, South $\mathsf{o_2}$, East $\mathsf{o_3}$, West $\mathsf{o_4}$) | (c) |
| $L_4$ | relative movement (stable 0, moving towards $-$, moving away from $+$) | (d) |
| $L_5$ | allowed orientations (lack of information $\varnothing$, none $\mathsf{o_0}$, North $\mathsf{o_1}$, South $\mathsf{o_2}$, East $\mathsf{o_3}$, West $\mathsf{o_4}$) | (e) |
| $L_6$ | qualitative latitude $((\mathsf{o_i}, \mathsf{d_j})$, where the orientations are North $\mathsf{o_1}$, and South $\mathsf{o_2}$ and the distances are zero $\mathsf{d_0}$, close $\mathsf{d_1}$, normal $\mathsf{d_1}$ and distant $\mathsf{d_3}$) | (f) |
| $L_7$ | qualitative longitude $((\mathsf{o_i}, \mathsf{d_j})$, where the orientations are East $\mathsf{o_3}$, and West $\mathsf{o_4}$ and the distances are zero $\mathsf{d_0}$, close $\mathsf{d_1}$, normal $\mathsf{d_1}$ and distant $\mathsf{d_3}$) | (g) |

The examples (a) ... (g) referred to in the previous table are the following:

(a) $(\mathsf{A_i}, \mathsf{A_j}; \ldots)$ object $\mathsf{A_i}$ is moving with respect to $\mathsf{A_j}$...

(b) $(\mathsf{A_i}, \mathsf{A_j}; \mathsf{v_2 v_3}; \ldots)$ object $\mathsf{A_i}$ is moving with respect to $\mathsf{A_j}$ with a normal or quick velocity ...

(c) $(\mathsf{A_i}, \mathsf{A_j}; \mathsf{v_2 v_3}; \mathsf{o_3}; \ldots)$ object $\mathsf{A_i}$ is moving with respect to $\mathsf{A_j}$ with a normal or quick velocity towards the East ...

(d) $(\mathsf{A_i}, \mathsf{A_j}; \mathsf{v_2 v_3}; \mathsf{o_3}; +, -; \ldots)$ object $\mathsf{A_i}$ will be moving with respect to $\mathsf{A_j}$ with a normal or quick velocity towards the East. $\mathsf{A_i}$ is moving away from $\mathsf{A_j}$, and $\mathsf{A_j}$ is moving towards $\mathsf{A_i}$ ...

(e) $(\mathsf{A_i}, \mathsf{A_j}; \mathsf{v_2 v_3}; \mathsf{o_3}; +, -; \mathsf{o_1 o_2 o_3}; \ldots)$ object $\mathsf{A_i}$ will be moving with respect to $\mathsf{A_j}$ with a normal or quick velocity towards the East. $\mathsf{A_i}$ is moving away from $\mathsf{A_j}$, and $\mathsf{A_j}$ is moving towards $\mathsf{A_i}$. $\mathsf{A_i}$ can move only to the North, South or East ...

(f) $(\mathsf{A_i}, \mathsf{A_j}; \mathsf{v_2 v_3}; \mathsf{o_3}; +, -; \mathsf{o_1 o_2 o_3}; \mathsf{o_1}, \mathsf{d_1 d_2}; \ldots)$ object $\mathsf{A_i}$ will be moving with respect to $\mathsf{A_j}$ with a normal or quick velocity towards the East. $\mathsf{A_i}$ is moving away from $\mathsf{A_j}$, and $\mathsf{A_j}$ is moving towards $\mathsf{A_i}$. $\mathsf{A_i}$ can move only to the North, South or East. The qualitative latitude of $\mathsf{A_i}$ with respect to $\mathsf{A_j}$ is that $\mathsf{A_i}$ is to the North at a close or normal distance with respect to $\mathsf{A_j}$ ...

(g) $(\mathsf{A_i}, \mathsf{A_j}; \mathsf{v_2 v_3}; \mathsf{o_3}; +, -; \mathsf{o_1 o_2 o_3}; \mathsf{o_1}, \mathsf{d_1 d_2}; \mathsf{o_3}, \mathsf{d_0})$ object $\mathsf{A_i}$ is moving with respect to $\mathsf{A_j}$ with a normal or quick velocity towards the East. $\mathsf{A_i}$ is moving away from $\mathsf{A_j}$, and $\mathsf{A_j}$ is moving towards $\mathsf{A_i}$. $\mathsf{A_i}$ can move only to the North, South or East. The qualitative latitude of $\mathsf{A_i}$ with respect to $\mathsf{A_j}$ is that $\mathsf{A_i}$ is to the North at a close or normal distance with respect to $\mathsf{A_j}$. The qualitative longitude of $\mathsf{A_i}$ with respect to $\mathsf{A_j}$ is that $\mathsf{A_i}$ neither to the East nor to the West with respect to $\mathsf{A_j}$.

The *composition* of a movement of $\mathsf{A_i}$ with respect to $\mathsf{A_j}$ and a movement of $\mathsf{A_j}$ with respect to $\mathsf{A_k}$ provides information about the movement of $\mathsf{A_i}$ with respect to $\mathsf{A_k}$. Notice that, as some of the information about the movement of $\mathsf{A_i}$ with respect to $\mathsf{A_j}$ is independent from $\mathsf{A_j}$, those components (namely $L_3, L_4$ and $L_6$) are directly inherited by the movement of $\mathsf{A_i}$ with respect to $\mathsf{A_k}$ for any $\mathsf{k}$.

The components of relative movement $L_4$ and qualitative latitude $L_6$ and longitude $L_7$ need information about the movement of $\mathsf{A_j}$ with respect to $\mathsf{A_k}$. For the composition of component $L_5$ we have to consider different cases depending on the qualitative latitude and longitude of the objects in question.

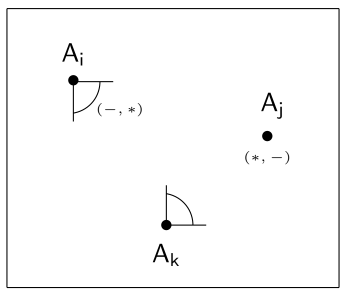


Figure 1: One of the cases for composition of $L_4$

Table 1: Composition of components of $L_4$.

| $\mathsf{A_iA_j} \backslash \mathsf{A_jA_k}$ | $*0$ | $*-$ | $*+$ |
|---|---|---|---|
| $0*$ | $00$ | $0\pm$ | $0\pm$ |
| $-*$ | $-0$ | $-\pm$ | $-\pm$ |
| $+*$ | $-0$ | $+\pm$ | $+\pm$ |

For instance, we show in Table 1 the composition for the case given in Figure 1 (the rest of cases are similar) where $* \in \{0, -, +\}$. The occurrence of $(-, *)$ in the component $L_4$ of the movement of $\mathsf{A_i}$ with respect to $\mathsf{A_j}$ determines a certain range in the angle of possible movements of $\mathsf{A_i}$, as indicated in Figure 1. Similarly, for $\mathsf{A_k}$.

The occurrence of $(-, *)$ in the component $L_4$ of the movement of $\mathsf{A_i}$ with respect to $\mathsf{A_j}$ determines a certain range in the angle of possible movements of

$\mathsf{A_i}$, as indicated in Figure 1. Similarly for $\mathsf{A_k}$, if the movement of $\mathsf{A_j}$ with respect to $\mathsf{A_k}$ is represented by $(*, -)$.

Finally, Table 2 gives information about how to get both the components $L_6$ (and similarly $L_7$) of a composition of a movement of $\mathsf{A_i}$ with respect to $\mathsf{A_j}$ and a movement of $\mathsf{A_j}$ with respect to $\mathsf{A_k}$. In Table 2, we assume the following constraints: $\mathsf{s, u, r, t} \neq 0$, $\mathsf{r} \neq \mathsf{t}$; moreover, if $\mathsf{s} < \mathsf{u}$ we write $\mathsf{d_{u'}} = \mathsf{d_{u-s}} \ldots \mathsf{d_u}$, and if $\mathsf{u} < \mathsf{s}$ we write $\mathsf{d_{s'}} = \mathsf{d_{s-u}} \ldots \mathsf{d_s}$. In addition, for $\mathsf{m} = \max\{\mathsf{s}, \mathsf{u}\}$, we write $\mathsf{d_{m'}} = \begin{cases} \mathsf{d_m d_{m+1}} & \text{if } \mathsf{m} < 3 \\ \mathsf{d_m} & \text{if } \mathsf{m} = 3 \end{cases}$

**Notation:** *We will denote hereafter the complete list in the component $l$ as $\mathcal{C}_l$. For instance, $\mathcal{C}_2 = \mathsf{v_0 v_1 v_2 v_3}$. In the case of qualitative latitude and longitude, we will write, for example, $\mathcal{C}_{6_1} = \mathsf{o_1 o_2}$ to represent the complete list for the first component of the qualitative latitude and, $\mathcal{C}_6 = (\mathcal{C}_{6_1}, \mathcal{C}_{6_2})$, being $\mathcal{C}_{6_2} = \mathsf{d_0 d_1 d_2 d_3}$.*

Table 2: Composition of components of $L_6$.

| $\mathsf{A_iA_j \backslash A_jA_k}$ | $\mathsf{o_r d_0}$ | $\mathsf{o_r d_u}$ | $\mathsf{o_t d_u}$ |
|---|---|---|---|
| $\mathsf{o_r d_0}$ | $\mathsf{o_r d_0}$ | $\mathsf{o_r d_u}$ | $\mathsf{o_t d_u}$ |
| $\mathsf{o_r d_s}$ | $\mathsf{o_r d_s}$ | $\mathsf{o_r d_{m'}}$ | $\begin{cases} \mathsf{o_t d_{u'}} & \text{if } \mathsf{s} < \mathsf{u} \\ \mathcal{C}_{6_1}\mathsf{d_0 d_1} & \text{if } \mathsf{s} = \mathsf{u} \neq 3 \\ \mathcal{C}_6 & \text{if } \mathsf{s} = \mathsf{u} = 3 \\ \mathsf{o_r d_{s'}} & \text{if } \mathsf{s} > \mathsf{u} \end{cases}$ |

For instance, in Figure 2 we compose a movement $\mathsf{A_i}$ with respect to $\mathsf{A_j}$ with qualitative latitude $\mathsf{o_1 d_s}$, with a movement of $\mathsf{A_j}$ with respect to $\mathsf{A_k}$ with qualitative latitude $\mathsf{o_2 d_u}$. The following possibilities hold for the qualitative latitude of the movement of $\mathsf{A_i}$ with respect to $\mathsf{A_k}$:

- If $\mathsf{s} = 1$ and $\mathsf{u} = 3$, the composition is $\mathsf{o_2 d_2 d_3}$.
- If $\mathsf{s} = 1$ and $\mathsf{u} = 2$ the composition is $\mathsf{o_2 d_1 d_2}$.
- If $\mathsf{s} = 2$ and $\mathsf{u} = 3$ the composition is $\mathsf{o_2 d_1 d_2 d_3}$.
- $\mathsf{s} = \mathsf{u} \neq 3$, then the composition is $\mathcal{C}_{6_1}\mathsf{d_0 d_1}$.
- $\mathsf{s} = \mathsf{u} = 3$, then the composition is $\mathcal{C}_6$.
- If $\mathsf{s} = 3$ and $\mathsf{u} = 1$, the composition is $\mathsf{o_1 d_2 d_3}$.
- If $\mathsf{s} = 2$ and $\mathsf{u} = 1$ the composition is $\mathsf{o_1 d_1 d_2}$.
- If $\mathsf{s} = 3$ and $\mathsf{u} = 2$ the composition is $\mathsf{o_1 d_1 d_2 d_3}$.

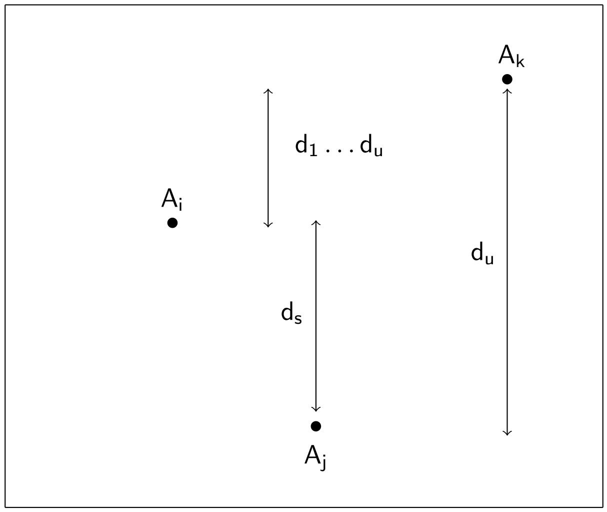


Figure 2: Composition of latitudes $\mathsf{o_1 d_s}$ with $\mathsf{o_2 d_u}$ ($\mathsf{s} < \mathsf{u}$)

The cases in which component $L_6$ (or $L_7$) is given by a list, Table 2 above is used to consider all the corresponding compositions. For instance, to compose $\mathsf{o_1 d_0 d_1}$ with $\mathsf{o_2 d_2}$, we use the previous table to compose $\mathsf{o_1 d_0}$ and $\mathsf{o_1 d_1}$, with $\mathsf{o_2 d_2}$. The result in this case is $\mathsf{o_2 d_1 d_2}$.

## 3. Our approach in different scenarios

In this section we use some real applications in the literature, and explain how our approach works on them. Some of these cases will be used as running examples on which the logic approach (to be introduced later) will be applied. In this section, we focus only on the specific notation introduced above; to begin with, we introduce the example below, which is inspired by those given in [9].

### *3.1. A chasing situation*

Consider the situation of Figure 3, where two policemen $\mathsf{A_1}$ and $\mathsf{A_2}$ are chasing a gangster $\mathsf{A_3}$. Suppose that $\mathsf{A_1}$ and $\mathsf{A_2}$ know their relative position with respect to each other, whereas only $\mathsf{A_2}$ has information about the movement of $\mathsf{A_3}$. Label

$$(\mathsf{A_3}, \mathsf{A_2}; \mathsf{v_3}; \mathsf{o_3}; +, -; \mathsf{o_2 o_3 o_4}; \mathsf{o_1}, \mathsf{d_1}; \mathsf{o_3}, \mathsf{d_0})$$

represents that $\mathsf{A_3}$ is moving with respect to $\mathsf{A_2}$ with a quick velocity ($\mathsf{v_3}$) towards East ($\mathsf{o_3}$), being $\mathsf{A_3}$ moving away from $\mathsf{A_2}$ (+), while $\mathsf{A_2}$ is moving towards $\mathsf{A_3}$ (−). Moreover, $\mathsf{A_3}$ can move only towards South, East and West ($\mathsf{o_2 o_3 o_4}$),

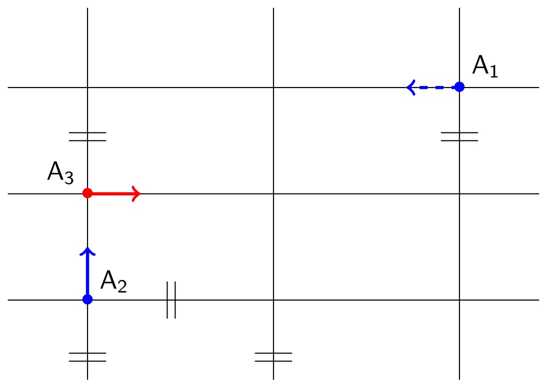


Figure 3: $A_1$ and $A_2$ chasing $A_3$

because the North street is a dead-end. $A_3$ is close to the North from $A_2$ $(o_1, d_1)$, and it is neither to the East nor to the West from $A_2$ $(o_3, d_0)$. Analogously, label

$$(A_2, A_1; v_3; o_1; -, 0; o_1o_4; o_2, d_2; o_4, d_2)$$

represents that $A_2$ is moving with respect to $A_1$ with a quick velocity towards North, being $A_2$ moving towards $A_1$, whereas $A_1$ is stable with respect to $A_2$. Moreover, $A_2$ can move only towards North, East and West. $A_2$ is at a normal distance to the South, and at a normal distance to the West with respect to $A_1$. In this case, using the notion of composition given above, the result is

$$(A_3, A_1; v_3; o_3; -, 0; o_2o_3o_4; o_2, d_1d_2; o_4, d_2).$$

As $A_1$ is chasing $A_3$, and the street to the South of $A_1$ is a dead-end, the movement of $A_1$ with respect to $A_3$ will be

$$(A_1, A_3; v_3; o_4; -, -; o_1o_3o_4; o_1, d_1d_2; o_3, d_2),$$

that is, a quick velocity towards the West, $A_1$ and $A_3$ moving towards each other, $A_1$ can move towards the North, East and West, $A_1$ is at a close or at a normal distance and to the North, and at a normal distance to the East with respect to $A_3$.

### *3.2. Collision avoidance*

We focus on the example about collision avoidance given in [9, pp. 15–16]. In Figure 4, we can see two situations (a) and (b) without real collision danger but which could trigger some collision detection systems. Firstly, let us focus on case (a) which, in our system, is specified by

$$(k, l; v_3; o_3; -, -; o_3o_4; \mathcal{C}_{6_1}, d_0; o_4, d_1) \text{ and } (l, k; v_3; o_4; -, -; o_3o_4; \mathcal{C}_{6_1}, d_0; o_3, d_1)$$

In this case, as $\mathsf{k}$ and $\mathsf{l}$ are moving towards each other, one can predict a future collision. Yet, if we consider that in the future both $\mathsf{k}$ and $\mathsf{l}$ will change their direction, the representation would be

$$(\mathsf{k},\mathsf{l};\mathsf{v}_3;\mathsf{o}_2;\mathcal{C}_{4_1},\mathcal{C}_{4_2};\mathsf{o}_1\mathsf{o}_2;\mathcal{C}_{6_1},\mathsf{d}_0;\mathsf{o}_4,\mathsf{d}_1) \text{ and } (\mathsf{l},\mathsf{k};\mathsf{v}_3;\mathsf{o}_2;\mathcal{C}_{4_1},\mathcal{C}_{4_2};\mathsf{o}_1\mathsf{o}_2;\mathcal{C}_{6_1},\mathsf{d}_0;\mathsf{o}_3,\mathsf{d}_1)$$

The fact that both objects will move to the South and they are at different longitude ($\mathsf{k}$ is to the West from $\mathsf{l}$) shows that there is no real collision danger because both objects will be moving in parallel directions.

On the other hand, situation (b) does not produce any collision provided that $\mathsf{k}$ is moving slower than $\mathsf{l}$. This can be expressed by

$$(\mathsf{k},\mathsf{l};\mathsf{v}_2;\mathsf{o}_3;-,+;\mathsf{o}_3\mathsf{o}_4;\mathcal{C}_{6_1},\mathsf{d}_0;\mathsf{o}_4,\mathsf{d}_1), \text{ and } (\mathsf{l},\mathsf{k};\mathsf{v}_3;\mathsf{o}_3;+,-;\mathsf{o}_3\mathsf{o}_4;\mathcal{C}_{6_1},\mathsf{d}_0;\mathsf{o}_3,\mathsf{d}_1)$$

In this case, the key information to detect that there is not real collision danger is the fact that the velocity of $\mathsf{k}$ is $\mathsf{v}_2$ and the velocity of $\mathsf{l}$ is $\mathsf{v}_3$, that is, the former is moving slower than the latter.

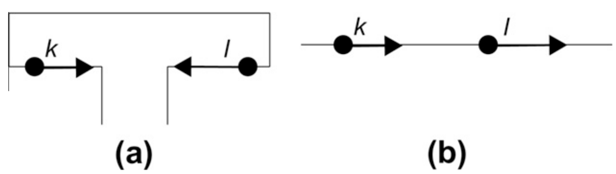


Figure 4: Two examples without collision danger

Let us consider now a different example, in which the situation is that of predicting collisions from an egocentric point of view, using the terminology of [15]. This means that all the information we have is about movements of objects relative to $x$, whereas the aim is to detect *any possible* collision: either collisions of $x$ with other objects, or collisions among other objects.

In Figure 5, the movements of $y$ and $z$ with respect to $x$ can be represented, respectively, by

$$\phi_1 = (y,x;\mathsf{v}_2;\mathsf{o}_4;-,+;\mathcal{C}_5;\mathsf{o}_2,\mathsf{d}_1;\mathsf{o}_3,\mathsf{d}_1), \text{ and}$$

$$\phi_2 = (x,z;\mathsf{v}_2;\mathsf{o}_1;-,-;\mathcal{C}_5;\mathsf{o}_2,\mathsf{d}_1;\mathsf{o}_3,\mathsf{d}_1).$$

Notice that $\phi_2$ suggests a collision danger between $x$ and $z$. By using the corresponding composition table, as introduced in the previous section, we obtain information about the movement of $y$ with respect to $z$, that is

$$\phi_3 = (y,z;\mathsf{v}_2;\mathsf{o}_4;-,-;\mathcal{C}_5;\mathsf{o}_2,\mathsf{d}_1\mathsf{d}_2;\mathsf{o}_3,\mathsf{d}_1\mathsf{d}_2)$$

which indicates also ad collision danger between $y$ and $z$, because $(-,-)$ means that $y$ and $z$ are moving towards each other, and they have the same velocity $\mathsf{v}_2$. We can deduce also the following information about the movement of $z$ with respect to $y$

$$\phi_4 = (z, y; \mathsf{v}_1\mathsf{v}_2\mathsf{v}_3; \mathsf{o}_2\mathsf{o}_3; -, -; \mathcal{C}_5; \mathsf{o}_1, \mathsf{d}_1\mathsf{d}_2; \mathsf{o}_4, \mathsf{d}_1\mathsf{d}_2)$$

Notice that, from the information given in $\phi_1$ and $\phi_2$, we can only say that velocity of $z$ is not zero and that $z$ is moving to the South or to the East, because, as stated by $\phi_2$, $z$ is *moving towards* $x$.

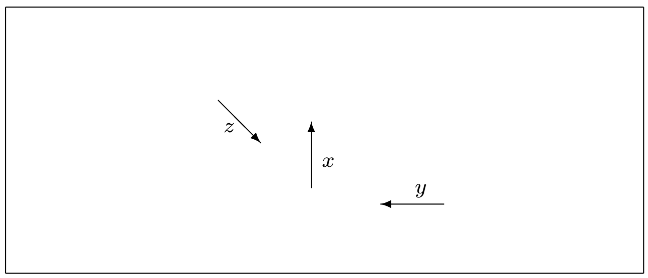


Figure 5: Collision detection among objects

It is remarkable that our approach has very important advantages with respect to [15]. In both cases above, the collision warning could be false, for example, depending on the velocities. This information is taken into account by our system; for example, in the case of Figure 5, if the velocity of $y$ is quick enough and the velocity of $z$ is slow, then the collision warning could be avoided. In this context, we could add also predictions of future locations of objects, in the line of [18]. Specific axioms for reasoning with collision avoidance will be presented in Section 5.4.

*3.3. Catching a ball*

We now consider the problem of intercepting a ball as presented in [2]. Assume the situation of Figure 6, where two robots $\mathsf{R}_1, \mathsf{R}_2$ are chasing a ball $\mathsf{B}$. The movement of the robot $\mathsf{R}_1$ with respect to the robot $\mathsf{R}_2$ is represented by

$$(\mathsf{R}_1, \mathsf{R}_2; \mathsf{v}_2; \mathsf{o}_3; -, +; \mathcal{C}_5; \mathsf{o}_1, \mathsf{d}_1; \mathsf{o}_4, \mathsf{d}_1)$$

the movement of robot $\mathsf{R}_2$ with respect to the ball $\mathsf{B}$ is represented by

$$(\mathsf{R}_2, \mathsf{B}; \mathsf{v}_2; \mathsf{o}_1; -, -; \mathcal{C}_5; \mathsf{o}_2, \mathsf{d}_2; \mathsf{o}_4, \mathsf{d}_1)$$

Following the tables presented above, we can compose both movements in order to obtain the movement of the robot $\mathsf{R}_1$ with respect to the ball $\mathsf{B}$, by

$$(\mathsf{R}_1, \mathsf{B}; \mathsf{v}_2; \mathsf{o}_3; \pm, -; \mathcal{C}_5; \mathsf{o}_2, \mathsf{d}_1\mathsf{d}_2; \mathsf{o}_4, \mathsf{d}_1\mathsf{d}_2)$$

In order to catch the ball, robots $\mathsf{R}_1$ and $\mathsf{R}_2$ could need to *modify* its velocity and orientation, as we will see in Example 1, after introducing the syntax and semantics of our proposed logic.

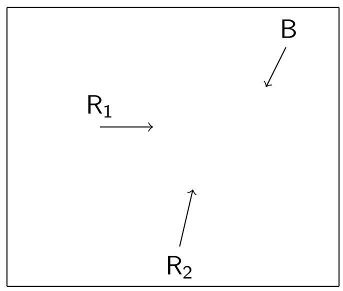


Figure 6: Catching a ball

## 4. Fuzzy qualitative representation of moving objects

In this section, we extend our previous approach [5], following the line of [34, 20]. As stated above, the consideration of fuzzy numbers allows us to combine both numerical (if any) and qualitative data.

Moreover, we apply fuzzy arithmetic operations [34] in order to obtain the composition of the movements. First, we consider fuzzy numbers in order to represent the each component of the movement. Recall that a fuzzy number $A$ is defined as a set

$$A = \{(x, \mu_A(x)) \mid x \in \mathbb{R}, \mu_A(x) \in [0, 1]\}$$

where $\mathbb{R}$ is the set of real numbers. We use the membership distribution of a trapezoidal fuzzy number given by the 4-tuple $[a, b, \alpha, \beta]$, where $a \leq b$ and $a \cdot b \geq 0$, see Figure 7. We impose the restriction $a \cdot b \geq 0$ because we want to clearly distinguish between positive and negative fuzzy numbers, as this is essential for interpreting the direction of the movements. Note that the restriction above allows considering "degenerated" cases such as $a = b = 0 = \alpha = \beta$.

In our case, the values of $a$ and $b$ will represent the milestones which determine each qualitative class.

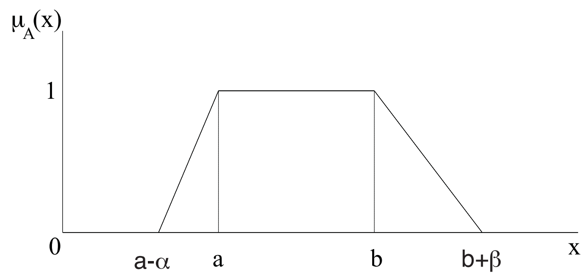


Figure 7: A trapezoidal fuzzy number.

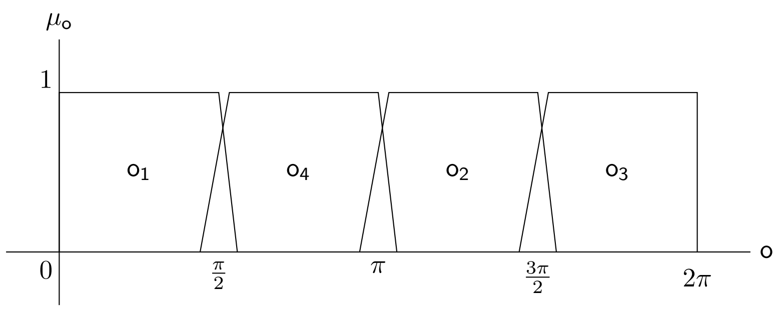


Figure 8: The fuzzy qualitative classes for orientation

For example, if we consider for the module of the velocity the qualitative classes $\mathsf{v}_0, \mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3$, and its values are normalised to the numeric range $[0, 1]$, then they could be represented as follows:

$$\mathsf{v}_0 = [0, 0, 0, 0]; \mathsf{v}_1 = [0, 0.2, 0, 0.2]; \mathsf{v}_2 = [0.4, 0.7, 0.1, 0.2]; \mathsf{v}_3 = [0.9, 1, 0.1, 0]$$

The fuzzy qualitative classes for orientation (none, E, N, W, S) are represented by:

$$\mathsf{o}_0 = [0, 0, 0, 0]; \quad \mathsf{o}_1 = [0, \frac{\pi}{2}, 0.1, 0.1]; \quad \mathsf{o}_4 = [\frac{\pi}{2} + 0.1, \pi, 0.1, 0.1]$$

$$\mathsf{o}_2 = [\pi + 0.1, \frac{3\pi}{2}, 0.1, 0.1]; \quad \mathsf{o}_3 = [\frac{3\pi}{2} + 0.1, 2\pi, 0.1, 0]$$

The arithmetic operations between fuzzy numbers introduced in Table 3 will be used on the components of movements defined above. For example, we can consider either the sum or difference of fuzzy orientations, in order to obtain some information about the composition of movements.

Table 3: Arithmetic operations with fuzzy numbers

| Operation | Result | Conditions |
|---|---|---|
| $-n$ | $(-d, -c, \delta, \gamma)$ | all $n$ |
| $\frac{1}{n}$ | $\left(\frac{1}{d}, \frac{1}{c}, \frac{\delta}{d(d+\delta)}, \frac{\gamma}{c(c-\gamma)}\right)$ | $n >_0 0, n <_0 0$ |
| $m+n$ | $(a+c, b+d, \tau+\gamma, \beta+\delta)$ | all $m, n$ |
| $m-n$ | $(a-d, b-c, \tau+\delta, \beta+\gamma)$ | all $m, n$ |
| $m \times n$ | $(ac, bd, a\gamma + c\tau - \tau\gamma, b\delta + d\beta + \beta\delta)$ | $m >_0 0, n >_0 0$ |
| | $(ad, bc, d\tau - a\delta + \tau\delta, -b\gamma + c\beta - \beta\gamma)$ | $m <_0 0, n >_0 0$ |
| | $(bc, ad, b\gamma - c\beta + \beta\gamma, -d\tau + a\delta - \tau\delta)$ | $m >_0 0, n <_0 0$ |
| | $bd, ac, -b\delta - d\beta - \beta\delta, -a\gamma - c\tau + \tau\gamma)$ | $m <_0 0, n <_0 0$ |
| | $m = [a, b, \tau, \beta], n = [c, d, \gamma, \delta]$ | |

We will consider the usual partial ordering $<_\alpha$ in the set of fuzzy numbers defined as follows. Given two fuzzy numbers $A, B$, we write $A <_\alpha B$ iff $x < y$, for every $x \in A_\alpha$, $y \in B_\alpha$, where

$$A_\alpha = \{x \in \mathbb{R} \mid \mu_A(x) > \alpha\}$$

In particular, we will use $<_0$, which corresponds to the relation *Before* between intervals [1].

The approximation of a fuzzy number $A'$ to a qualitative value $A$ can be determined by choosing $A$ such that $d(A, A')$ is the smallest among all the distances between the fuzzy number $A$ and all the fuzzy qualitative classes. There is a number of defuzzification methods in the literature that serve well for this purpose.

The main advantage of our fuzzy qualitative approach is that it allows us to use fuzzy arithmetic operations in order to obtain the composition of movements. Hence, Table 2 for composition of components $L_6$ and $L_7$ can be simplified as can be seen in Table 4.

Moreover, we will use the fuzzy partial ordering defined above to compare velocities, which will be useful, for example, for avoiding collisions.

Let us consider the example about collision avoidance in Figure 4(b), a real collision danger depends on the velocities of both $\mathsf{k}$ and $\mathsf{l}$. There is indeed a real collision danger whenever $\mathsf{l}$ is moving slower than $\mathsf{k}$. We represent both movements as follows:

$$(\mathsf{k}, \mathsf{l}; \mathsf{v}; \mathsf{o}_3; -, +; \mathsf{o}_3\mathsf{o}_4; \mathcal{C}_{6_1}, \mathsf{d}_0; \mathsf{o}_4, \mathsf{d}_1), \text{ and } (\mathsf{l}, \mathsf{k}; \mathsf{v}'; \mathsf{o}_3; +, -; \mathsf{o}_3\mathsf{o}_4; \mathcal{C}_{6_1}, \mathsf{d}_0; \mathsf{o}_3, \mathsf{d}_1)$$

Table 4: Using fuzzy operations for composition of $L_6$ and $L_7$.

| $\mathsf{A_iA_j}\backslash\mathsf{A_jA_k}$ | $\mathsf{o_rd_0}$ | $\mathsf{o_rd_u}$ | $\mathsf{o_td_u}$ |
|---|---|---|---|
| $\mathsf{o_rd_0}$ | $\mathsf{o_rd_0}$ | $\mathsf{o_rd_u}$ | $\mathsf{o_td_u}$ |
| $\mathsf{o_rd_s}$ | $\mathsf{o_rd_s}$ | $\mathsf{o_r(d_s+d_u)}$ | $\begin{cases} \mathsf{o_t(d_u-d_s)} & \text{if } \mathsf{s \leq u} \\ \mathsf{o_r(d_s-d_u)} & \text{if } \mathsf{s \geq u} \end{cases}$ |

In this case, we can take advantage of the partial ordering between fuzzy numbers $<_0$, and state that there is a real collision danger whenever $\mathsf{v}' <_0 \mathsf{v}$.

## 5. The logic $\mathsf{PDL}^{\mathsf{F}}_{\mathsf{M}}$

### *5.1. Syntax*

The language of logic $\mathsf{PDL}^{\mathsf{F}}_{\mathsf{M}}$ consists of a set of formulas $\Phi$ and a set of programs $\Pi$, which are defined recursively on disjoint sets $\Phi_0$ and $\Pi_0$, respectively. $\Phi_0$ is called the set of *atomic formulas* which can be thought of as abstractions of properties of states. Similarly, $\Pi_0$ is called the set of *atomic programs* which are intended to represent basic instructions.

**Formulas:**

- $\Phi_0 = \mathbb{V} \cup L$, where $\mathbb{V}$ is a denumerable set consisting of propositional variables and $L = L_1 \times \ldots \times L_7$, intended to represent atomic labels.
- If $\varphi$ and $\psi$ are formulas and $a$ is a program, then $\varphi \to \psi$ (propositional implication), $\perp$ (propositional falsity) and $[a]\varphi$ (program necessity) are also formulas. As usual, $\vee$ and $\wedge$ represent logical disjunction and conjunction, respectively; whereas $\langle a \rangle$ represents program possibility.

An atomic label $x = (x_1; x_2; x_3; (x_4^1, x_4^2); x_5; (x_6^1, x_6^2); (x_7^1, x_7^2)) \in L$ is said to be *simple* if components $x_2, x_3, x_4^1, x_4^2, x_6^1, x_6^2, x_7^1, x_7^2$ are singletons.

An non-simple atomic label will be seen disjunctively in terms of its simple components. For example, $x = (\mathsf{A_i}, \mathsf{A_j}; \mathsf{v_1}; \mathsf{o_2}; -, +; \mathsf{o_2o_3}; \mathsf{o_1}\underline{\mathbf{d_1d_2d_3}}; \mathsf{o_3d_1})$ can be decomposed as the disjunction of the following simple atomic labels

$$(\mathsf{A_i}, \mathsf{A_j}; \mathsf{v_1}; \mathsf{o_2}; -, +; \mathsf{o_2o_3}; \mathsf{o_1}\underline{\mathbf{d_1}}; \mathsf{o_3d_1})$$
$$(\mathsf{A_i}, \mathsf{A_j}; \mathsf{v_1}; \mathsf{o_2}; -, +; \mathsf{o_2o_3}; \mathsf{o_1}\underline{\mathbf{d_2}}; \mathsf{o_3d_1})$$
$$(\mathsf{A_i}, \mathsf{A_j}; \mathsf{v_1}; \mathsf{o_2}; -, +; \mathsf{o_2o_3}; \mathsf{o_1}\underline{\mathbf{d_3}}; \mathsf{o_3d_1})$$

The set of resulting disjuncts for an atomic label $x$ will be denoted as $\mathtt{decomp}(x)$.

**Programs:**

- The set $\Pi_0$ of *specific* programs is defined as follows:

$$\begin{aligned}\Pi_0 \quad = \quad & \{\mathtt{rev}_x \mid x \in L\} \cup \{\otimes_{x,y} \mid x, y \in L\} \cup \\ & \cup \{\mathtt{Dec}^{\mathsf{s}}_x, \mathtt{Man}^{\mathsf{s}}_x, \mathtt{Inc}^{\mathsf{s}}_x \mid \mathsf{s} \in \{0,1,2,3,4\}, x \in L\}\end{aligned}$$

- If $a$ and $b$ are programs and $\varphi$ is a formula, then $(a; b)$ ("do $a$ followed by $b$"), $a \cup b$ ("do either $a$ or $b$, nondeterministically"), $a^*$ ("repeat $a$ a nondeterministically chosen finite number of times") and $\varphi?$ ("proceed if $\varphi$ is true, else fail") are also programs.

The intuitive meaning of programs $\mathtt{rev}_x$ is considered to be the *reverse* of the movement $x$, that is, if $x$ represents a movement of $\mathsf{A_i}$ with respect to $\mathsf{A_j}$, then $\mathtt{rev}_x$ is the movement of $\mathsf{A_j}$ with respect to $\mathsf{A_i}$. In addition, $\otimes_{x,y}$ is *compose* the movement labeled by $x$, with the movement labeled by $y$. Moreover, programs $\mathtt{Dec}^{\mathsf{s}}_x$, $\mathtt{Man}^{\mathsf{s}}_x$, and $\mathtt{Inc}^{\mathsf{s}}_x$ for $\mathsf{s} \in \{0,1,2,3,4\}$ have the intuitive meaning of *modifying* the velocity and orientation of the movement labeled by $x$, specifically:

- $\mathtt{Dec}^{\mathsf{s}}_x$ means *decrease* the velocity *and modify* the orientation towards $\mathsf{o_s}$.

  That is, $\mathtt{Dec}^{\mathsf{0}}_s$ means *decrease* the velocity *and maintain* the orientation of the movement while, for example, $\mathtt{Dec}^{\mathsf{3}}_i$ means *decrease* the velocity *and modify* the orientation towards $\mathsf{o_3}$, that is, towards the East.

- Similarly, $\mathtt{Man}^{\mathsf{s}}_x$ means *maintain* the velocity *and modify* the orientation according to $\mathsf{s}$.

- Finally, $\mathtt{Inc}^{\mathsf{s}}_x$ means *increase* the velocity *and modify* the orientation according to $\mathsf{s}$.

*5.2. Semantics*

The *semantics* of $\mathsf{PDL^F_M}$ is defined as follows. A *model* $\mathcal{M}$ is a tuple $(W, m)$ where $W$ is a nonempty set of states. Each element $u \in W$ is to be understood as a state of an object moving with respect to another object and is labeled by elements of $L$.

The meaning function $m$ is required to fulfill the following:

- $m(p) \subseteq W$, for every propositional variable,
- If $x$ is an atomic label, then $m(x) = \displaystyle\bigcup_{y \in \mathtt{decomp}(x)} m(y) \subseteq W$
- for all $u \in W$, there exists $x \in L$ such that $u \in m(x)$.
- $m(x) \subseteq (W \smallsetminus m(y))$, where $x = (x_1; \ldots; x_7)$ and $y = (y_1; \ldots; y_7)$ with $x_1 \neq y_1$.

- $m(a) \subseteq W \times W$, for all atomic program $a$.

We define now the semantics of the specific programs in $\Pi_0$.

- If $x$ represents a movement of $\mathsf{A_i}$ with respect to $\mathsf{A_j}$, then

  $m(\mathtt{rev}_x)(m(x)) \subseteq m(y)$, where $y$ represents a movement of $\mathsf{A_j}$ with respect to $\mathsf{A_i}$.

- $m(\otimes_{x,y})(m(x)) \subseteq m(z)$

  where[3] $x = (\mathsf{A_i}, \mathsf{A_j}; x_2; x_3; x_4; x_5; x_6; x_7)$, $y = (\mathsf{A_j}, \mathsf{A_k}; y_2; y_3; y_4; y_5; y_6; y_7)$ and $z = (\mathsf{A_i}, \mathsf{A_k}; x_2; x_3; x_4^\tau; x_5; x_6^\tau; x_7^\tau)$, where $x_l^\tau$ for $l = 4, 6, 7$ is defined by the composition Tables 1 and 4.

  For every $\mathsf{s} \in \{0, 1, 2, 3, 4\}$, and an atomic label $x = (x_1; \ldots; x_7)$:

- $m(\mathtt{Dec}^{\mathsf{s}}_x)(m(x)) \subseteq m(y)$, where $y = (y_1; \ldots; y_7)$, being $y_1 = x_1$,

  $$y_2 = \begin{cases} \mathsf{v}_{\mathsf{k_1}-1} \ldots \mathsf{v}_{\mathsf{k_s}-1} & \text{if } x_2 = \mathsf{v}_{\mathsf{k_1}} \ldots \mathsf{v}_{\mathsf{k_s}}, \mathsf{k_1} > 0 \\ \mathsf{v}_0 \mathsf{v}_{\mathsf{k_1}-1} \ldots \mathsf{v}_{\mathsf{k_s}-1} & \text{if } x_2 = \mathsf{v}_0 \mathsf{v}_{\mathsf{k_1}} \ldots \mathsf{v}_{\mathsf{k_s}} \end{cases} \quad \text{and } y_3 = \mathsf{o_s}.$$

  In this case we say that $y$ is $x$**-decreasing**.

- $m(\mathtt{Man}^{\mathsf{s}}_x)(m(x)) \subseteq m(y)$, where $y = (y_1; \ldots; y_7)$, being $y_1 = x_1$, $y_2 = x_2$ and $y_3 = \mathsf{o_s}$.

- $m(\mathtt{Inc}^{\mathsf{s}}_x)(m(x)) \subseteq m(y)$, where $y = (y_1; \ldots; y_7)$, being $y_1 = x_1$,

  $$y_2 = \begin{cases} \mathsf{v}_{\mathsf{k_1}+1} \ldots \mathsf{v}_{\mathsf{k_s}+1} & \text{if } x_2 = \mathsf{v}_{\mathsf{k_1}} \ldots \mathsf{v}_{\mathsf{k_s}}, \mathsf{k_s} < 3 \\ \mathsf{v}_{\mathsf{k_1}+1} \ldots \mathsf{v}_{\mathsf{k_s}+1} \mathsf{v}_3 & \text{if } x_2 = \mathsf{v}_{\mathsf{k_1}} \ldots \mathsf{v}_{\mathsf{k_s}} \mathsf{v}_3 \end{cases} \quad \text{and } y_3 = \mathsf{o_s}$$

  In this case we say that $y$ is $x$**-increasing**.

Notice that the previous definition formalizes the intuitive meaning of $\mathtt{Dec}^{\mathsf{s}}_x$ as a binary relation such that $u$ is related to $v$ iff $v$ gives the description of a movement obtained by *decreasing* the velocity and modifying the orientation towards $\mathsf{o_s}$. Similarly for $\mathtt{Man}^{\mathsf{s}}_x$ and $\mathtt{Inc}^{\mathsf{s}}_x$.

Finally, if $\varphi$ and $\psi$ are formulas and $a, b$ are programs, then we have the following:

- $m(\varphi \to \psi) = (W \smallsetminus m(\varphi)) \cup m(\psi)$

- $m(\bot) = \varnothing$

- $m([a]\varphi) = \{w \in W : \text{for all } v \in W, \text{ if } (w, v) \in m(a) \text{ then } v \in m(\varphi)\}$

[3]Note that the left part of the inclusion represents a relation, $m(\otimes_{x,y})$, applied to a set, $m(x)$, with the usual meaning of the set of all the elements which are related to some element in $m(x)$.

- $m(a \cup b) = m(a) \cup m(b)$
- $m(a; b) = m(a); m(b)$
- $m(a^*) = m(a)^*$ (reflexive and transitive closure of relation $m(a)$).
- $m(\varphi?) = \{(w, w) \mid w \in m(\varphi)\}$

Given a model $\mathcal{M} = (W, m)$, a formula $\varphi$ is *true* in $u \in W$ whenever we have that $u \in m(\varphi)$. We say that $\varphi$ is *satisfiable* if there exists $u \in W$ such as $\varphi$ is true in $u$. Moreover, $\varphi$ is *valid in a model* $\mathcal{M} = (W, m)$ if $\varphi$ is true in all $u \in W$, that is, if $m(\varphi) = W$. Finally, $\varphi$ is *valid* if $\varphi$ is valid in all models.

The informal meaning of some formulas is given as follows. Let $p$ be any propositional formula, then $\langle x? \rangle\, p$ is true in $u$ iff $u$ represents a movement labeled by $x$, and $p$ is true in $u$. Formula $[\otimes_{x,y}; \otimes_{y,z}]\, p$ is true in $u$ iff for every movement $u'$ obtained by composing $u$ (labeled by $x$) with a movement labeled by $y$, followed by a composition with a movement labeled by $z$, $p$ is true in $u'$.

**Example 1.** *In order to emphasize the expressivity of our logic, we consider again the example* Catching a ball *presented in Section 3.3. The movement of robot* $\mathsf{R}_2$ *with respect to the ball* $\mathsf{B}$ *(see Figure 9) can be represented by:*

$$\varphi_1 = (\mathsf{R}_2, \mathsf{B}; \mathsf{v}_2; \mathsf{o}_1; -, -; \mathcal{C}_5; \mathsf{o}_2, \mathsf{d}_2; \mathsf{o}_4, \mathsf{d}_1)$$

*In order to catch the ball, the following formula has to be true:* $(\varphi_1?; \mathtt{Man}^3_{\varphi_1})^*; \neg\varphi_1?$. *The meaning of this formula is:* while *the movement of the ball with respect to the robot is given by* $\varphi_1$, do $\mathtt{Man}^3_{\varphi_1}$ *that is, maintain the velocity and modify the orientation towards the East.*

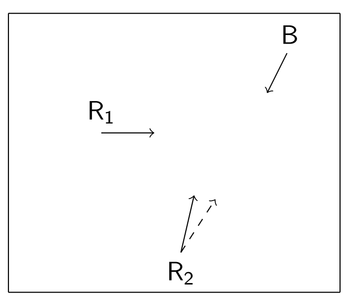


Figure 9: Catching a ball. Correction of the movement towards the East

*Consider now the situation of Figure 10, where there is a real collision danger between robots* $\mathsf{R}_1$ *and* $\mathsf{R}_2$*, while they are trying to catch the ball* $\mathsf{B}$*. Let us denote by* $\varphi_2$ *the movement of* $\mathsf{R}_2$ *with respect to* $\mathsf{R}_1$*, then the following formula has to be true* $(\varphi_2?; \mathtt{Inc}^3_{\varphi_2})$*. The previous formula means that* if *the movement* $\mathsf{R}_2$ *with respect to* $\mathsf{R}_1$ *is represented by* $\varphi_2$*, then* $\mathtt{Inc}^3_{\varphi_2}$*, that is, increase the velocity and modify the orientation towards the East in order to catch the ball.*

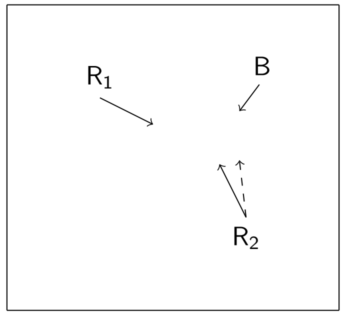


Figure 10: Catching a ball and collision avoidance

Notice that, in the previous example, we use the advantages of PDL for expressing programming commands such as *while ... do* is an improvement with respect to the IF-THEN rules used in [27, 28].

From a syntactical point of view, the conditions reflecting the required properties have to be included as axioms of our system. This situation is considered in the following section.

*5.3. Axiom system*

The following axiom system is intended to deal with the required properties presented in the previous section.

**Specific axiom schemata**:

For every $x, y, z \in L$:

**E:** $\bigvee_{x \in L} x$

where $x$ is a simple atomic label.

**U:** $x \to \neg y$

where $x = (x_1; \ldots; x_7)$ and $y = (y_1; \ldots; y_7)$ with $x_1 \neq y_1$, and both $x, y$ are simple.

**D:** $x \leftrightarrow \bigvee_{y \in \mathtt{decomp}(x)} y$

**Rev** $x \to [\mathtt{rev}_x]y$

where $x = (\mathsf{A_i}, \mathsf{A_j}; x_2; x_3; (x_4^1, x_4^2); x_5; (x_6^1, x_6^2); (x_7^1, x_7^2))$, and

$$y = (\mathsf{A_j}, \mathsf{A_i}; \mathcal{C}_2; \mathcal{C}_3; (x_4^2, x_4^1); \varnothing; (-x_6^1, x_6^2); (-x_7^1, x_7^2))$$

where $-x_l^1$ is the opposite orientation of $x_l^1$.

**Comp:** $x \to [\otimes_{x,y}]z$

where $x = (\mathsf{A_i}, \mathsf{A_j}; x_2; x_3; x_4; x_5; x_6; x_7)$, $y = (\mathsf{A_j}, \mathsf{A_k}; y_2; y_3; y_4; y_5; y_6; y_7)$, $z = (\mathsf{A_i}, \mathsf{A_k}; x_2; x_3; x_4^\tau; x_5; x_6^\tau; x_7^\tau)$, and $x_l^\tau$ for $l = 4, 6, 7$ is defined by the composition tables.

For every $\mathsf{s} \in \{0, 1, 2, 3, 4\}$, we define:

**Dec** $x \to [\mathtt{Dec}^{\mathsf{s}}_x]y$

where $y$ is $x$-decreasing.

**Man** $x \to [\mathtt{Man}^{\mathsf{s}}_x]y$

where $y = (y_1; \dots; y_7)$, being $y_1 = x_1$, $y_2 = x_2$, and $y_3 = \mathsf{o_s}$.

**Inc** $x \to [\mathtt{Inc}^{\mathsf{s}}_x]y$

where $y$ is $x$-increasing.

The previous axioms have the following intuitive meaning:

- **E** means that every state is labeled by some element of $L$.
- **U** means that every state represents the movement of a specific object with respect to another specific object.
- **D** represents the disjunctive nature of our labels.
- **Rev** collects information about the movement of $\mathsf{A_j}$ with respect to $\mathsf{A_i}$ from the information of the movement of $\mathsf{A_i}$ with respect to $\mathsf{A_j}$.
- **Comp** gives the information about the composition of a movement of $\mathsf{A_i}$ with respect to $\mathsf{A_j}$, with a movement of $\mathsf{A_j}$ with respect to $\mathsf{A_k}$, collecting this information from the composition tables.
- **Dec**, **Man** and **Inc** describe the modification of a movement in order to either decreasing, maintaining or increasing its velocity and towards a fixed orientation.

The rest of axioms are those specific to PDL.

**Axiom schemata for PDL**:

**A1** All instances of tautologies of the propositional calculus.

**A2** $[a](\varphi \to \psi) \to ([a]\varphi \to [a]\psi)$

**A3** $[a](\varphi \wedge \psi) \to ([a]\varphi \wedge [a]\psi)$

**A4** $[a \cup b]\varphi \to ([a]\varphi \vee [b]\varphi)$

**A5** $[a;b]\varphi \rightarrow [a][b]\varphi$

**A6** $[\varphi?]\psi \rightarrow (\varphi \rightarrow \psi)$

**A7** $(\varphi \wedge [a][a^*]\varphi) \rightarrow [a^*]\varphi$

**A8** $(\varphi \wedge [a^*](\varphi \rightarrow [a]\varphi)) \rightarrow [a^*]\varphi$ (induction axiom)

**Inference Rules**:
**(MP)** $\varphi, \varphi \rightarrow \psi \vdash \psi$ (*Modus Ponens*) **(G)** $\varphi \vdash [a]\varphi$ (generalization)

*5.4. Specific axioms for collision avoidance*

We use here the expressiveness of our approach in order to introduce some specific axioms for collision avoidance. Let us consider the case where the movement of $\mathsf{A_i}$ with respect to $\mathsf{A_j}$ is given by

$$x = (\mathsf{A_i}, \mathsf{A_j}; \mathcal{C}_2; \mathcal{C}_3; -, -; \mathcal{C}_5; \mathcal{C}_{6_1}, \mathsf{d_0 d_1}; \mathcal{C}_{7_1}, \mathsf{d_0 d_1})$$

that is, $\mathsf{A_i}, \mathsf{A_j}$ are moving towards each other, $\mathsf{A_i}$ is allowed to move in any direction, and both objects are at zero or close latitude and longitude. This situation represents a collision danger.

To ensure the collision avoidance, if we denote by $y = y_1 \vee y_2 \vee y_3$, for

$$y_1 = (\mathsf{A_i}, \mathsf{A_j}; \mathcal{C}_2; \mathcal{C}_3; +, -; \mathcal{C}_5; \mathcal{C}_{6_1}, \mathsf{d_0 d_1}; \mathcal{C}_{7_1}, \mathsf{d_0 d_1})$$

$$y_2 = (\mathsf{A_i}, \mathsf{A_j}; \mathcal{C}_2; \mathcal{C}_3; -, +; \mathcal{C}_5; \mathcal{C}_{6_1}, \mathsf{d_0 d_1}; \mathcal{C}_{7_1}, \mathsf{d_0 d_1})$$

$$y_3 = (\mathsf{A_i}, \mathsf{A_j}; \mathcal{C}_2; \mathcal{C}_3; +, +; \mathcal{C}_5; \mathcal{C}_{6_1}, \mathsf{d_0 d_1}; \mathcal{C}_{7_1}, \mathsf{d_0 d_1})$$

the following family of formulas, for $\mathsf{s} \in \{0, 1, 2, 3, 4\}$, has to be an axiom schema for collision avoidance:

$$x?; (\mathtt{Dec}^{\mathsf{s}}_x; (\neg y?; \mathtt{Dec}^{\mathsf{s}}_x)^*; y?)$$

which means *if* there is a collision danger (for the movement represented by $x$), *then* decrease the velocity and modify the orientation of the movement (represented by $\mathtt{Dec}^{\mathsf{s}}_x$) *until* the collision danger disappears because either one of the objects is moving away from the other one, or both objects are moving away form the other one (represented by $y$).

The full set of specific axioms for collision avoidance can be obtained depending on the relative position of $\mathsf{A_i}$ with respect to $\mathsf{A_j}$, which would allow us to choose the specific value of $s$ in $y$, that is, the direction given to the modification of the orientation. For instance, if

$$x = (\mathsf{A_i}, \mathsf{A_j}; \mathcal{C}_2; \mathcal{C}_3; -, -; \mathcal{C}_5; \mathsf{o_1}, \mathsf{d_0 d_1}; \mathsf{o_4}, \mathsf{d_0 d_1})$$

then the value of $\mathsf{s}$ in the previous formula would be 1, because, as $\mathsf{A_i}$ is to the North $\mathsf{o_1}$ and to the West $\mathsf{o_4}$ from $\mathsf{A_j}$. In order to avoid the collision, we decrease the velocity and change the orientation of the movement to the North (represented by $\mathtt{Dec}_x^{\mathsf{1}}$). Similarly, we obtain each specific axiom depending on this position.

## 6. Soundness, Completeness and Decidability

In order to prove the soundness of our system, we give the following result.

**Lemma 1.** *All the axioms are valid formulas and all the inference rules preserve validity.*

*Proof.* The proofs of validity of the axiom schemata **A1**, ..., **A8** are standard in PDL. The proofs of validity of the specific axiom schemata are very similar. As a way of example, let us consider axioms **D** and **Dec**.

The validity of axiom **D** is proved as follows: Given any model $(W, m)$, we have

$$m(x) = \bigcup_{y \in \mathtt{decomp}(x)} m(y)$$

Therefore

$$m(x) = m(\bigvee_{y \in \mathtt{decomp}(x)} y)$$

which proves the validity of axiom **D**. For proving **Dec**: $x \to [\mathtt{Dec}_x^{\mathsf{s}}]y$, where $y$ is $x$-decreasing, take any model $(W, m)$. Consider $u \in W$ such that $u \in m(x)$, we have to prove that $u \in m([\mathtt{Dec}_x^{\mathsf{s}}]y)$. For this, consider any $v \in W$ such that $(u, v) \in m(\mathtt{Dec}_x^{\mathsf{s}})$, that is, $v \in m(\mathtt{Dec}_x^{\mathsf{s}})(m(x))$. From the semantic condition $m(\mathtt{Dec}_x^{\mathsf{s}})(m(x)) \subseteq m(y)$ stated in Section 5.2, we have that $v \in m(y)$, and this proves that $u \in m([\mathtt{Dec}_x^{\mathsf{s}}]y)$ and, as a consequence, the validity of axiom **Dec**.

On the other hand, it is a trivial task to check that rules **(MP)** and **(G)** preserve validity.

As a consequence, we have the soundness of our system as follows.

**Theorem 1.** *For every formula $\varphi$, if $\varphi$ is a theorem then $\varphi$ is a valid formula.*

For a solution of the satisfiability problem for our logic we can prove the *small model property* following the pattern established in [17]. Modifications of Fisher-Lander Closure in order to get a finite filtration of a model are trivial and for details of the filtration technique we refer to that work. So we have the following result.

**Theorem 2.** *Let $\varphi$ a satisfiable formula, then $\varphi$ is satisfied in a model with no more than $2^{|\varphi|}$ states, where $|\varphi|$ is the length of the formula $\varphi$.*

Now we are concerned with the Completeness of our logic. To this end, we build a *nonstandard model* from maximal consistent sets of formulas [17]. Then the Filtration Lemma for non standard models, can be used to collapse this model into a finite *standard* model. A *nonstandard model* is any structure $\mathcal{N} = (N, m_{\mathcal{N}})$ such as it is a model as defined previously in every respect, except that, for every program $a$, $m_{\mathcal{N}}(a^*)$ needs not be the reflexive and transitive closure of $m_{\mathcal{N}}(a)$, but only a reflexive and transitive relation which contains $m_{\mathcal{N}}(a)$.

We define a nonstandard model $(N, m_{\mathcal{N}})$ as follows: $N$ contains all the maximal consistent sets of formulas of our logic and $m_{\mathcal{N}}$ is defined, for every formula $\varphi$ and every program $a$, by:

$$m_{\mathcal{N}}(\varphi) = \{u \mid \varphi \in u\}; \quad m_{\mathcal{N}}(a) = \{(u, v) \mid \text{for all } \varphi, \text{ if } [a]\varphi \in u \text{ then } \varphi \in v\}$$

Using the previous definition, all the properties for nonstandard models are satisfied, even the ones for our specific atomic programs, as we can see in the following result.

**Lemma 2.** *$(N, m_{\mathcal{N}})$ verifies the required properties for non-standard models.*

*Proof.* As a way of example, let us prove some of the specific properties of models presented in Section 5.2.

Let us prove firstly that if $x$ is an atomic label, then

$$m_{\mathcal{N}}(x) = \bigcup_{y \in \texttt{decomp}(x)} m_{\mathcal{N}}(y)$$

For every $u \in W$, we have that $u \in m_{\mathcal{N}}(x)$, which implies, by definition of $m_{\mathcal{N}}$, that $x \in u$. By axiom schema **D**, it holds $\displaystyle\bigvee_{y \in \texttt{decomp}(x)} y \in u$, that is, $u \in m_{\mathcal{N}}(\displaystyle\bigvee_{y \in \texttt{decomp}(x)} y)$. The other implication is similar.

Let us prove now the specific property $m_{\mathcal{N}}(\texttt{Dec}^{\texttt{s}}_x)(m_{\mathcal{N}}(x)) \subseteq m_{\mathcal{N}}(y)$, where $y$ is $x$-decreasing. Suppose $u \in m_{\mathcal{N}}(\texttt{Dec}^{\texttt{s}}_x)(m_{\mathcal{N}}(x))$, which means that there exists $v \in m_{\mathcal{N}}(x)$ such that $(v, u) \in m_{\mathcal{N}}(\texttt{Dec}^{\texttt{s}}_x)$. Notice that $v \in m_{\mathcal{N}}(x)$ means that $x \in v$, and using now the axiom schema **Dec**, we obtain $[\texttt{Dec}^{\texttt{s}}_x]y \in v$. From $(v, u) \in m_{\mathcal{N}}(\texttt{Dec}^{\texttt{s}}_x)$, and using the definition of $m_{\mathcal{N}}$, we get $u \in m_{\mathcal{N}}(y)$, which ends the proof of this specific property.

The proof of the rest of properties, can be done similarly.

Now, we can give the following completeness result.

**Theorem 3.** *For every formula $\varphi$, if $\varphi$ is valid then $\varphi$ is a theorem.*

*Proof.* We need to prove that if $\varphi$ is consistent, then it is satisfied. If $\varphi$ is consistent, it is contained in a maximal consistent set $u$, which is a state of the nonstandard model constructed above. By the Filtration Lemma for nonstandard models, $\varphi$ is satisfied in a state corresponding to $u$ of the filtration model.

From Theorems 1, 2 and 3, we have the following result.

**Theorem 4.** *The logic* $\mathsf{PDL}_{\mathsf{M}}^{\mathsf{F}}$ *is sound, complete and decidable.*

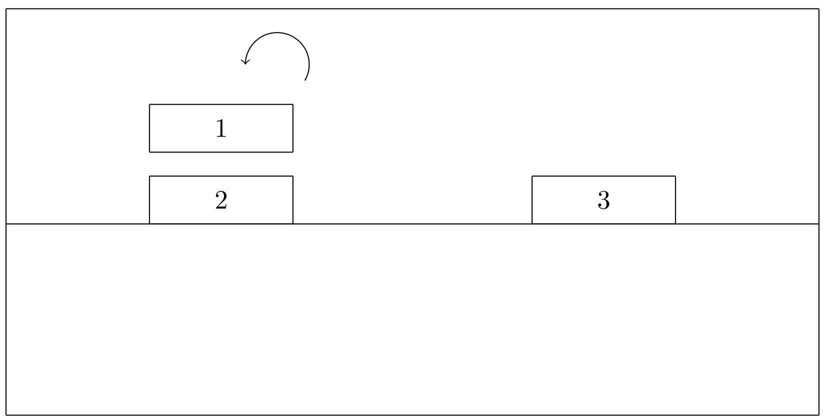


Figure 11: Parking 1

We conclude this section with one more example of application of our approach, this time to model a method for parallel parking. In this example, we could see the expresivity of PDL for using programming commands as IF...THEN, DO...UNTIL, etc.

Suppose car 1 is parking between cars 2 and 3, and our reference system is in the back end of car 2, see Figure 11. To begin with, car 1 has to move parallel to car 2 until both back ends coincide, that is, until the qualitative latitude East-West of car 1 with respect to car 2 is represented by $x_7 = (\mathsf{o}_1, \mathsf{d}_0)$. Hence, if the movement of 1 with respect to 2 is represented by

$$\varphi_1 = (\mathsf{A}_1, \mathsf{A}_2; \mathsf{v}_1; \mathsf{o}_3; -, 0; \mathsf{o}_3\mathsf{o}_4; \mathsf{o}_1, \mathsf{d}_1; \mathsf{o}_4, \mathsf{d}_1)$$

the following formula has to be true $\varphi_2?; (\neg\varphi_3?; \varphi_2?)^*; \varphi_3?$, being

$$\varphi_2 = (\mathsf{A}_1, \mathsf{A}_2; \mathsf{v}_1; \mathsf{o}_3; -, 0; \mathsf{o}_3\mathsf{o}_4; \mathsf{o}_1, \mathsf{d}_1; \mathsf{o}_4, \mathsf{d}_1)$$

$$\varphi_3 = (\mathsf{A}_1, \mathsf{A}_2; \mathsf{v}_1; \mathsf{o}_3; -, 0; \mathsf{o}_3\mathsf{o}_4; \mathsf{o}_1, \mathsf{d}_1; \mathsf{o}_3, \mathsf{d}_0)$$

which means *repeat* $\varphi_2$? *until* $\varphi_3$, that is, car 1 has to maintain its movement until its back end coincides with the back end of car 2. In this moment, the car has to change abruptly its direction totally to the West, as shown in Figure 11. This situation can be represented by the formula $\varphi_4?;\Delta_{\mathsf{o}_4}$, which means that *if* $\psi_{\mathsf{P}}$ *then* $\Delta_{\mathsf{o}_4}$, being

$$\varphi_4 = (\mathsf{A}_1, \mathsf{A}_2; \mathsf{v}_1; \mathsf{o}_3; -, 0; \mathsf{o}_3\mathsf{o}_4; \mathsf{o}_1, \mathsf{d}_1; \mathsf{o}_3, \mathsf{d}_0)$$

and $\Delta_{\mathsf{o}_\mathsf{j}}$, $\mathsf{j} \in \{1, \dots 4\}$ new programs representing and abrupt and complete change of direction towards $\mathsf{o}_\mathsf{j}$. After that, car 1 has to continue moving slowly until its position is close to and angle of $\pi/4$ radians with respect to car 2, see Figure 12. Now, car 2 has to change its direction again, but now completely to the East, and continue slowly until either car 2 is totally parallel to the pavement or it is very close to car 3. After that, car 1 would be *perfectly* parked between cars 2 and 3. The previous statements imply the truth of the following three formulas: $\mathtt{Man}^0_1; (\neg\chi?; \mathtt{Man}^0_1)^*; \chi?$, meaning that maintain the velocity and orientation *until* the angle of car 2 with respect to car 1 is $\pi/4$; formula $\chi?;\Delta_{\mathsf{o}_3}$ means that *if* the angle of car 2 with respect to car 1 is $\pi/4$ *then* change totally the direction towards the East; and $\mathtt{Man}^0_1; (\neg(\nu \vee \mu)?; \mathtt{Man}^0_1)^*; (\nu \vee \mu)?$, meaning maintain the velocity and orientation *until* either car 1 is parallel to car 2 or car 1 is very close to car 3, being:

$$\chi = (\mathsf{A}_1, \mathsf{A}_2; \mathsf{v}_1; \mathsf{o}_3; -, 0; \mathcal{C}_5; \mathsf{o}_2, \mathsf{d}_1; \mathsf{o}_3, \mathsf{d}_1)$$

similarly to $\chi$, and

$$\nu = (\mathsf{A}_1, \mathsf{A}_2; \mathsf{v}_1; \mathsf{o}_3; -, 0; \mathcal{C}_5; \mathcal{C}_{6_1}, \mathsf{d}_0\mathsf{d}_1; \mathsf{o}_3, \mathsf{d}_1)$$

$$\mu = (\mathsf{A}_1, \mathsf{A}_3; \mathsf{v}_1; \mathsf{o}_3; -, 0; \mathcal{C}_5; \mathcal{C}_{6_1}, \mathsf{d}_0\mathsf{d}_1; \mathsf{o}_4, \mathsf{d}_0\mathsf{d}_1)$$

## 7. Conclusions and Future Work

We presented a PDL framework for reasoning with fuzzy qualitative movement which entitles us to manage both qualitative and quantitative information, and consequently, to obtain more accurate results. Some of the advantages of PDL have been exploited and explained on the basis of some real examples from the literature, such as the use of programming commands as *while ... do* and *repeat ... until*, which enrich the expressivity or our approach.

As a future work, we consider the application of our approach to more real situations such a moving robots or the design of automatic driving systems for cars. Last, but not least, we consider the construction of a theorem prover for our logic and the study of its complexity, in the line of [13, 14].

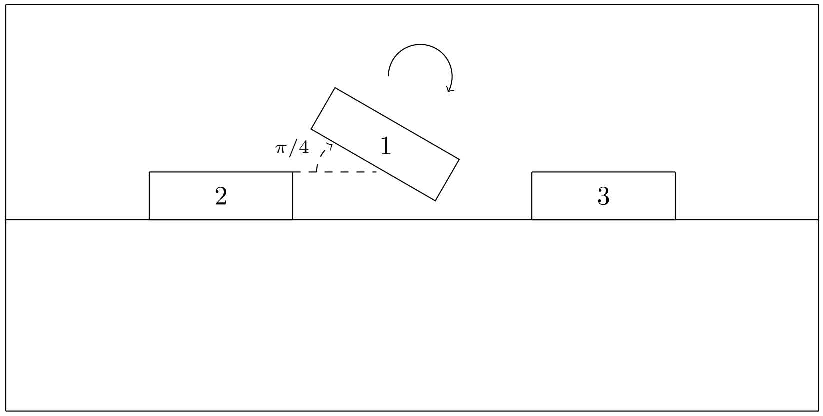


Figure 12: Parking 2

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