Fuzzy Logic Programming via Multilattices [1]

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Abstract

We investigate the use of multilattices as the set of truth-values underlying a general fuzzy logic programming framework. On the one hand, some theoretical results about ideals of a multilattice are presented in order to provide an ideal-based semantics; on the other hand, a restricted semantics, in which interpretations assign elements of a multilattice to each propositional symbol, is presented and analysed.

Key words: Fuzzy logic programming, multilattices, fixed point semantics

1 Introduction

The problem of generalising the structure of the underlying set of truth-values for fuzzy logic programming has attracted the attention of a number of researchers in the recent years.

On the one hand, weakening the structure of the underlying set of truth-values for logic programming has been studied extensively: There are approaches which are based either on the structure of lattice (residuated lattice [6, 21] or multi-adjoint lattice [16]) with the aim of providing a common framework for monotonic extensions of logic programming, or more restrictive structures, such as bilattices [7,14], specially suited for the treatment of non-monotonicity, or trilattices [13], in which points can be ordered according to truth, information, or precision. There have been even more general structures such as algebraic domains [19].

On the other hand, one can also find some attempts aiming at weakening the restrictions imposed on a (complete) lattice, namely, the "existence of least

[1] Partially supported by Spanish MEC projects TIC2003-09001-C02-01 and TIN2006-15455-C03-01.

upper bounds and greatest lower bounds" is relaxed to the "existence of *minimal* upper bounds and *maximal* lower bounds". In this direction, Benado [1] and Hansen [9] proposed definitions of a structure so-called multilattice.

More recently, another formalisation of the notion of multilattice was introduced [3, 4] as a theoretical tool to deal with some problems in the theory of mechanised deduction in temporal logics. This kind of structure also arises in the research area concerning fuzzy extensions of logic programming: for instance, one of the hypotheses of the main termination result for sorted multi-adjoint logic programs [5] can be weakened only when the underlying set of truth-values is a multilattice (the question of providing a counter-example on a lattice remains open).

A typical example of multilattice can be obtained by interlacing two or more given chains, for instance as in Fig 1 below. From an applicative point of view regarding logic programming, when constructing one program, the assignment of weights to each fact or rule could be done by asking a set of experts, and each of the given chains may be interpreted as the scoring labels used by each of these experts; for instance, values a and c would correspond to Expert 1 and b and d to Expert 2 (maximum and minimum are identified). The underlying idea is to consider the value of any element of the multilattice modulo the interlace, this means that we can think of the chain consisting of $[\bot, \{a, b\}, \{c, d\}, \top]$, and get the added benefit of the traceability of the result, that is, to check to what extent the information provided by an expert has been useful for a particular computation. Obviously, the reference to M6 has been only to provide a simple example, a much more interesting example can be obtained by interlacing two or more chains $[0, 0.1, 0.2, \dots, 0.9, 1]$

As far as we know, the only attempt to use multilattices in the context of extended fuzzy logic programming has been [15], which is extended in several directions in this work. The aim of this paper is two-fold: on the one hand, we study some mathematical properties of multilattices, including new conditions under which fixed points exist, together with an introduction to the theory of ideals of multilattices, in which some new results are presented (particularly those related to submultilattices) and linked to previous results in the literature, in order to provide (some) ideal-based semantics of monotonic logic programs valued on a multilattice; on the other hand, a restricted semantics, in which interpretations assign elements of a multilattice to each propositional symbol, is presented and analysed. The study of the restricted semantics generates a number of interesting algebraic problems among which we concentrate in the development of a (non-deterministic) fixed point theory. Last but not least, we improve the result about the existence of minimal models in [15], and we prove here that any model is lower bounded by a minimal model.

The structure of the paper is as follows: In Section 2 the definition and pre-

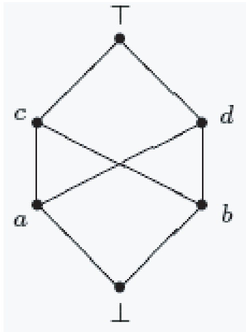


Fig. 1. The multilattice M6.

liminary theoretical results about multilattices are introduced, including the selection of desirable computational properties of our multilattices, and its theory of ideals; later, the syntax of our extended logic programs are presented in Section 3, together with the ideal-based semantics and the restricted semantics, both based on the fixed points of the immediate consequence operator. The last section contains conclusions and prospects for future work.

2 Preliminary results about multilattices

Recall that a complete lattice is a poset such that the set of upper (lower) bounds of every subset has a unique minimal (maximal) element, that is, a *minimum* (*maximum*). In a multilattice, this property is relaxed in the sense that minimal elements for the set of upper bounds should exist, but the uniqueness condition is dropped.

Definition 1 *A* complete multilattice *is a partially ordered set, $\langle M, \leq \rangle$, such that for every subset $X \subseteq M$, the set of upper (lower) bounds of X has minimal (maximal) elements, which are called* multi-suprema *(*multi-infima*).*

Note that, by definition, it follows that the sets multinf(X) and multisup(X) are *antichains* (non-empty sets consisting of pair-wise incomparable elements).

Example 2 *The simplest proper complete multilattice is shown in Fig. 1. Note that the elements a and b do not have a least upper bound, but two minimal upper bounds, namely c and d.*

It is remarkable that, under suitable conditions, the set of fixed points of a mapping from M to M does have a minimum and a maximum.

Definition 3 *A mapping $f : P \longrightarrow Q$ between two posets is said to be* isotone *if $x \leq y$ implies $f(x) \leq f(y)$; a mapping $g : P \longrightarrow P$ is* inflationary *if $x \leq g(x)$ for all $x \in P$.*

Theorem 4 *Let $f: M \longrightarrow M$ be an isotone and inflationary mapping on a multilattice, then its set of fixed points is non-empty and has a minimum element.*

PROOF. Let us write $X = \{x \mid f(x) = x\}$, this set is nonempty since inflation forces $\top$ to be a fixed point; now, consider $a \in \text{multinf}(X)$, a maximal lower bound of X, and let us prove that a is a fixed point of f.

As a is a lower bound for all $x \in X$, we have $a \leq x$ and, by isotonicity, $f(a) \leq f(x) = x$ for all $x \in X$ (the equality follows by definition of X); thus, $f(a)$ is also a lower bound of X. Moreover, by inflation, we have $a \leq f(a)$; thus, as a is maximal, we have $f(a) = a$, and a is a fixed point. Hence, we have $a \in X$.

We have proved that multi-infima of the non-empty set of fixed points are fixed points. Now, let us show that there cannot be two different multi-infima.

Consider $a, b \in \text{multinf}(X)$, and recall that we have just proved that $a, b \in X$. As both are lower bounds of X, then $a \leq b$ and $b \leq a$. Thus, multinf(X) is a singleton consisting of the minimum element of X, that is, the minimum fixed point. $\square$

As by assumption, our sets will not necessarily have a supremum but a *set* of multi-suprema, we will need to work with some ordering between subsets of posets. Three different (pre-)orderings are usually considered in the literature, the Hoare ordering, the Smyth ordering and the Egli-Milner ordering:

Definition 5 *Consider $X, Y \subseteq 2^M$:*

- *$X \sqsubseteq_H Y$ iff for all $x \in X$ exists $y \in Y$ such that $x \leq y$.*
- *$X \sqsubseteq_S Y$ iff for all $y \in Y$ exists $x \in X$ such that $x \leq y$.*
- *$X \sqsubseteq_{EM} Y$ iff $X \sqsubseteq_H Y$ and $X \sqsubseteq_S Y$*

2.1 Some desirable properties of multilattices

Regarding computational properties of multilattices, it is interesting to impose certain conditions on the sets of upper (lower) bounds of a given set X. Specifically, we would like to ensure that any upper (lower) bound is greater (less) than a minimal (maximal); this condition enables to work on the set of multi-suprema (multi-infima) as a set of “generators” of the bounds of X. The formalisation of these concepts is given as follows:

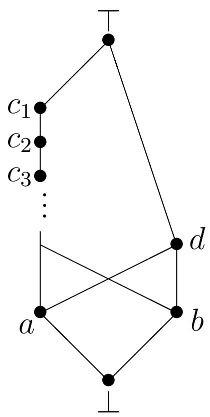


Fig. 2. A non-coherent multilattice.

Definition 6 *A multilattice is said to be* coherent *if the following set of inequalities hold for all $X \subseteq M$:*

$$LB(X) \sqsubseteq_{EM} \text{multinf}(X) \qquad \text{multisup}(X) \sqsubseteq_{EM} UB(X)$$

where $UB(X)$ (resp. $LB(X)$) denotes the set of upper (lower) bounds of X.

Note that in the two items above, one part of the Egli-Milner ordering is trivial, since any multi-infimum is a lower bound and any multi-supremum is an upper bound. It is not difficult to provide examples of non-coherent multilattices:

Example 7 *A non-coherent multilattice is shown in Fig. 2, where*

$$UB(\{a,b\}) = \{\top, d\} \cup \{c_n \mid n \in \mathbb{N}\}$$

in which element d is minimal in $UB(\{a,b\})$; however, the elements c_n fail to be greater than one minimal upper bound.

Another reasonable condition to require on a multilattice is that it should not contain infinite sets of mutually incomparable elements (antichains) since, semantically, it makes little sense to consider infinitely many incomparable truth-values. Coherent multilattices without infinite antichains have interesting computational properties: to begin with, recall that the sets of multi-suprema or multi-infima for totally ordered subsets (also called *chains*) always have a supremum and an infimum.

Lemma 8 *Let M be a coherent multilattice without infinite antichains, then any chain in M has a supremum and an infimum.*

PROOF. Let $\{x_i\}_{i \in I} \subset M$ be a chain and, assume that $a, b \in \text{multisup}(\{x_i\})$. We will show that there is an element $c \in \text{multinf}(\{a,b\})$ which is an upper bound of the chain.

As there are no infinite antichains in M, the set $\text{multinf}(\{a, b\})$ is finite, and we can write

$$\text{multinf}(\{a, b\}) = \{c_1, \ldots, c_n\}$$

If $n = 1$, as any x_i is a lower bound of $\{a, b\}$, by the hypothesis of consistency we would have $x_i \leq c_1$ for all $i \in I$.

If $n > 1$, by contradiction, assume that no c_j, with $j = 1, \ldots, n$, is an upper bound of the chain; then, for all j we choose an element x_j which is not upper bounded by c_j. Now, as $\{x_i\}$ is a chain, let us consider the greatest of $x_1, \ldots, x_n$, say x_{j_0}. By consistency, there is c_k which is greater than x_{j_0}, but then

$$x_k \leq x_{j_0} \leq c_k$$

which would contradict the choice of x_k.

To summarise, we have proved the existence of $c \in \text{multinf}(\{a, b\})$ which, moreover, is an upper bound of the chain. Now, $c \in \text{multinf}(\{a, b\})$ implies the inequalities $c \leq a$ and $c \leq b$; on the other hand, as c is also an upper bound of $\{x_i\}$ and a and b are multi-suprema of $\{x_i\}$, then $a \leq c$ and $b \leq c$, resulting that $a = b = c$, which proves that $\text{multisup}(\{x_i\})$ is a singleton, hence the supremum of the chain.

The proof for the infimum is similar. □

All the hypotheses are necessary for the existence of supremum and infimum of chains; in particular, the condition on infinite antichains cannot be dropped.

Example 9 *The poset in Fig. 3 is a coherent multilattice; however, the set of upper bounds of the increasing sequence $\{x_n\}$ does not have a minimum, but two minimals, namely, a and b.*

An important consequence of the choice of coherent multilattices and without infinite antichains is the following result, alternative to Theorem 4, concerning the existence and reachability of least fixed points of mappings $f\colon M \longrightarrow M$.

Theorem 10 *Let M be a coherent multilattice without infinite antichains and $f\colon M \longrightarrow M$ an isotone map then f has a least fixed point which can be reached by (transfinite) iterations of f.*

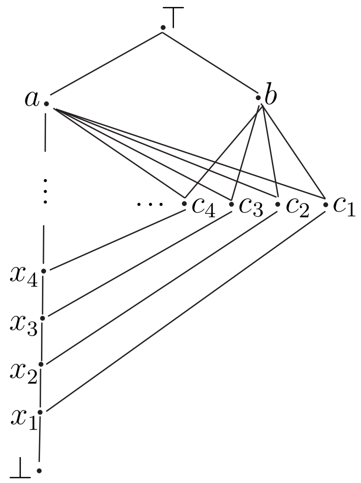


Fig. 3. Coherent multilattice with infinite antichains.

PROOF. Isotonicity guarantees the construction, starting from $\bot$, of an increasing sequence for every successor ordinal. Now, by Lemma 8, every chain has a supremum, so it makes sense to define the extension of the sequence for limit ordinals in the usual manner. The rest of the proof follows that of Knaster-Tarski for complete lattices. □

Note that the same result does not hold when changing inflation for isotonicity. Although it is trivial that an inflationary function has fixed points, it is not always the case that a least one should exist. Moreover, even if a least fixed point exists it might happen that the sequence defined above does not converge to it.

Example 11 *Let us consider the map $f\colon M6 \longrightarrow M6$ defined as follows:*

$$f(\bot) = a, \quad f(a) = d, \quad f(b) = c, \quad f(c) = c, \quad f(d) = \top, \quad f(\top) = \top$$

Obviously, f is inflationary but non-isotonic, and has a least fixed point (the element c); however, the successive iterations from $\bot$ lead to the fixed point $\top$.

We will assume in the rest of the paper that our underlying multilattices are complete, coherent and without infinite antichains.

2.2 Ideals of multilattices

As stated in the introduction, we are interested in studying the set of ideals of a multilattice, so that we can define an ideal-semantics as in [12]; however, the definition of ideal in a multilattice is not canonical. For instance, one can find the notion of s-ideals introduced by Rachůnek, or the l-ideals of Burgess, or the m-ideals given by Johnston [10,17]. In this section, we study the differences between the various definitions and propose some new alternatives.

To begin with, let us recall the definition of ideal of a lattice:

Definition 12 *A nonempty subset D of a lattice L is said to be an ideal of L if it is downward closed and for all $a, b \in D$ we have that $a \vee b \in D$.*

For a multilattice, as stated above, at least three extensions of the concept of ideal can be found.

Definition 13 *Given a multilattice M and a non-empty subset D of M, we say that D is:*

- *An* s-ideal *if and only if it is downward closed and for every $a, b \in D$ we have that $UB(\{a, b\}) \cap D \neq \varnothing$.*
- *An* l-ideal *if and only if for every $a, b \in D$ we have that $LB(UB(\{a, b\})) \subseteq D$.*
- *An* m-ideal *if and only if for every $a, b \in D$ such that* $\sup\{a, b\}$ *exists we have that $LB(\sup\{a, b\}) \subseteq D$.*

It is not difficult to check that, in the particular case of a lattice, all definitions above collapse to the usual definition of ideal of a lattice. Moreover, for the general case of a multilattice M, if $\mathfrak{I}_\alpha(M)$ denotes the set of α-ideals, where $\alpha \in \{l, m, s\}$, we clearly have

$$\mathfrak{I}_s(M) \subseteq \mathfrak{I}_l(M) \subseteq \mathfrak{I}_m(M). \tag{1}$$

Recall the following interesting relationship between sublattices and ideals of a lattice, see [2, 8].

Lemma 14 *Let A be a sublattice of a lattice L, then A is an ideal of L if and only if for every $a \in A$ and $x \in L$ we have that $a \wedge x \in A$.*

In order to obtain a similar result for multilattices, we introduce the definition of submultilattice of a multilattice M.

Definition 15 *Let M be a multilattice and $\varnothing \neq B \subseteq M$, we say that B is a* submultilattice *of M if it inherits the multilattice structure when using the restriction of the multisuprema and multinfima to B.*

Notice that there are two reasonable possibilities of considering the restriction of the operators multisup and multinf:

- A submultilattice B of M is said to be *full* (or f-submultilattice) if for every $a, b \in B$ we have that all the multisuprema and multinfima of $\{a, b\}$ in M are in B.
- A submultilattice B of M is said to be *restricted* (or r-submultilattice) if for every $a, b \in B$ we have that at least one multisupremum and one multinfimum of $\{a, b\}$ in M is in B.

It is obvious that every f-submultilattice is a r-submultilattice, but not vice versa. The following lemma gives us a relationship between (f- or) r-submultilattices and each type of ideal:

Lemma 16 *Let B be a (f- or) r-submultilattice of multilattice M then, B is an (s- or l- or) m-ideal if an only if for every $b \in B$ and $x \in M$ we have that* $\mathrm{multinf}\{b, x\} \subseteq B$.

PROOF. Let B be a r-submultilattice B, for the case of f-submultilattices the proof is equal.

Firstly, we suppose that B is an (s- or l- or) m-ideal then the property is clearly satisfied since every such ideal is downward closed.

Reciprocally, by the chain of inclusions $\mathfrak{I}_s(M) \subseteq \mathfrak{I}_l(M) \subseteq \mathfrak{I}_m(M)$, it is enough to prove that B is an s-ideal, that is, B is downward closed and for every $a, b \in B$ we have that $UB(\{a, b\}) \cap B \neq \varnothing$.

It is straightforward that B is downward closed, since for $b \in B$, and every $x \in M$ such that $x \leq b$ we have $\mathrm{multinf}\{b, x\} = \{x\} \subseteq B$ by our hypothesis.

Now, let $a, b \in B$, as B is a r-submultilattice, we have that at least an element c of $\mathrm{multisup}\{a, b\}$ is in B, hence $c \in UB(\{a, b\}) \cap B \neq \varnothing$. □

From the previous result one could be tempted to say that all three types of ideals for multilattices coincide. This is not true, as the inclusions in (1) are, in general, strict:

- In $M6$ we have that $\{\bot, a, b\}$ is an l-ideal but it is not an s-ideal.
- In the multilattice of Fig. 4 we have that $\{\bot, a, b\}$ is an m-ideal but it is not an l-ideal.

An interesting consequence of Lemma 16 is that, in difference with what happens with lattices, not every l-ideal or m-ideal is necessarily a (f- or) r-submultilattice, otherwise all three types of ideals would coincide. However, every s-ideal of a coherent multilattice is a r-submultilattice but not necessarily a f-submultilattice.

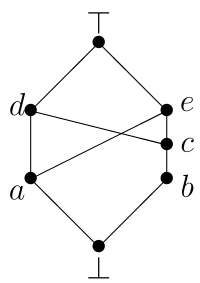


Fig. 4.

Lemma 17 *Let D be a s-ideal of a coherent multilattice M, then D is a r-submultilattice.*

PROOF. Given $a, b \in D$, we must prove that $\text{multisup}\{a, b\} \cap D \neq \varnothing$ and $\text{multinf}\{a, b\} \cap D \neq \varnothing$.

The last one is trivial from the downward closed property. For the other one, as D is a s-ideal, we consider $c \in UB(\{a, b\}) \cap D$ then, from the coherent of M, there exists $d \in \text{multisup}(\{a, b\})$ such that $d \leq c$ and, as D is downward closed and $c \in D$, we obtain that $d \in \text{multisup}(\{a, b\}) \cap D$. $\square$

In the following example we show that the result above does not hold if the initial multilattice is not coherent.

Example 18 *In the non-coherent multilattice shown previously in Fig. 2, the subset $D = \{\bot, a, b, c_1, c_2, c_3, \ldots\}$ is an s-ideal, but we can check easily that $\text{multisup}\{a, b\} \cap D = \varnothing$, hence it is not a submultilattice.*

Example 19 *An s-ideal is not always a f-submultilattice, for example in the (coherent) multilattice $M6$ we have that $\{\bot, a, b, c\}$ is an s-ideal but not a f-submultilattice.*

Another interesting property on the framework of ideals of lattices is the following result, which relates the kernel of a join-homomorphism (the inverse image of $\bot$) with ideals.

Theorem 20 (Birkhoff [2]) *If $\Phi \colon L_1 \longrightarrow L_2$ is an join-homomorphism between lattices then $\ker(\Phi)$ is an ideal.*

In the statement, a join-homomorphism between lattices is an application which preserves joins. The extension to the framework of multilattices is the following:

Definition 21 *Let M_1 and M_2 be multilattices; a mapping $\Phi \colon M_1 \longrightarrow M_2$ is said to be a multisup-homomorphism if for every $B \subseteq M_1$ and $b \in \text{multisup}(B)$,* *we have that* $\Phi(b) \in \text{multisup}(\Phi(B))$.

With these definitions we can translate the theorem above to multilattices.

Theorem 22 *If* $\Phi\colon M_1 \longrightarrow M_2$ *is a multisup-homomorphism between multilattices, then* $\ker(\Phi)$ *is an (s- or l- or) m-ideal.*

PROOF. By the chain of inclusions (1) we only have to prove the result for s-ideals. Let $a, b \in \ker(\Phi)$ and let us prove that $UB(\{a,b\}) \cap \ker(\Phi) \neq \varnothing$.

As $a, b \in \ker(\Phi)$ we have that $\Phi(a) = \Phi(b) = \bot$ so $\text{multisup}\{\Phi(a), \Phi(b)\} = \bot$. Since Φ is a multisup-homomorphism we have that if $x \in \text{multisup}\{a, b\}$, then

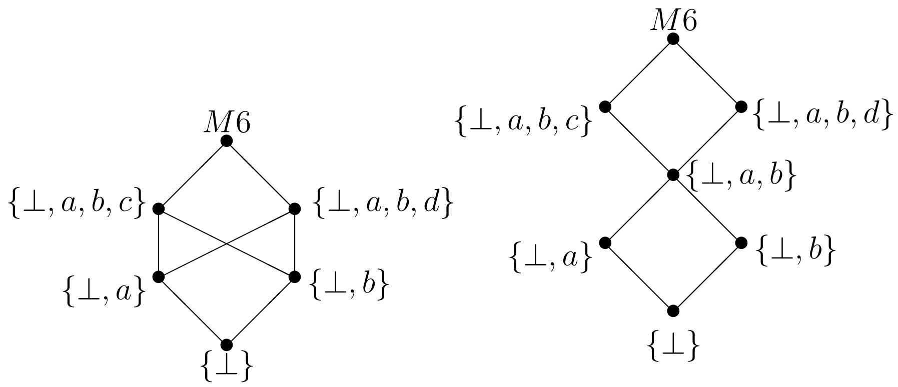
$$\Phi(x) \in \text{multisup}\{\Phi(a), \Phi(b)\} = \bot$$

hence $x \in \ker(\Phi)$ and therefore $x \in UB(\{a,b\}) \cap \ker(\Phi)$. □

Now that we have shown that the proposed definitions of ideals for a multilattice are friendly to most of the properties of ideals of a lattice, let us concentrate now on the algebraic structure of the sets of every type of ideal.

Assuming that the set of ideals $\mathfrak{I}_\alpha(M)$ is ordered by set-inclusion, then it is easy to check that $\mathfrak{I}_l(M)$ and $\mathfrak{I}_m(M)$ are complete lattices. On the other hand, $\mathfrak{I}_s(M)$ is a complete multilattice (provided that M is complete and infinite antichains do not exist) [10].

Example 23 *For M6, the complete multilattice of its s-ideals is depicted on the left of the picture below, and turns out to be isomorphic to M6. The complete lattices of l-ideal and m-ideals coincide, since in this case there is no difference between l- and m- ideals, and the result is depicted on the right part of the picture:*



That the set of s-ideals of M6 coincides with M6 is not just a coincidence, but an instance of the following result:

Proposition 24 *Let M be a finite multilattice, then M and $\mathfrak{I}_s(M)$ are order-isomorphic, i.e. there exists a bijective mapping which is both order-preserving and order-reflecting.*

PROOF. Let us define an order-isomorphism $\Phi\colon M \longrightarrow \mathfrak{I}_s(M)$. Given an element $a \in M$, define $\Phi(a) = D_a = \{x \in M \mid x \leq a\}$. It is not difficult to check that D_a is the least s-ideal containing a, so Φ is well-defined.

Let us prove that Φ is bijective: If $a \neq b$ it is clear that $D_a \neq D_b$, hence Φ is one-to-one; on the other hand, to prove that Φ is onto it suffices to check that any s-ideal in M has a maximum.

As $D \subseteq M$ is finite, it has maximal elements. Consider two maximal elements c and d then, by definition of s-ideal, there exists $e \in UB(\{c, d\}) \cap D$. This means that $c, d \leq e$, by maximality of c and d we would have $c = e = d$, as a result D is a finite set with just one maximal element, which is its maximum. $\square$

3 Extended logic programs

In this section we provide a first approximation of the definition of an extended logic programming paradigm in which the underlying set of truth-values is assumed to have structure of multilattice. The proposed schema is an extension of the monotonic and residuated logic programs of [6]. The definition of logic program is given, as usual, as a set of rules and facts.

Definition 25 *An* extended logic program *is a set $\mathbb{P}$ of rules of the form $A \leftarrow \mathcal{B}$ such that:*

(1) A is a propositional symbol of Π, and
(2) $\mathcal{B}$ is a formula of $\mathfrak{F}$ built from propositional symbols and elements of M by using isotone operators.

The structure of complete lattice of some sets of ideals enables us to provide an ideal-based fixpoint semantics for extended programs in terms of the Knaster-Tarski theorem. This will be done in the following section.

3.1 Ideal fix-point semantics

We will consider α-ideals (where α denotes either l or m, since $\mathfrak{I}_l(M)$ and $\mathfrak{I}_m(M)$ form a complete lattice), and our interpretations will attach an ideal to any propositional symbol:

Definition 26 *An* interpretation *is a mapping* $I\colon \Pi \to \mathfrak{I}_\alpha(M)$*. The set of all interpretations is denoted* $\mathcal{I}$*.*

The ordering $\leq$ of the truth-values M can be extended point-wise to the set of interpretations $\mathcal{I}$ as usual; and also endows $\mathcal{I}$ with a complete lattice structure.

Now, at this point we cannot apply the homomorphic extension theorem in order to extend the definition of interpretation to any formula occurring in the body of our rules[2] as usual, since our connective operators are usually interpreted as functions in the multilattice M, not between ideals. Thus, we explicitly define how to extend a given interpretation to any body-formula:

- Given I and $a \in M$, we define $\hat{I}(a)$ as the least α-ideal containing a.
- Given I, atoms $A_1, \dots, A_n$, and any isotone n-ary function @ we define $\hat{I}(@(A_1, \dots, A_n))$ as

$$\biguplus \Big\{ @(a_1, \dots, a_n) \mid a_i \in I(A_i), i \in \{1, \dots, n\} \Big\}$$

where $\biguplus$ denotes the least α-ideal generated by a subset.

Note that, with the definition above, @ can be seen as an operator in $\mathfrak{I}_\alpha$ sending the ideals $I(A_1), \dots, I(A_n)$ to the ideal defined above.

Remark 27 *It is worth to recall that, as a consequence of* $\mathfrak{I}_s(M)$ *not being a lattice, the operator* $\biguplus$ *is not defined for s-ideals: for instance, in M6 there is not a least ideal containing* $\{a, b\}$*, but two minimal ideals.*

A rule of an extended logic program is satisfied whenever the truth-value of the head of the rule is greater or equal than the truth-value of its body. Formally:

Definition 28 *Given an interpretation* I*, a rule* $A \leftarrow \mathcal{B}$ *is* satisfied *by* I *if and only if* $\hat{I}(\mathcal{B}) \subseteq I(A)$*. An interpretation* I *is said to be a* model *of an extended logic program* $\mathbb{P}$ *if and only if all rules in* $\mathbb{P}$ *are satisfied by* I*, then we write* $I \models \mathbb{P}$*.*

After defining the (two) model semantics based on the (l- and m-) ideals of a multilattice, we can proceed with the development of a fixpoint semantics and study possible similarities with the ideal fixed semantics based on lattices.

The usual definition of immediate consequences operator $T_{\mathbb{P}}$, can be translated to the context of our ideal-based semantics as follows:

Definition 29 *Given an extended logic program* $\mathbb{P}$*, an interpretation* I *and a propositional symbol* A*; the immediate operator of consequences* $T_{\mathbb{P}}$ *is defined*

[2] We will refer to these formulas as *body-formulas*.

about I and A as

$$T_{\mathbb{P}}(I)(A) = \biguplus \left(\bigcup_{A \leftarrow \mathcal{B} \in \mathbb{P}} \hat{I}(\mathcal{B}) \right)$$

As we are working on a lattice structure we can apply the results of [16], and obtain that:

(1) $T_{\mathbb{P}}$ is an isotone operator.
(2) An interpretation I is a model if and only if $T_{\mathbb{P}}(I) \leq I$.
(3) $T_{\mathbb{P}}$ is a continuous operator if and only if all the operators @ involved in the bodies are continuous (as operators in $\mathfrak{I}_\alpha$).
(4) If $T_{\mathbb{P}}$ is continuous, then it has a least fixed point that can be attained after at most ω iterations.

Obviously, the least ideal-based model for a program might be different depending on whether we are using either l-ideals or m-ideals.

Example 30 *Let us consider in Figure 4 the following program:*

$A \leftarrow a \qquad A \leftarrow b$

We have that the least m-ideal is $I_m(A) = \{\bot, a, b\}$, but its least l-ideal is $I_l(A) = \{\bot, a, b, c\}$.

The previous example shows that the use of different type of ideals can lead to different least models. Anyway, by the chain of inclusions (1), it is always the case that the least model provided by the m-ideals is smaller or equal than that provided by the s-ideals.

The question of choosing one type of ideal or the other seems to be conditioned by the particular application in mind; however, in order to provide some theoretical support for helping a suitable choice of the type of ideal to work with, it seems reasonable to develop an alternative semantics directly based on the underlying multilattice, and study its relationship with the ideal-based semantics.

3.2 Restricted fix-point semantics

In this section, we aim at providing a semantics directly based on the underlying multilattice of truth-values. This means that the definition of interpretation has to be modified in that an interpretation will be considered as an assignment of truth-values to every propositional symbol in the language.

Definition 31 *A (restricted)* interpretation *is a mapping $I\colon \Pi \to M$. The set* *of all interpretations is denoted* $\mathcal{I}$.

Note that by the unique homomorphic extension theorem, any interpretation I can be uniquely extended to the whole set of formulas (the extension will be denoted as $\hat{I}$). The ordering $\leq$ of the truth-values M can be extended point-wise to the set of interpretations as usual.

A rule of an extended logic program is satisfied whenever the truth-value of the head of the rule is greater or equal than the truth-value of its body. Formally:

Definition 32 *Given an interpretation I, a rule $A \leftarrow \mathcal{B}$ is* satisfied *by I iff $\hat{I}(\mathcal{B}) \leq I(A)$. An interpretation I is said to be a* model *of an extended logic program $\mathbb{P}$ iff all rules in $\mathbb{P}$ are satisfied by I, then we write $I \models \mathbb{P}$.*

Example 33 *Let us consider the following program on the multilattice M6:*

$$E \leftarrow A \qquad E \leftarrow B \qquad A \leftarrow a \qquad B \leftarrow b$$

It is easy to check that the interpretation I defined as $I(E) = \top$, $I(A) = a$, $I(B) = b$ is a model of the program.

Every extended program $\mathbb{P}$ has the top interpretation $\triangledown$ as a model; regarding minimal models, it is possible to prove the following technical lemma.

Lemma 34 *A chain of models $\{I_k\}_{k \in K}$ of $\mathbb{P}$ has an infimum in $\mathcal{I}$ which is a model of $\mathbb{P}$.*

PROOF. Given a propositional symbol A, the existence of $\inf_k\{I_k(A)\}$ is guaranteed by Lemma 8, thus we can safely define an interpretation I_ω as follows:

$$I_\omega(A) = \inf_{k \in K}\{I_k(A)\}$$

Now, let us show that I_ω is a model of $\mathbb{P}$:

Given a rule $A \leftarrow @[B_1, \ldots, B_n]$ in $\mathbb{P}$, where @ denotes the composition of the operators occurring in the body of the rule, and the B_i's are the variables occurring in it; by isotonicity of @ we obtain the following chain of inequalities for all $i \in K$:

$$\hat{I}_i(\mathcal{B}) = @[I_i(B_1), ..., I_i(B_n)] \geq @\left[\inf_{k \in K}\{I_k(B_1)\}, ..., \inf_{k \in K}\{I_k(B_n)\}\right] = \hat{I}_\omega(\mathcal{B})$$

As I_i is a model for all i we obtain:

$$I_i(A) \geq \hat{I}_i(\mathcal{B}) \geq \hat{I}_\omega(\mathcal{B})$$

thus, by definition of infimum, we have

$$I_\omega(A) = \inf_{k \in K}\{I_k(A)\} \geq \hat{I}_\omega(\mathcal{B})$$

so I_ω is a model of $\mathbb{P}$. □

Theorem 35 *There exist minimal models for any extended logic program $\mathbb{P}$. Moreover, if I is a model of $\mathbb{P}$ then there exists a minimal model of $\mathbb{P}$, m, such that $m \sqsubseteq I$.*

PROOF. Let $\mathcal{M}$ be the set of models of $\mathbb{P}$. By Zorn's lemma, we only have to prove that any chain in $\mathcal{M}$ is lower bounded, but this follows from the previous lemma since the infimum of a chain of models is also a model.

For the second result consider I to be a non-minimal model, then there exists another model I_1 such that $I_1 \sqsubseteq I$. If I_1 is minimal we have finished, otherwise there exists a model I_2 such that $I_2 \sqsubseteq I_1 \sqsubseteq I$, and so on. Following this process we build a chain of models, therefore by Kuratowski's lemma (a chain of elements of a poset is contained in a maximal chain) on the poset of models, there should exist a maximal chain of models C containing our chain, but by Lemma 34 we have that C has an infimum m which is a model but C was maximal so m is a minimal model and $m \sqsubseteq I$. □

Example 36 *Continuing with the program in the previous example, it is easy to check that the program does not have a minimum model but two minimal ones:*

$$\begin{array}{ll} I_1(E) = c & I_2(E) = d \\ I_1(A) = a & I_2(A) = a \\ I_1(B) = b & I_2(B) = b \end{array}$$

An interesting technical problem arises when trying to extend the definition of the immediate consequences operators to the framework of multilattice-based logic programs. One of the several possible approaches to provide a fixed point semantics for the extended logic programs is presented and analysed.

The main theoretical tool for the study of the fixed point semantics of programming languages is Knaster-Tarski theorem in some of its constructive versions, although some other fixed point theorems are also of use, see [11].

Given a logic program $\mathbb{P}$ valued on a *lattice*, the operator $T_\mathbb{P}: \mathcal{I} \to \mathcal{I}$, maps interpretations to interpretations, and can be defined by considering

$$T_\mathbb{P}(I)(A) = \sup\{\hat{I}(\mathcal{B}) \mid A \leftarrow \mathcal{B} \in \mathbb{P}\}$$

Note that all the suprema involved in the definition do exist provided that we are assuming a complete lattice structure on the underlying set of truth-values; however, this needs not hold for a multilattice.

In order to work this problem out, we consider the following definition

Definition 37 *Given an extended logic program $\mathbb{P}$, an interpretation I and a propositional symbol A; we can define*

$$T_\mathbb{P}(I)(A) = \text{multisup}\Big(\{I(A)\} \cup \{\hat{I}(\mathcal{B}) \mid A \leftarrow \mathcal{B} \in \mathbb{P}\}\Big)$$

Some properties of this definition of the $T_\mathbb{P}$ operator are stated below, where $\sqsubseteq_S$ denotes the Smyth-ordering between subsets of a poset:

Lemma 38 *If $I \sqsubseteq J$, then $T_\mathbb{P}(I)(A) \sqsubseteq_S T_\mathbb{P}(J)(A)$ for all propositional symbol A.*

PROOF. Let us write X_I to denote the set $\{I(A)\} \cup \{\hat{I}(\mathcal{B}) \mid A \leftarrow \mathcal{B} \in \mathbb{P}\}$, then the hypothesis states that $X_J^\uparrow \subseteq X_I^\uparrow$, where the $\uparrow$ denotes the upwards-closure of a set. Now, consider $b \in T_\mathbb{P}(J)(A)$, then b is an element of $X_J^\uparrow \subseteq X_I^\uparrow$; thus, by consistency, considering any minimal a of $X_I^\uparrow$ below b leads to the existence of an element $a \in T_\mathbb{P}(I)(A)$. □

The definition of $T_\mathbb{P}$ proposed above generates some coherence problems, in that the resulting 'value' is not an element, but a subset of the multilattice. A possible solution to this problem would be to consider a *choice function* $()^*$ which, given an interpretation, for any propositional symbol A selects an element in $T_\mathbb{P}(I)(A)$; this way, $T_\mathbb{P}(I)^*$ represents actually an interpretation.

Regarding particular properties of the composition of the $T_\mathbb{P}$ operator with suitable choice functions, the first property one can obtain, directly from the definition, is that the composition leads to an inflationary operator.

Lemma 39 *Given an interpretation I and a choice function $()^*$, then $I(A) \leq T_\mathbb{P}(I)^*(A)$ for all propositional symbol A.*

PROOF. Straightforward. □

Note that, for some choice functions, the resulting operator ${T_{\mathbb{P}}}^*$ might not be monotone in the set of interpretations, since it can lead to incomparable interpretations; the multilattice M6 of Fig. 1 can be used to construct a counter-example.

Example 40 *Consider the following program with just two facts*

$$A \leftarrow a \qquad A \leftarrow b$$

and consider interpretations $I(A) = \bot$ *and* $J(A) = c$*; obviously* $I \sqsubseteq J$*.*

Now, we have that $T_{\mathbb{P}}(I)(A) = \{c, d\}$ *and* $T_{\mathbb{P}}(J)(A) = \{c\}$*. Thus, the choice function* $()^*$ *which selects* d *in* $T_{\mathbb{P}}(I)(A)$ *generates incomparable interpretations* $T_{\mathbb{P}}(I)^*$ *and* $T_{\mathbb{P}}(J)^*$*.*

We are interested in computing models of our extended programs by successive iteration of ${T_{\mathbb{P}}}^*$. Therefore, we should characterise the models of $\mathbb{P}$ in terms $T_{\mathbb{P}}$. The following result, which characterises the models of our extended programs in terms of properties of $T_{\mathbb{P}}$, can be proved:

Proposition 41 *The four statements below are equivalent:*

(1) I *is a model of* $\mathbb{P}$*.*
(2) $T_{\mathbb{P}}(I)(A) = \{I(A)\}$ *for all* $A \in \Pi$*.*
(3) $T_{\mathbb{P}}(I)^* = I$ *for all choice function.*
(4) $I \in T_{\mathbb{P}}(I)$*,*[3] *(i.e. I is a fixed point of* $T_{\mathbb{P}}$ *as a non-deterministic operator).*

PROOF.
$(1 \Rightarrow 2)$. Let us assume that I is a model of $\mathbb{P}$; then, we have that $I(A) \geq \hat{I}(\mathcal{B})$ for all rule $A \leftarrow \mathcal{B} \in \mathbb{P}$. This implies that $I(A)$ is the maximum of the set

$$\{I(A)\} \cup \{\hat{I}(\mathcal{B}) \mid A \leftarrow \mathcal{B} \in \mathbb{P}\}$$

hence, the only multi-supremum.

$(2 \Rightarrow 1)$. The hypothesis implies that $I(A)$ is an upper bound of

$$\{\hat{I}(\mathcal{B}) \mid A \leftarrow \mathcal{B} \in \mathbb{P}\}$$

as a result, $I(A) \geq \hat{I}(\mathcal{B})$ for all rule $A \leftarrow \mathcal{B} \in \mathbb{P}$ and $I \models \mathbb{P}$.

[3] Abusing notation this means that $I(A) \in T_{\mathbb{P}}(I)(A)$ for all $A \in \Pi$.

$(2 \Leftrightarrow 3 \Leftrightarrow 4)$. Trivial. $\square$

Regarding the iterated application of the $T_{\mathbb{P}}$ operator, the use of choice functions is essential. Let us consider a model I, that is, a fixed point of $T_{\mathbb{P}}$, then for all propositional variable A, we have that $T_{\mathbb{P}}(I)(A) = \{I(A)\}$. Lemma 38 guides us in the choice after each application of $T_{\mathbb{P}}$ as follows:

- For the base case, we have[4] $\triangle \sqsubseteq I$, then $T_{\mathbb{P}}(\triangle)(A) \sqsubseteq_S T_{\mathbb{P}}(I)(A) = \{I(A)\}$. This means that there exists an element $m_1(A) \in T_{\mathbb{P}}(\triangle)(A)$ such that

  $$m_1(A) \leq I(A)$$

  This way we obtain an interpretation m_1 satisfying $m_1 \sqsubseteq I$ such that for any propositional variable A, $m_1(A)$ is an element of $T_{\mathbb{P}}(\triangle)(A)$.
- This argument applies also to any successor ordinal: given $m_k \sqsubseteq I$, there exists an element $m_{k+1}(A) \in T_{\mathbb{P}}(m_k)(A)$ such that

  $$m_k(A) \leq m_{k+1}(A) \leq I(A)$$

  where the first inequality holds by the definition of $T_{\mathbb{P}}$ and the second inequality follows from Lemma 38.
- For a limit ordinal α, Lemma 8 states that for all A the increasing sequence $\{m_n(A)\}$ has a supremum, which is considered, by definition, to be $m_\alpha(A)$.

As a result of the discussion above we obtain that we can choose suitable elements in the sets generated by the application of $T_{\mathbb{P}}$ in such a way that we can construct a transfinite sequence of interpretations m_k satisfying

$$m_1 \sqsubseteq m_2 \sqsubseteq \cdots \sqsubseteq m_k \sqsubseteq \cdots \sqsubseteq I$$

Note that the sequence of interpretations above, can be interpreted as the Kleene sequence which allows to reach the least fixed point of $T_{\mathbb{P}}$ in the classical case.

Interestingly enough, if I is a minimal model of $\mathbb{P}$, the previous sequence of interpretations can be proved to converge to I.

Theorem 42 *Let I be a minimal model of $\mathbb{P}$, then the previous construction leads to a Kleene-like sequence $\{m_\lambda\}$ which converges to I.*

PROOF. A cardinality-based argument suffices to show that $\{m_\lambda\}$ is eventually constant and equal to I:

[4] Here, as usual, $\triangle$ denotes the minimum interpretation.

Let β be the least ordinal greater than the cardinal of the set of interpretations, for all $\lambda < \beta$ we can consider the interpretation m_λ and, thus, define the map $h: \beta \longrightarrow \mathcal{I}$ assigning m_λ to λ.

If the transfinite sequence were strictly increasing, then h would be injective, obtaining a contradiction with the choice of β. As a result, we have proved the existence of an ordinal α such that $m_\alpha = m_{\alpha+1}$.

Recall that, by definition, we have $m_\alpha \sqsubseteq I$ and $m_{\alpha+1} \in T_{\mathbb{P}}(m_\alpha)$, therefore $m_\alpha \in T_{\mathbb{P}}(m_\alpha)$ and, by Proposition 41, m_α is a model of $\mathbb{P}$. By minimality of I we have that $m_\alpha = I$. □

Example 43 *Continuing with Example 36, let us consider the minimal model I_1, and let us construct a sequence of approximating interpretations as stated in the theorem above.*

	$\triangle$	$T_{\mathbb{P}}(\triangle)$	m_1	$T_{\mathbb{P}}(m_1)$	m_2
A	$\perp$	$\{a\}$	a	$\{a\}$	a
B	$\perp$	$\{b\}$	b	$\{b\}$	b
E	$\perp$	$\{\perp\}$	$\perp$	$\{c, d\}$	c

4 Conclusions and future work

A mathematical study of the theory multilattices, including the selection of desirable computational properties of our multilattices, and its theory of ideals has been presented, with the aim of providing an extension of fuzzy logic programming. Ideal-based semantics and the restricted semantics, both based on the fixed points of the immediate consequence operator have been introduced. We have obtained some initial and encouraging results for the restricted semantics: specifically, the existence of minimal models below any model of an extended program, and that any minimal model can be attained by some Kleene-like sequence.

A number of theoretical problems still remain to be investigated in the future: for instance, the relationship between the ideal-based semantics and the restricted semantics, or the constructive nature of minimal models (is it possible to construct suitable choice functions which generate convergent sequence of interpretations with limit a minimal model?). Possible answers could be based on a general theory of fixed points, relying on some of the ideas related to fixed points in partially ordered sets [18] or, perhaps, in fuzzy extensions of Tarski's theorem [20].

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