

On Multi-Adjoint Concept Lattices Based on Heterogeneous Conjunctions

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Abstract

Sets of attributes and objects in fuzzy formal concept analysis are usually different and, hence, it might not make sense to evaluate them on the same carrier. In this context, the operators used to obtain the concept lattice could be defined by associating different lattices to attributes and objects; several reasons exist for which we need to evaluate the sets of attributes and objects in the same carrier. Following this direction, we introduce a new definition of a concept lattice, where objects and attributes are evaluated on the same lattice L , although operators evaluating objects and attributes in different carriers are used. Moreover, we study the relationship between this new concept lattice and the alternative one which can be obtained directly by using different carriers for the sets of attributes and objects.

Key words: Formal concept analysis, multi-adjoint lattices, Galois connection.

1. Introduction

Formal concept analysis, introduced by Wille in the decade of 1980 [37], arose as a mathematical theory for qualitative data analysis and, currently, has become an interesting research topic both on its mathematical foundations [34, 23, 18, 35] and on its multiple applications [21, 30, 3, 16, 15].

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The initial, Boolean, approach was soon extended following ideas coming from different frameworks: fuzzy set theory [4, 9, 31, 26]; possibility theory [13]; fuzzy logic reasoning [5, 14, 2]; from rough set theory [25, 32, 36]; some integrated approaches such as fuzzy and rough [33], or rough and domain theory [22], or rough and grey sets [38].

Recently, a new fuzzy framework has been introduced which is more general and flexible than other fuzzy extensions, see [28]. One of its most interesting features is that it allows for considering extremely general conjunctors, which need not be either commutative or associative, to build formal concepts.

In the topic of fuzzy concept analysis it is reasonable to consider the particular nature of objects and attributes and, hence, the fuzzy subsets corresponding to either of them should be evaluated in different underlying lattices. In spite of the usefulness of considering different carrier sets for objects and attributes, it is convenient to recall that, sometimes, it could be interesting to “soften” this framework. For instance, given two experts who are consulted in order to evaluate a knowledge system, they could believe that the carriers associated to the set of objects and attributes should not be different, or some of them believe that the attributes should be evaluated on L_1 and some others believe that they should be evaluated on L_2 and, once the evaluation is finished, the results should be homogenized. An elegant solution could be to embed both L_1 and L_2 into a new set L , and to obtain a new concept lattice \mathfrak{M}_L , by considering the sets of objects and attributes evaluated in the *same* lattice, albeit using the operators which evaluate objects and attributes in different carriers.

This work introduces the notion of L -connected lattices, with the aim of embedding two given lattices L_1 and L_2 , into a third one L , satisfying certain properties. Using this notion, it is possible to obtain formal concepts which are not directly affected by the underlying evaluation of objects and attributes on two different carriers. The idea is to evaluate the fuzzy sets for extension and intension of a formal concept in the lattice L , which embeds both original lattices L_1 and L_2 . In this paper, we obtain several results which prove the coherence of this embedding, in the sense that the original concept lattice and the homogenized concept lattice are isomorphic; as a result, we obtain that the proposed transformation between both frameworks is sound. An interesting by-product of this approach is the possibility of using suitable modifications of the different algorithms recently developed for obtaining concept lattices on specific contexts in which objects and at-

tributed have a common carrier [7, 8, 24]. Last but not least, it worth to remark that the structures introduced in this paper generalize the ones given in the framework of fuzzy concept lattices with hedges [11, 10, 19].

2. P -connected posets

The main notion in this section is the definition of P -connection between two complete lattices. As we will see later, this condition allows for somehow conciliating the different values generated when considering a non-commutative conjunctive in the construction of a concept lattice.

Definition 1. Given the posets (P_1, \leq_1) , (P_2, \leq_2) and (P, \leq) , we say that P_1 and P_2 are P -connected if there exist non-decreasing mappings $\psi_1: P_1 \rightarrow P$, $\phi_1: P \rightarrow P_1$, $\psi_2: P_2 \rightarrow P$ and $\phi_2: P \rightarrow P_2$ verifying that $\phi_1(\psi_1(x)) = x$, and $\phi_2(\psi_2(y)) = y$, for all $x \in P_1$, $y \in P_2$.

Note that this definition generalizes the well-known approach of concept lattices with hedges. To see this, we can simply use Krajčí's approach [19] to prove that concept lattices with hedges were a particular case of generalized concepts lattices: in our case L_1 and L_2 are the fixpoints of the two hedges, ϕ_1 are the hedge for objects, ϕ_2 are the hedge for attributes, ψ_1 and ψ_2 be the identities.

Example 2. Any pair of posets (P_1, \leq_1) , (P_2, \leq_2) with top elements \top_1 and \top_2 , respectively, are $P_1 \times P_2$ -connected, where $P_1 \times P_2$ is the Cartesian product with the pairwise ordering, by considering the mappings ϕ_i as the projections π_i , and ψ_1 , ψ_2 as the inclusions defined as $\psi_1(x) = (x, \top_2)$, $\psi_2(y) = (\top_1, y)$, for all $x \in P_1$, $y \in P_2$. \square

A more complex example is presented below:

Example 3. Assume that, in order to perform the evaluation of a product, we have to assign one value out of four possible ones. We ask two experts to collaborate in this task and, only when collecting the feedback from each expert, we notice that one expert has considered the ordering of values as in Fig. 1, whereas the other has considered that in Fig. 2. In both cases, the expert chose a suitable poset for her evaluation.

In order to unify both evaluations, we want to embed the posets⁴ in Figs. 1 and 2 into another one, for example, we might consider that given in Fig. 3.

⁴Note that these posets are indeed lattices.

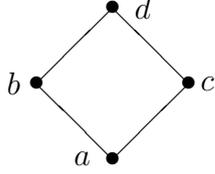


Figure 1: Poset (L_1, \preceq_1)



Figure 2: Poset (L_2, \preceq_2)

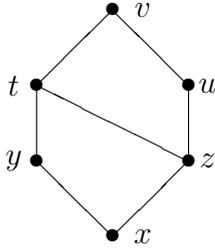


Figure 3: Poset (L, \preceq)

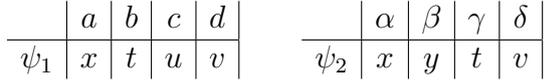


Figure 4: Definition of ψ_1 and ψ_2

We can define two mappings $\psi_1: L_1 \rightarrow L$, $\psi_2: L_2 \rightarrow L$ as in Fig. 4; moreover, there exist several possibilities for the mappings $\phi_1: L \rightarrow L_1$, $\phi_2: L \rightarrow L_2$ in order to satisfy the properties in Definition 1, one of them is shown below:



As a result, L_1 and L_2 are L -connected. □

Example 4. A different example arises considering the posets $([0, 1]_2, \leq)$ and $([0, 1]_4, \leq)$, where $[0, 1]_n$ is a regular partition of $[0, 1]$ into n pieces, for instance $[0, 1]_2 = \{0, 1/2, 1\}$, $[0, 1]_4 = \{0, 1/4, 2/4, 3/4, 1\}$.

We have that $[0, 1]_2, [0, 1]_4$ are $[0, 1]$ -connected, under the usual ordering, considering the mappings ψ_1, ψ_2 as the inclusions $\psi_1(x) = x, \psi_2(y) = y$, for all $x \in L_1, y \in L_2$; and ϕ_1, ϕ_2 defined as $\phi_1(t) = \lceil 2 \cdot t \rceil / 2, \phi_2(t) = \lceil 4 \cdot t \rceil / 4$, where $\lceil _ \rceil$ is the ceiling function. For example, if $t = 0.55$, $\phi_1(0.55) = 1, \phi_2(0.55) = 3/4$. □

3. L -connected lattices and formal concept analysis

To begin with, in order to define a new concept lattice where the objects and attributes are evaluated on the same lattice L , we will recall the definition of adjoint triple, multi-adjoint frame and context, and the multi-adjoint concept lattice.

3.1. Recalling the basics of the theory of adjoint triples

Assuming a conjunctive defined on the product $P_1 \times P_2$ directly provides two different ways of generalising the well-known adjoint property between a t-norm and its residuated implication [1, 29], depending on which argument is fixed.

Definition 5. Let (P_1, \leq_1) , (P_2, \leq_2) , (P_3, \leq_3) be posets, and consider mappings $\&: P_1 \times P_2 \rightarrow P_3$, $\diagup: P_3 \times P_2 \rightarrow P_1$, $\diagdown: P_3 \times P_1 \rightarrow P_2$, then $(\&, \diagup, \diagdown)$ is an *adjoint triple* with respect to P_1, P_2, P_3 if:

1. $\&$ is order-preserving in both arguments.
2. \diagup and \diagdown are order-preserving in the consequent and order-reversing in the antecedent.
3. $x \leq_1 z \diagup y$ iff $x \& y \leq_3 z$ iff $y \leq_2 z \diagdown x$, where $x \in P_1$, $y \in P_2$ and $z \in P_3$.

As mentioned in [6], the definition above may be shortened (in fact, condition 3 is sufficient), but we prefer to stress on the two first conditions since they will be often used hereafter.

More general examples than the classical Gödel, product and Łukasiewicz connectives were introduced in [28], where non-commutative and non-associative conjunctives were considered on regular partitions of $[0, 1]$ together with their corresponding adjoint implications. In the following example, an adjoint triple will be presented with respect to the posets (L_1, \preceq_1) , (L_2, \preceq_2) , given in Example 3, and the unit interval $[0, 1]$, with the classical order.

Example 6. Let (L_1, \preceq_1) , (L_2, \preceq_2) be the lattices given in Example 3 and the operator $\&: L_1 \times L_2 \rightarrow [0, 1]$, defined in Table 1.

This operator has two adjoint implications $\diagup: [0, 1] \times L_2 \rightarrow L_1$, $\diagdown: [0, 1] \times L_1 \rightarrow L_2$, which are defined in Table 2.

Either using the definition or the results given in [29], we obtain that the triple $(\&, \diagup, \diagdown)$ is an adjoint triple. \square

Table 1: Definition of $\&$

$\&$	α	β	γ	δ
a	0	0	0	0
b	0	0	$3/4$	$3/4$
c	0	$1/2$	$1/2$	1
d	0	$1/2$	$3/4$	1

Table 2: Definition of \swarrow and \nwarrow

\swarrow	α	β	γ	δ	\nwarrow	a	b	c	d
1	d	d	d	d	1	δ	δ	δ	δ
$[3/4, 1)$	d	d	d	b	$[3/4, 1)$	δ	δ	γ	γ
$[1/2, 3/4)$	d	d	c	a	$[1/2, 3/4)$	δ	β	γ	β
$[0, 1/2)$	d	b	a	a	$[0, 1/2)$	δ	β	α	α

The general theory of formal concept analysis requires the underlying posets to have the structure of a lattice. Therefore, we will assume hereafter that we are working on lattices instead of on posets.

The multi-adjoint framework allows the existence of several adjoint triples for a given triplet of lattices.

Definition 7. A *multi-adjoint frame* \mathcal{L} is a tuple

$$(L_1, \preceq_1, L_2, \preceq_2, P, \leq, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$$

where (L_1, \preceq_1) and (L_2, \preceq_2) are complete lattices, (P, \leq) is a poset and, for all $i \in \{1, \dots, n\}$, $(\&_i, \swarrow^i, \nwarrow_i)$ is an adjoint triple with respect to L_1, L_2, P . A multi-adjoint frame is denoted as $(L_1, L_2, P, \&_1, \dots, \&_n)$.

Definition 8. Let $(L_1, L_2, P, \&_1, \dots, \&_n)$ be a multi-adjoint frame, a *multi-adjoint context* is a tuple (A, B, R, σ) such that A and B are non-empty sets (usually interpreted as attributes and objects, respectively), R is a P -fuzzy relation $R: A \times B \rightarrow P$ and $\sigma: B \rightarrow \{1, \dots, n\}$ is a mapping which associates any element in B with some particular adjoint triple in the frame.⁵

⁵A similar theory could be developed by considering a mapping $\tau: A \rightarrow \{1, \dots, n\}$ which associates any element in A with some particular adjoint triple in the frame.

Once we have fixed a multi-adjoint frame and a context for that frame, we can define the following mappings $\uparrow^\sigma: L_2^B \rightarrow L_1^A$ and $\downarrow^\sigma: L_1^A \rightarrow L_2^B$ which can be seen as generalisations of those given in [20]:⁶

$$g^{\uparrow^\sigma}(a) = \inf\{R(a, b) \swarrow^{\sigma(b)} g(b) \mid b \in B\} \quad (1)$$

$$f^{\downarrow^\sigma}(b) = \inf\{R(a, b) \searrow_{\sigma(b)} f(a) \mid a \in A\} \quad (2)$$

It is not difficult to show that these two arrows generate a Galois connection [28], whose definition is recalled below:

Definition 9. Let (P_1, \leq_1) and (P_2, \leq_2) be posets, a pair (\uparrow, \downarrow) of mappings $\downarrow: P_1 \rightarrow P_2$, $\uparrow: P_2 \rightarrow P_1$ forms a *Galois connection* between P_1 and P_2 if and only if:

1. \uparrow and \downarrow are order-reversing.
2. $x \leq_1 x^{\downarrow\uparrow}$ for all $x \in P_1$.
3. $y \leq_2 y^{\uparrow\downarrow}$ for all $y \in P_2$.

A *multi-adjoint concept*, as it is often the case in the different frameworks of formal concept analysis, is a pair $\langle g, f \rangle$ satisfying that $g \in L_2^B$, $f \in L_1^A$ and that $g^{\uparrow^\sigma} = f$ and $f^{\downarrow^\sigma} = g$; with $(\uparrow^\sigma, \downarrow^\sigma)$ being the Galois connection defined above.

Definition 10. The *multi-adjoint concept lattice* associated to a multi-adjoint frame $(L_1, L_2, P, \&_1, \dots, \&_n)$ and a context (A, B, R, σ) is the set

$$\mathfrak{M} = \{\langle g, f \rangle \mid g \in L_2^B, f \in L_1^A \text{ and } g^{\uparrow^\sigma} = f, f^{\downarrow^\sigma} = g\}$$

in which, given $\langle g, f \rangle, \langle h, k \rangle \in \mathfrak{M}$, the ordering is defined by $\langle g, f \rangle \preceq \langle h, k \rangle$ if and only if $g \preceq_2 h$ (equivalently $k \preceq_1 f$).

In [28], a detailed construction of a multi-adjoint concept lattice was presented, and it was proved that the ordering just defined above actually provides \mathfrak{M} with the structure of a complete lattice.

In the following example a concept will be obtained from a fuzzy subset of objects using the adjoint triple of Example 6.

Example 11. Let (L_1, \preceq_1) , (L_2, \preceq_2) be the lattices given in Example 3, the adjoint triple $(\&, \swarrow, \searrow)$, where $\&: L_1 \times L_2 \rightarrow [0, 1]$, $\swarrow: [0, 1] \times L_2 \rightarrow L_1$ and $\searrow: [0, 1] \times L_1 \rightarrow L_2$ were introduced in Example 6, the frame $(L_1, L_2, [0, 1], \&)$ and the context (A, B, R, σ) , where σ is constant, $A =$

Table 3: Fuzzy relation between objects and attributes

R	b_1	b_2	b_3
a_1	0.8	0.6	0.1
a_2	0.2	0.3	0.9

$\{a_1, a_2\}$, $B = \{b_1, b_2, b_3\}$ and the fuzzy relation R is defined in Table 3. We will write (\uparrow, \downarrow) instead of $(\uparrow^\sigma, \downarrow^\sigma)$.

Given the fuzzy subset of objects $g: B \rightarrow L_2$, defined as $g(b_1) = \alpha$, $g(b_2) = \beta$ and $g(b_3) = \gamma$, the least concept that ‘‘contains’’ g is $(g^{\uparrow\downarrow}, g^\uparrow)$, which is obtained as follows:

$$\begin{aligned}
 g^\uparrow(a_1) &= \inf\{R(a_1, b_j) \swarrow^{\sigma(b)} g(b_j) \mid b_j \in B\} \\
 &= \inf\{0.8 \swarrow \alpha, 0.6 \swarrow \beta, 0.1 \swarrow \gamma\} = \inf\{d, d, a\} = a \\
 g^\uparrow(a_2) &= \inf\{0.2 \swarrow \alpha, 0.3 \swarrow \beta, 0.9 \swarrow \gamma\} = \inf\{d, d, d\} = d
 \end{aligned}$$

This mapping is used to compute $g^{\uparrow\downarrow}$:

$$\begin{aligned}
 g^{\uparrow\downarrow}(b_1) &= \inf\{R(a_i, b_1) \nwarrow_{\sigma(b)} g^\uparrow(a_i) \mid a_i \in A\} \\
 &= \inf\{0.8 \nwarrow a, 0.2 \nwarrow d\} = \inf\{\delta, \alpha\} = \alpha \\
 g^{\uparrow\downarrow}(b_2) &= \inf\{0.6 \nwarrow a, 0.3 \nwarrow d\} = \inf\{\delta, \beta\} = \beta \\
 g^{\uparrow\downarrow}(b_3) &= \inf\{0.1 \nwarrow a, 0.9 \nwarrow d\} = \inf\{\delta, \gamma\} = \gamma
 \end{aligned}$$

□

3.2. Concept lattices on L -connected lattices

In the following paragraphs, we define a pair of mappings on which the new concept lattice structure will be built in order to evaluate the objects and attributes on the same lattice L .

Given a complete lattice (L, \preceq) such that L_1 and L_2 are L -connected, a multi-adjoint frame $(L_1, L_2, P, \&_1, \dots, \&_n)$, and a context (A, B, R, σ) , we can define the mappings⁷ $\uparrow^{c\sigma}: L^B \rightarrow L^A$ and $\downarrow^{c\sigma}: L^A \rightarrow L^B$ defined for all

⁶We follow the common usage to denote the set of mappings from A to L as L^A .

⁷The subscript c refers to the L -connection, since we are using the mappings ϕ_j and ψ_j ; on the other hand, σ is needed to refer the particular choice of adjoint triple for a given b .

$g \in L^B$ and $f \in L^A$ as follows:

$$g^{\uparrow c\sigma}(a) = \psi_1(\inf\{R(a, b) \swarrow^{\sigma(b)} \phi_2(g(b)) \mid b \in B\}) \quad (3)$$

$$f^{\downarrow c\sigma}(b) = \psi_2(\inf\{R(a, b) \nwarrow_{\sigma(b)} \phi_1(f(a)) \mid a \in A\}) \quad (4)$$

Note that these definitions can be related to those given in [28] in that, for each adjoint triple $(\&, \swarrow, \nwarrow)$ of the multi-adjoint frame, we can define the mappings $\&^*: L \times L \rightarrow P$, $\swarrow^*: P \times L \rightarrow L$ and $\nwarrow_*: P \times L \rightarrow L$ for all $x, y \in L$ and $z \in P$ as follows:

$$\begin{aligned} x \&^* y &= \phi_1(x) \& \phi_2(y) & z \swarrow^* y &= \psi_1(z \swarrow \phi_2(y)) \\ & & & z \nwarrow_* x &= \psi_2(z \nwarrow \phi_1(x)) \end{aligned}$$

which, under the requirements $t \leq \psi_1(\phi_1(t))$ and $t \leq \psi_2(\phi_2(t))$, for all $t \in L$, forms another adjoint triple $(\&^*, \swarrow^*, \nwarrow_*)$. Under the additional assumption that the mappings ψ_j are inf-preserving, the mappings $\uparrow^{c\sigma}: L^B \rightarrow L^A$ and $\downarrow^{c\sigma}: L^A \rightarrow L^B$ can be rewritten as

$$g^{\uparrow c\sigma}(a) = \inf\{R(a, b) \swarrow^{*b} g(b) \mid b \in B\} \quad (5)$$

$$f^{\downarrow c\sigma}(b) = \inf\{R(a, b) \nwarrow_{*b} f(a) \mid a \in A\} \quad (6)$$

and coincide with the Galois connection introduced in [28], which is associated to the new frame $(L, L, P, \&_1^*, \dots, \&_n^*)$. As our construction of the new concept lattice does not require either of the requirements above, the proposed framework is *strictly more general* than the previous one. Hence, the requirements $t \leq \psi_1(\phi_1(t))$ and $t \leq \psi_2(\phi_2(t))$, for all $t \in L$, and that the mappings ψ_j are inf-preserving, will not be assumed.

Example 12. Mappings ψ_1 and ϕ_1 , defined in Example 3, verify the conditions in Definition 1 and ψ_1 is not inf-preserving since

$$\begin{aligned} \psi_1(\inf\{b, c\}) &= \psi_1(a) = x \\ \inf\{\psi_1(b), \psi_1(c)\} &= \inf\{t, u\} = z \end{aligned}$$

and, certainly, $x \neq z$.

Moreover, ψ_1 and ϕ_1 do not satisfy the inequality $t \leq \psi_1(\phi_1(t))$, for all $t \in L$. For example, the element $\psi_1(\phi_1(z)) = \psi_1(a) = x$ is obtained, which is not greater than z . \square

Summarizing, expressions (3) and (4) do not coincide with those given in [28], because they are not defined directly from a residuated implication, although the mappings ψ_1 , ψ_2 , ϕ_1 and ϕ_2 are involved as well. Indeed, *these mappings do not form a Galois connection*.

The next proposition shows that these mappings are order-reversing but do not verify conditions 2 and 3 of Definition 9.

Proposition 13. *Let $(L_1, L_2, P, \&_1, \dots, \&_n)$ be a multi-adjoint frame, where L_1 and L_2 are L -connected, and a context (A, B, R, σ) .*

1. *The mappings $\uparrow^{\sigma}, \downarrow^{\sigma}$ are order-reversing.*
2. *For each $g \in L^B$, $f \in L^A$, the following inequalities are obtained:
 $(\psi_2 \circ \phi_2 \circ g) \preceq g^{\uparrow^{\sigma}\downarrow^{\sigma}}$, $(\psi_1 \circ \phi_1 \circ f) \preceq f^{\downarrow^{\sigma}\uparrow^{\sigma}}$.*

PROOF. To improve readability, we will write $(\uparrow^c, \downarrow^c)$ instead of $(\uparrow^{\sigma}, \downarrow^{\sigma})$ and \swarrow^b, \searrow_b instead of $\swarrow^{\sigma(b)}, \searrow_{\sigma(b)}$.

1. \uparrow^c and \downarrow^c are order-reversing. If $f_1, f_2 \in L^A$, $f_1 \preceq f_2$, then $\phi_1(f_1(a)) \preceq_1 \phi_1(f_2(a))$ for all $a \in A$, because ϕ_1 is increasing. Now, as the implications are order-reversing in the second argument we obtain that:

$$R(a, b) \searrow_b \phi_1(f_2(a)) \preceq_2 R(a, b) \searrow_b \phi_1(f_1(a))$$

for all $a \in A$ and $b \in B$. Therefore, since ψ_2 is increasing, by the infimum property we obtain that

$$\begin{aligned} f_2^{\downarrow^c}(b) &= \psi_2(\inf\{R(a, b) \searrow_b \phi_1(f_2(a)) \mid a \in A\}) \\ &\preceq \psi_2(\inf\{R(a, b) \searrow_b \phi_1(f_1(a)) \mid a \in A\}) \\ &= f_1^{\downarrow^c}(b) \end{aligned}$$

for all $b \in B$. The proof for \uparrow^c is similar.

2. We need to prove that, given $g \in L^B$, then $(\psi_2 \circ \phi_2 \circ g) \preceq g^{\uparrow^c\downarrow^c}$. We begin from the definition of \uparrow^c :

$$g^{\uparrow^c}(a) = \psi_1(\inf\{R(a, b) \swarrow^b \phi_2(g(b)) \mid b \in B\})$$

for all $a \in A$. Now, applying the mapping ϕ_1 , we obtain:

$$\phi_1(g^{\uparrow^c}(a)) = \inf\{R(a, b) \swarrow^b \phi_2(g(b)) \mid b \in B\}$$

for all $a \in A$. Therefore, given $a \in A$ and $b \in B$ the next chain of inequalities holds by adjointness:

$$\begin{aligned}\phi_1(g^{\uparrow c}(a)) \preceq_1 R(a, b) \swarrow^b \phi_2(g(b)) &\iff \phi_1(g^{\uparrow c}(a)) \&_b \phi_2(g(b)) \leq R(a, b) \\ &\iff \phi_2(g(b)) \preceq_2 R(a, b) \nwarrow_b \phi_1(g^{\uparrow c}(a))\end{aligned}$$

As these inequalities hold for all $a \in A$, by applying properties of the infimum, we obtain that, for all $b \in B$

$$\phi_2(g(b)) \preceq_2 \inf\{R(a, b) \nwarrow_b \phi_1(g^{\uparrow c}(a)) \mid a \in A\}$$

Thus, $(\psi_2 \circ \phi_2 \circ g)(b) \preceq g^{\uparrow c \downarrow c}(b)$ for all $b \in B$, because ψ_2 is increasing.

Finally, a similar argument proves that $(\psi_1 \circ \phi_1 \circ f) \preceq f^{\downarrow c \uparrow c}$ for all $f \in L^A$. \square

The pair $(\uparrow^c, \downarrow^c)$ is not a Galois connection since $\downarrow^c \uparrow^c$ is not a closure operator, as the following example shows:

Example 14. Continuing with Example 3, consider the mapping $f \in L^A$ defined by $f(a_1) = z$, $f(a_2) = x$. Then, $f^{\downarrow c \uparrow c}$ is obtained as follows:

$$\begin{aligned}f^{\downarrow c}(b_1) &= \psi_2(\inf\{R(a_i, b_1) \nwarrow_{b_1} \phi_1 \circ f(a_i) \mid a_i \in A\}) \\ &= \psi_2(\inf\{0.8 \nwarrow a, 0.2 \nwarrow a\}) = \psi_2(\inf\{\delta, \delta\}) = \psi_2(\delta) = v \\ f^{\downarrow c}(b_2) &= \psi_2(\inf\{0.6 \nwarrow a, 0.3 \nwarrow a\}) = \psi_2(\inf\{\delta, \delta\}) = \psi_2(\delta) = v \\ f^{\downarrow c}(b_3) &= \psi_2(\inf\{0.1 \nwarrow a, 0.9 \nwarrow a\}) = \psi_2(\inf\{\delta, \delta\}) = \psi_2(\delta) = v \\ f^{\downarrow c \uparrow c}(a_1) &= \psi_1(\inf\{R(a_1, b_j) \swarrow^{b_j} \phi_2(f^{\downarrow c}(b_j)) \mid b_j \in B\}) \\ &= \psi_1(\inf\{0.8 \swarrow \delta, 0.6 \swarrow \delta, 0.1 \swarrow \delta\}) \\ &= \psi_1(\inf\{b, a, a\}) = \psi_1(a) = x \\ f^{\downarrow c \uparrow c}(a_2) &= \psi_1(\inf\{0.2 \swarrow \delta, 0.3 \swarrow \delta, 0.9 \swarrow \delta\}) \\ &= \psi_1(\inf\{a, a, b\}) = \psi_1(a) = x\end{aligned}$$

Therefore, as $f(a_1) = z$ and $f^{\downarrow c \uparrow c}(a_1) = x$, the inequality $f \preceq f^{\downarrow c \uparrow c}$ does not hold. \square

As a consequence of the previous example, the operators $\uparrow^c \downarrow^c$ and $\downarrow^c \uparrow^c$ are not closure operators, since they do not satisfy the property of extensivity.

However, although \uparrow^c and \downarrow^c do not form a Galois connection, and $\uparrow^c \downarrow^c$, $\downarrow^c \uparrow^c$ are not closure operators, they have properties on which we still can provide the definition of a concept in this extended framework.

Lemma 15. *Let $(L_1, L_2, P, \&_1, \dots, \&_n)$ be a multi-adjoint frame, where L_1 and L_2 are L -connected, and (A, B, R, σ) be a context, then the following equalities hold:*

$$\begin{aligned} (\psi_1 \circ \phi_1 \circ f)^{\downarrow^c} &= f^{\downarrow^c} & \psi_2 \circ \phi_2 \circ (f)^{\downarrow^c} &= f^{\downarrow^c} \\ (\psi_2 \circ \phi_2 \circ g)^{\uparrow^c} &= g^{\uparrow^c} & \psi_1 \circ \phi_1 \circ (g)^{\uparrow^c} &= g^{\uparrow^c} \end{aligned}$$

PROOF. We will prove just equalities $(\psi_1 \circ \phi_1 \circ f)^{\downarrow^c} = f^{\downarrow^c}$ and $\psi_2 \circ \phi_2 \circ (f)^{\downarrow^c} = f^{\downarrow^c}$, the rest follow analogously.

In order to prove the first equality, the following chain of equalities can be obtained for all $b \in B$:

$$\begin{aligned} (\psi_1 \circ \phi_1 \circ f)^{\downarrow^c}(b) &= \psi_2(\inf\{R(a, b) \frown_b \phi_1 \circ (\psi_1 \circ \phi_1 \circ f)(a) \mid a \in A\}) \\ &\stackrel{(*)}{=} \psi_2(\inf\{R(a, b) \frown_b (\phi_1 \circ f)(a) \mid a \in A\}) \\ &= f^{\downarrow^c}(b) \end{aligned}$$

where $(*)$ is obtained by Definition 1.

The second equality follows by a similar chain of equalities:

$$\begin{aligned} \psi_2 \circ \phi_2 \circ (f)^{\downarrow^c}(b) &= \psi_2 \circ \phi_2(\psi_2(\inf\{R(a, b) \frown_b (\phi_1 \circ f)(a) \mid a \in A\})) \\ &\stackrel{(*)}{=} \psi_2(\inf\{R(a, b) \frown_b (\phi_1 \circ f)(a) \mid a \in A\}) \\ &= f^{\downarrow^c}(b) \end{aligned}$$

where $(*)$ is obtained by Definition 1. □

As a consequence of the previous result, we obtain the following

Proposition 16. *Let $(L_1, L_2, P, \&_1, \dots, \&_n)$ be a multi-adjoint frame, where L_1 and L_2 are L -connected, and a context (A, B, R, σ) , then $f^{\downarrow^c} = f^{\downarrow^c \uparrow^c \downarrow^c}$, $g^{\uparrow^c} = g^{\uparrow^c \downarrow^c \uparrow^c}$, for all $g \in L^B$ and $f \in L^A$.*

PROOF. To begin with, from Proposition 13(2), we have that $(\psi_1 \circ \phi_1 \circ f) \preceq f^{\downarrow^c \uparrow^c}$, now applying the order-reversing mapping \downarrow^c we get the inequality $(f^{\downarrow^c \uparrow^c})^{\downarrow^c} \preceq (\psi_1 \circ \phi_1 \circ f)^{\downarrow^c}$; finally, by Lemma 15, $f^{\downarrow^c \uparrow^c \downarrow^c} \preceq f^{\downarrow^c}$ holds.

On the other hand, again by Lemma 15 and Proposition 13(2), we have that $f^{\downarrow^c} = \psi_2 \circ \phi_2 \circ f^{\downarrow^c} \preceq (f^{\downarrow^c})^{\uparrow^c \downarrow^c}$. As a result, the equality $f^{\downarrow^c} = f^{\downarrow^c \uparrow^c \downarrow^c}$ is obtained.

The other equality follows similarly. □

On a frame where (L_1, \preceq_1) and (L_2, \preceq_2) are L -connected, the result above allows for defining a new concept lattice by following the usual construction: a *concept* is a pair $\langle g^*, f^* \rangle$ satisfying $g^* \in L^B$, $f^* \in L^A$ and that $(g^*)^{\uparrow c} = f^*$ and $(f^*)^{\downarrow c} = g^*$.⁸

Definition 17. Given the complete lattices (L_1, \preceq_1) , (L_2, \preceq_2) and (L, \preceq) , where L_1 and L_2 are L -connected, the *multi-adjoint L -connected concept lattice* associated to a multi-adjoint frame $(L_1, L_2, L, \&_1, \dots, \&_n)$ and context (A, B, R, σ) is the set

$$\mathfrak{M}_L = \{ \langle g^*, f^* \rangle \mid \langle g^*, f^* \rangle \text{ is a concept} \}$$

in which the ordering is defined by $\langle g_1^*, f_1^* \rangle \leq \langle g_2^*, f_2^* \rangle$ if and only if $g_1^* \preceq g_2^*$ (equivalently $f_2^* \preceq f_1^*$).

The pair (\mathfrak{M}_L, \leq) is clearly a poset and, by Proposition 16, their elements can be obtained by applying $\uparrow c$ and $\downarrow c$ to a fuzzy subset of objects or attributes, respectively, as in the classical case.

Example 18. Let (L_1, \preceq_1) , (L_2, \preceq_2) and (L, \preceq) be the lattices given in Example 3 and the multi-adjoint frame presented in Example 11.

Given the fuzzy subset of objects $g^*: B \rightarrow L$, defined as $g^*(b_1) = u$, $g^*(b_2) = x$ and $g^*(b_3) = x$, the least concept “containing” g^* is $((g^*)^{\uparrow c \downarrow c}, (g^*)^{\uparrow c})$, which is obtained as follows:

$$\begin{aligned} (g^*)^{\uparrow c}(a_1) &= \psi_1(\inf\{R(a_1, b_j) \swarrow^{\sigma(b)} \phi_2(g^*(b_j)) \mid b_j \in B\}) \\ &= \psi_1(\inf\{0.8 \swarrow \delta, 0.6 \swarrow \alpha, 0.1 \swarrow \alpha\}) \\ &= \psi_1(\inf\{b, d, d\}) = \psi_1(b) = t \\ (g^*)^{\uparrow c}(a_2) &= \psi_1(\inf\{0.2 \swarrow \delta, 0.3 \swarrow \alpha, 0.9 \swarrow \alpha\}) \\ &= \psi_1(\inf\{a, d, d\}) = \psi_1(a) = x \\ (g^*)^{\uparrow c \downarrow c}(b_1) &= \psi_2(\inf\{R(a_i, b_1) \nwarrow_{\sigma(b)} \phi_1 \circ (g^*)^{\uparrow}(a_i) \mid a_i \in A\}) \\ &= \psi_2(\inf\{0.8 \nwarrow b, 0.2 \nwarrow a\}) = \psi_2(\inf\{\delta, \delta\}) = \psi_2(\delta) = v \\ (g^*)^{\uparrow c \downarrow c}(b_2) &= \psi_2(\inf\{0.6 \nwarrow b, 0.3 \nwarrow a\}) = \psi_2(\inf\{\beta, \delta\}) = \psi_2(\beta) = y \\ (g^*)^{\uparrow c \downarrow c}(b_3) &= \psi_2(\inf\{0.1 \nwarrow b, 0.9 \nwarrow a\}) = \psi_2(\inf\{\beta, \delta\}) = \psi_2(\beta) = y \end{aligned}$$

□

⁸We include $*$ as a superscript in this new construction so that we can distinguish this new approach from that in [28]. Note that, in order to simplify the notation, references to σ have been omitted.

Note, however, that (\mathfrak{M}_L, \leq) is not a complete lattice as in the classical case, since it is not closed under supremum, that is, given $\langle g_1^*, f_1^* \rangle$ and $\langle g_2^*, f_2^* \rangle$, the supremum should be defined as:

$$\langle g_1^*, f_1^* \rangle \vee \langle g_2^*, f_2^* \rangle = \langle (g_1^* \vee g_2^*)^{\uparrow c \downarrow c}, f_1^* \wedge f_2^* \rangle$$

but the element in the right hand side need not be a concept of \mathfrak{M}_L , as the following example shows.

Example 19. Let us denote the concept in Example 18 as $\langle (f_1^*)^{\downarrow c}, f_1^* \rangle$, and let us write $\langle (f_2^*)^{\downarrow c}, f_2^* \rangle$ to denote the one given by the fuzzy subset of objects $g_2^*: B \rightarrow L$, defined as $g_2^*(b_1) = y$, $g_2^*(b_2) = t$ and $g_2^*(b_3) = x$.

We will explicitly obtain the mapping f_2^* and prove that $f_1^* \wedge f_2^* \neq (f_1^* \wedge f_2^*)^{\downarrow c \uparrow c}$.

$$\begin{aligned} f_2^*(a_1) = (g_2^*)^{\uparrow c}(a_1) &= \psi_1(\inf\{R(a_1, b_j) \swarrow^b \phi_2(g_2^*(b_j)) \mid b_j \in B\}) \\ &= \psi_1(\inf\{0.8 \swarrow \beta, 0.6 \swarrow \gamma, 0.1 \swarrow \alpha\}) \\ &= \psi_1(\inf\{d, c, d\}) = \psi_1(c) = u \\ f_2^*(a_2) = (g_2^*)^{\uparrow c}(a_2) &= \psi_1(\inf\{0.2 \swarrow \beta, 0.3 \swarrow \gamma, 0.9 \swarrow \alpha\}) \\ &= \psi_1(\inf\{b, c, d\}) = \psi_1(a) = x \end{aligned}$$

Therefore, $(f_1^* \wedge f_2^*)(a_1) = t \wedge u = z$ and $(f_1^* \wedge f_2^*)(a_2) = x \wedge x = x$, which is the mapping assumed in Example 14. Thus, the equality $f_1^* \wedge f_2^* = (f_1^* \wedge f_2^*)^{\downarrow c \uparrow c}$ does not hold. \square

In the rest of this section we focus on alternative suitable definition of meet \wedge and a join \vee operators such that $(\mathfrak{M}_L, \wedge, \vee)$ will be a complete lattice.

To begin with, some technical lemmas are needed.

Lemma 20. *Given the complete lattices (L_1, \preceq_1) , (L_2, \preceq_2) and (L, \preceq) , where L_1 and L_2 are L -connected, and $x, x' \in L_1$, $y, y' \in L_2$, we have that*

$$\begin{aligned} x \wedge_1 x' &= \phi_1(\psi_1(x) \wedge \psi_1(x')) & y_1 \wedge_2 y' &= \phi_2(\psi_2(y) \wedge \psi_2(y')) \\ x \vee_1 x' &= \phi_1(i_1(x) \vee \psi_1(x')) & y_1 \vee_2 y' &= \phi_2(\psi_2(y) \vee \psi_2(y')) \end{aligned}$$

where $\wedge_1, \wedge_2, \wedge, \vee_1, \vee_2, \vee$, are the meet and join operators defined on L_1 , L_2 and L , respectively.

PROOF. On the one hand, we have that $x \wedge_1 x'$ is less or equal to x and x' , and hence $\psi_1(x \wedge_1 x')$ is less or equal to $\psi_1(x)$ and $\psi_1(x')$. By definition of infimum, the inequality $\psi_1(x \wedge_1 x') \preceq \psi_1(x) \wedge \psi_1(x')$ holds and, applying ϕ_1 now, the inequality $x \wedge_1 x' \preceq_1 \phi_1(\psi_1(x) \wedge \psi_1(x'))$ is obtained.

On the other hand, we have $\psi_1(x) \wedge \psi_1(x')$ is less or equal to $\psi_1(x)$ and $\psi_1(x')$, now applying ϕ_1 , we obtain $\phi_1(\psi_1(x) \wedge \psi_1(x')) \preceq_1 x \wedge_1 x'$, and the first equality holds.

The proof for the rest of equalities follows analogously. \square

Lemma 21. *Let $(L_1, L_2, P, \&_1, \dots, \&_n)$ be a multi-adjoint frame, where L_1 and L_2 are L -connected, a context (A, B, R, σ) , and a family $\langle g_i^*, f_i^* \rangle$ of concepts of \mathfrak{M}_L , for i running in an index set Λ , then*

$$\left(\bigwedge_{i \in \Lambda} g_i^* \right)^{\uparrow c \downarrow^c} = \psi_2 \circ \phi_2 \left(\bigwedge_{i \in \Lambda} g_i^* \right), \quad \left(\bigwedge_{i \in \Lambda} f_i^* \right)^{\uparrow c \downarrow^c} = \psi_1 \circ \phi_1 \left(\bigwedge_{i \in \Lambda} f_i^* \right)$$

PROOF. Only the first equality will be proved, the second follows similarly.

Since $\bigwedge_{i \in \Lambda} g_i^* \preceq g_i^*$, then $(\bigwedge_{i \in \Lambda} g_i^*)^{\uparrow c \downarrow^c} \preceq (g_i^*)^{\uparrow c \downarrow^c}$, for all $i \in \Lambda$. Now, as g_i^* is the extension of a concept, $(g_i^*)^{\uparrow c \downarrow^c} = g_i^*$ is verified, for all $i \in \Lambda$. Therefore, by the infimum property, the inequality $(\bigwedge_{i \in \Lambda} g_i^*)^{\uparrow c \downarrow^c} \preceq \bigwedge_{i \in \Lambda} g_i^*$ holds, which is considered together with Proposition 13 and Lemma 15, to obtain the following chain of inequalities.

$$\left(\bigwedge_{i \in \Lambda} g_i^* \right)^{\uparrow c \downarrow^c} = \psi_2 \circ \phi_2 \left(\left(\bigwedge_{i \in \Lambda} g_i^* \right)^{\uparrow c \downarrow^c} \right) \preceq \psi_2 \circ \phi_2 \left(\bigwedge_{i \in \Lambda} g_i^* \right) \preceq \left(\bigwedge_{i \in \Lambda} g_i^* \right)^{\uparrow c \downarrow^c}$$

Thus, the inequalities above are equalities and we have the result. \square

The following theorem defines meet and join operators which will provide \mathfrak{M}_L a complete lattice structure.

Theorem 22. *Given complete lattices (L_1, \preceq_1) , (L_2, \preceq_2) and (L, \preceq) , where L_1 and L_2 are L -connected, a context (A, B, R, σ) , and a multi-adjoint frame $(L_1, L_2, L, \&_1, \dots, \&_n)$, the multi-adjoint L -connected concept lattice \mathfrak{M}_L is actually a complete lattice with the meet and join operators $\wedge, \vee: \mathfrak{M}_L \times \mathfrak{M}_L \rightarrow \mathfrak{M}_L$ defined below, for all $\langle g_1^*, f_1^* \rangle, \langle g_2^*, f_2^* \rangle \in \mathfrak{M}_L$,*

$$\begin{aligned} \langle g_1^*, f_1^* \rangle \wedge \langle g_2^*, f_2^* \rangle &= \langle \psi_2 \circ \phi_2(g_1^* \wedge g_2^*), (f_1^* \vee f_2^*)^{\downarrow^c \uparrow c} \rangle \\ \langle g_1^*, f_1^* \rangle \vee \langle g_2^*, f_2^* \rangle &= \langle (g_1^* \vee g_2^*)^{\uparrow c \downarrow^c}, \psi_1 \circ \phi_1(f_1^* \wedge f_2^*) \rangle \end{aligned}$$

PROOF. Commutative and idempotent laws follow directly from the respective properties of the infimum and supremum defined on L_1 and L_2 . In order to prove the lattice structure, we have just to prove the associative and the absorption laws.

Consider $\langle g_1^*, f_1^* \rangle, \langle g_2^*, f_2^* \rangle$ and $\langle g_3^*, f_3^* \rangle \in \mathfrak{M}_L$, then

$$g_j^*(b) = (f_j^*)^{\downarrow c}(b) = \psi_2(\inf\{R(a, b) \frown_b (\phi_1 \circ f_j^*)(a) \mid a \in A\})$$

for all $j \in \{1, 2, 3\}$ and $b \in B$, hence there exist mappings g_1, g_2, g_3 defined on L_2^B such that $g_j^* = \psi_2(g_j)$, for all $j \in \{1, 2, 3\}$.

Associativity wrt \wedge : the fuzzy extent of $(\langle g_1^*, f_1^* \rangle \wedge \langle g_2^*, f_2^* \rangle) \wedge \langle g_3^*, f_3^* \rangle$ is

$$\begin{aligned} \psi_2 \circ \phi_2(\psi_2 \circ \phi_2(g_1^* \wedge g_2^*) \wedge g_3^*) &= \psi_2 \circ \phi_2(\psi_2 \circ \phi_2(\psi_2(g_1) \wedge \psi_2(g_2)) \wedge \psi_2(g_3)) \\ &= \psi_2 \circ \phi_2(\psi_2(g_1 \wedge_2 g_2) \wedge \psi_2(g_3)) \\ &= \psi_2((g_1 \wedge_2 g_2) \wedge_2 g_3) \end{aligned}$$

Similarly, the fuzzy extent of $\langle g_1^*, f_1^* \rangle \wedge (\langle g_2^*, f_2^* \rangle \wedge \langle g_3^*, f_3^* \rangle)$ is

$$\psi_2 \circ \phi_2(g_1^* \wedge \psi_2 \circ \phi_2(g_2^* \wedge g_3^*)) = \psi_2(g_1 \wedge_2 (g_2 \wedge_2 g_3))$$

Thus, the associativity law w.r.t. \wedge is proved by the associativity law of the meet operator \wedge_2 defined on L_2 . Associativity wrt \vee follows similarly.

Absorption law of \wedge w.r.t \vee . To begin with, the equality $(g_1^* \vee g_2^*)^{\uparrow c \downarrow c} = \psi_2(g_1 \vee_2 g_2)$ will be proved.

$$\begin{aligned} (g_1^* \vee g_2^*)^{\uparrow c}(a) &= \psi_1(\inf\{R(a, b_j) \swarrow^{b_j} \phi_2((g_1^* \vee g_2^*)(b_j)) \mid b_j \in B\}) \\ &\stackrel{(*)}{=} \psi_1(\inf\{R(a, b_j) \swarrow^{b_j} \phi_2((\psi_2(g_1) \vee \psi_2(g_2))(b_j)) \mid b_j \in B\}) \\ &= \psi_1(\inf\{R(a, b_j) \swarrow^{b_j} (g_1 \vee_2 g_2)(b_j) \mid b_j \in B\}) \\ &= \psi_1((g_1 \vee_2 g_2)^{\uparrow}(a)) \end{aligned}$$

where $a \in A$ and $(*)$ is obtained by Lemma 20.

As a consequence, the equalities below follow:

$$\begin{aligned} (g_1^* \vee g_2^*)^{\uparrow c \downarrow c}(b) &= \psi_2(\inf\{R(a_j, b) \frown_b \phi_1((g_1^* \vee g_2^*)^{\uparrow c}(a_j)) \mid a_j \in A\}) \\ &= \psi_2(\inf\{R(a_j, b) \frown_b \phi_1(\psi_1((g_1 \vee_2 g_2)^{\uparrow c}(a_j))) \mid a_j \in A\}) \\ &= \psi_2(\inf\{R(a_j, b) \frown_b (g_1 \vee_2 g_2)^{\uparrow c}(a_j) \mid a_j \in A\}) \\ &= (\psi_2 \circ (g_1 \vee_2 g_2)^{\uparrow \downarrow})(b) \end{aligned}$$

for all $b \in B$. Therefore, we have that $(g_1^* \vee g_2^*)^{\uparrow c \downarrow c} = \psi_2((g_1 \vee_2 g_2)^{\uparrow \downarrow}) = \psi_2(g_1 \vee_2 g_2)$, where the last equality is given by the definition of the join operator in \mathfrak{M} .

Now, using the comment above and Lemma 20, the fuzzy extent of $\langle g_1^*, f_1^* \rangle \wedge (\langle g_1^*, f_1^* \rangle \vee \langle g_2^*, f_2^* \rangle)$ is

$$\begin{aligned} \psi_2 \circ \phi_2(g_1^* \wedge (g_1^* \vee g_2^*))^{\uparrow c \downarrow c} &= \psi_2 \circ \phi_2(\psi_2(g_1) \wedge \psi_2(g_1 \vee_2 g_2)) \\ &= \psi_2(g_1 \wedge_2 (g_1 \vee_2 g_2)) \\ &= \psi_2(g_1) = g_1^* \end{aligned}$$

The other absorption law can be proved analogously.

Therefore, $(\mathfrak{M}_L, \wedge, \vee)$ is an algebraic lattice. As a consequence, an ordering on \mathfrak{M}_L , which will be denoted \preceq_L , can be defined from \wedge and \vee .

In order to prove that $(\mathfrak{M}_L, \preceq_L)$ is a complete lattice [12], given a family of concepts $\{\langle g_i^*, f_i^* \rangle\}_{i \in \Lambda}$ in \mathfrak{M}_L we will prove that its infimum exists (the existence of supremum follows similarly).

If $\langle (\bigwedge_{i \in \Lambda} g_i^*)^{\uparrow c \downarrow c}, (\bigwedge_{i \in \Lambda} f_i^*)^{\uparrow c} \rangle$ is the infimum, then, by Lemma 21, we obtain:

$$\bigwedge_{i \in \Lambda} \langle g_i^*, f_i^* \rangle = \langle \psi_2 \circ \phi_2(\bigwedge_{i \in \Lambda} g_i^*), (\bigwedge_{i \in \Lambda} f_i^*)^{\uparrow c} \rangle$$

Hence, we only need to prove that $\langle (\bigwedge_{i \in \Lambda} g_i^*)^{\uparrow c \downarrow c}, (\bigwedge_{i \in \Lambda} f_i^*)^{\uparrow c} \rangle$ is the infimum.

As $(\bigwedge_{i \in \Lambda} g_i^*) \preceq g_i^*$, then $(\bigwedge_{i \in \Lambda} g_i^*)^{\uparrow c \downarrow c} \preceq (g_i^*)^{\uparrow c \downarrow c}$. Now, since g_i^* is the fuzzy extent of a concept, Lemma 15 provides $(\bigwedge_{i \in \Lambda} g_i^*)^{\uparrow c \downarrow c} \preceq_L g_i^*$, for each $i \in \Lambda$.

In order to prove that $\langle (\bigwedge_{i \in \Lambda} g_i^*)^{\uparrow c \downarrow c}, (\bigwedge_{i \in \Lambda} f_i^*)^{\uparrow c} \rangle$ is the greatest lower bound, we will assume that $\langle g^*, f^* \rangle \in \mathfrak{M}_L$ is another lower bound of the family of concepts, that is, $\langle g^*, f^* \rangle \preceq_L \langle g_i^*, f_i^* \rangle$, for all $i \in \Lambda$. Equivalently, the equalities

$$\langle g^*, f^* \rangle = \langle g^*, f^* \rangle \wedge \langle g_i^*, f_i^* \rangle = \langle \psi_2 \circ \phi_2(g^* \wedge g_i^*), (f^* \vee f_i^*)^{\downarrow c \uparrow c} \rangle$$

are obtained. Hence, $g^* = \psi_2 \circ \phi_2(g^* \wedge g_i^*) \preceq \psi_2 \circ \phi_2(g_i^*) = g_i^*$, because $\psi_2 \circ \phi_2$ is order-preserving, by Lemma 15 and the definition of \preceq_L . Therefore, $g^* \preceq (\bigwedge_{i \in \Lambda} g_i^*) \preceq g_i^*$ and, from the order-preserving property of $\psi_2 \circ \phi_2$, the following chain of inequalities are obtained:

$$g^* = \psi_2 \circ \phi_2(g^* \wedge g_i^*) \preceq \psi_2 \circ \phi_2(g^*) \preceq \psi_2 \circ \phi_2(\bigwedge_{i \in \Lambda} g_i^*) = (\bigwedge_{i \in \Lambda} g_i^*)^{\uparrow c \downarrow c}$$

where the last equality is given by Lemma 21.

Finally, the fuzzy extent of $\langle g^*, f^* \rangle \wedge \langle (\bigwedge_{i \in \Lambda} g_i^*)^{\uparrow c \downarrow c}, (\bigwedge_{i \in \Lambda} f_i^*)^{\uparrow c} \rangle$ is given by $\psi_2 \circ \phi_2(g^* \wedge (\bigwedge_{i \in \Lambda} g_i^*)^{\uparrow c \downarrow c}) = \psi_2 \circ \phi_2(g^*) = g^*$, by the definition of \wedge , the comment above and Lemma 15.

Thus, the inequality $\langle g^*, f^* \rangle \preceq_L \langle (\bigwedge_{i \in \Lambda} g_i^*)^{\uparrow c \downarrow c}, (\bigwedge_{i \in \Lambda} g_i^*)^{\uparrow c} \rangle$ holds and, consequently, there exist the meet (and join) of each non-empty set of \mathfrak{M}_L .

The proof is finished by showing that $(\mathfrak{M}_L, \preceq_L)$ has bottom and top elements, which can be easily expressed as

$$\perp = \bigwedge \{ \langle g^*, f^* \rangle \mid \langle g^*, f^* \rangle \in \mathfrak{M}_L \} \quad \top = \bigvee \{ \langle g^*, f^* \rangle \mid \langle g^*, f^* \rangle \in \mathfrak{M}_L \}$$

□

As a result of the previous statements, two orderings have been defined on \mathfrak{M}_L , although only \preceq_L makes \mathfrak{M}_L to be a complete lattice. The following proposition prepares the next section, in which both orderings will be compared.

Proposition 23. *Given the set of concept \mathfrak{M}_L , and the orders defined above, \leq and \preceq_L , we obtain that: if $\langle g_1^*, f_1^* \rangle, \langle g_2^*, f_2^* \rangle \in \mathfrak{M}_L$, such that $\langle g_1^*, f_1^* \rangle \leq \langle g_2^*, f_2^* \rangle$, then $\langle g_1^*, f_1^* \rangle \preceq_L \langle g_2^*, f_2^* \rangle$.*

PROOF. As we stated in the proof of Theorem above, if $\langle g_1^*, f_1^* \rangle$ and $\langle g_2^*, f_2^* \rangle \in \mathfrak{M}_L$, then $g_1^* = (f_1^*)^{\downarrow c}$, $g_2^* = (f_2^*)^{\downarrow c}$ and there exist mappings g_1, g_2 defined on $(L_2)^B$ such that $g_1^* = \psi_2(g_1)$, $g_2^* = \psi_2(g_2)$. Hence, the following chain of equalities is obtained:

$$\begin{aligned} \langle g_1^*, f_1^* \rangle \wedge \langle g_2^*, f_2^* \rangle &= \langle \psi_2 \circ \phi_2(g_1^* \wedge g_2^*), (g_1^* \wedge g_2^*)^{\uparrow c} \rangle \\ &= \langle \psi_2 \circ \phi_2(\psi_2(g_1) \wedge \psi_2(g_2)), (g_1^* \wedge g_2^*)^{\uparrow c} \rangle \\ &\stackrel{(1)}{=} \langle \psi_2(g_1 \wedge_2 g_2), (g_1^* \wedge g_2^*)^{\uparrow c} \rangle \\ &= \langle \psi_2(g_1), (g_1^* \wedge g_2^*)^{\uparrow c} \rangle \\ &\stackrel{(2)}{=} \langle g_1^*, (g_1^* \wedge g_2^*)^{\uparrow c} \rangle \\ &\stackrel{(3)}{=} \langle g_1^*, (g_1^*)^{\uparrow c} \rangle \end{aligned}$$

where (1) is given by Lemma 20 and (2) is obtained since $\psi_2(g_1) = g_1^* \preceq g_2^* = \psi_2(g_2)$ and, applying ϕ , we have $g_1 \preceq_2 g_2$. (3) is satisfied by the definition of concept.

4. Comparison between (\mathfrak{M}, \preceq) and $(\mathfrak{M}_L, \preceq_L)$

In this section, we establish a comparison between the concept lattices $(\mathfrak{M}_L, \preceq_L)$ (defined above) and (\mathfrak{M}, \preceq) (defined in [28]). We will consider a fixed context (A, B, R, σ) , a frame $(L_1, L_2, L, \&_1, \dots, \&_n)$, where L_1

and L_2 are L -connected, and the corresponding multi-adjoint concept lattices (\mathfrak{M}, \preceq) and $(\mathfrak{M}_L, \preceq_L)$.

Firstly we will prove, in the following result, that each concept $\langle g, f \rangle \in \mathfrak{M}$ determines a concept in \mathfrak{M}_L .

Proposition 24. *If $\langle g, f \rangle \in \mathfrak{M}$, then the mappings $g^*: B \rightarrow L$, $f^*: A \rightarrow L$, defined as $g^* = \psi_2 \circ g$, $f^* = \psi_1 \circ f$, form a concept of the multi-adjoint concept lattice $(\mathfrak{M}_L, \preceq_L)$.*

PROOF. The equalities $(\psi_2 \circ g)^{\uparrow c} = (\psi_1 \circ f)$ and $(\psi_1 \circ f)^{\downarrow c} = (\psi_2 \circ g)$ have to be checked. We will prove just the first one, as the second follows similarly.

Given $a \in A$, as $\langle g, f \rangle \in \mathfrak{M}$, we obtain the following chain of equalities:

$$\begin{aligned} (\psi_2 \circ g)^{\uparrow c}(a) &= \psi_1(\inf\{R(a, b) \swarrow^b \phi_2((\psi_2 \circ g)(b)) \mid b \in B\}) \\ &= \psi_1(\inf\{R(a, b) \swarrow^b g(b) \mid b \in B\}) \\ &= \psi_1(g^\uparrow(a)) \\ &= \psi_1(f(a)) \\ &= (\psi_1 \circ f)(a) \end{aligned}$$

Hence, $(\psi_2 \circ g)^{\uparrow c} = (\psi_1 \circ f)$. □

Example 25. Considering the concept $(g^{\uparrow\downarrow}, g^\uparrow) \in \mathfrak{M}$ given in Example 11, and the mappings in Example 3, by Proposition 24, the pair $(\psi_2 \circ g^{\uparrow\downarrow}, \psi_1 \circ g^\uparrow)$ is a concept of the multi-adjoint L -connected concept lattice $(\mathfrak{M}_L, \preceq_L)$.

For example, the mapping $\psi_1 \circ g^\uparrow: A \rightarrow L$ is defined as: $(\psi_1 \circ g^\uparrow)(a_1) = x$ and $(\psi_1 \circ g^\uparrow)(a_2) = v$. □

Now, given a mapping $g: B \rightarrow L_2$, we have two possible ways to construct the smallest concept in \mathfrak{M}_L “containing” g :

- Considering the mapping $\psi_2 \circ g \in L^B$ and obtaining the corresponding concept in \mathfrak{M}_L , that is, $\langle (\psi_2 \circ g)^{\uparrow c \downarrow c}, (\psi_2 \circ g)^{\uparrow c} \rangle$.
- Obtaining the corresponding concept in \mathfrak{M} and, by Proposition 24, considering the concept $\langle \psi_2 \circ g^{\uparrow\downarrow}, \psi_1 \circ g^\uparrow \rangle$ in \mathfrak{M}_L .

The following proposition states that the two constructions above coincide.

Proposition 26. *Given a mapping $g: B \rightarrow L_2$, the concepts $\langle (\psi_2 \circ g)^{\uparrow c \downarrow c}, (\psi_2 \circ g)^{\uparrow c} \rangle$ and $\langle \psi_2 \circ g^{\uparrow\downarrow}, \psi_1 \circ g^\uparrow \rangle$ coincide.*

PROOF. It is sufficient to prove that $(\psi_2 \circ g)^{\uparrow c} = \psi_1 \circ g^{\uparrow}$, and this follows from the proof of Proposition 24. \square

Similarly, we obtain a concept of \mathfrak{M} from each concept of \mathfrak{M}_L , and the two possible constructions of the smallest concept “containing” $g^*: B \rightarrow L$ coincide.

Proposition 27. *If $\langle g^*, f^* \rangle \in \mathfrak{M}_L$, then the mappings $g: B \rightarrow L_2$, $f: A \rightarrow L_1$, defined as: $g = \phi_2 \circ g^*$, $f = \phi_1 \circ f^*$, form a concept of the multi-adjoint concept lattice \mathfrak{M} . Moreover, given a mapping $g^*: B \rightarrow L$, the concepts $\langle (\phi_2 \circ g^*)^{\uparrow \downarrow}, (\phi_2 \circ g^*)^{\uparrow} \rangle$ and $\langle \phi_2 \circ (g^*)^{\uparrow c \downarrow}, \phi_1 \circ (g^*)^{\uparrow c} \rangle$ coincide.*

PROOF. First of all, we need to prove that $g^{\uparrow} = f, f^{\downarrow} = g$. We will prove just the first equality (the second follows similarly).

Given $a \in A$, as $\langle g^*, f^* \rangle \in \mathfrak{M}_L$, we obtain the following chain of equalities:

$$\begin{aligned} f^*(a) &= (g^*)^{\uparrow c}(a) \\ &= \psi_1(\inf\{R(a, b) \swarrow^b \phi_2(g^*(b)) \mid b \in B\}) \\ &= \psi_1(\inf\{R(a, b) \swarrow^b g(b) \mid b \in B\}) \\ &= \psi_1(g^{\uparrow}(a)) \end{aligned}$$

Hence, applying ϕ_1 on both sides, we obtain

$$f(a) = \phi_1(f^*(a)) = \phi_1(\psi_1(g^{\uparrow}(a))) = g^{\uparrow}(a)$$

Finally, from the chain of equalities above, we conclude that $(\phi_2 \circ g^*)^{\uparrow} = \phi_1 \circ (g^*)^{\uparrow c}$, since

$$(\phi_2 \circ g^*)^{\uparrow}(a) = g^{\uparrow}(a) = \phi_1(\psi_1(g^{\uparrow}(a))) = \phi_1(f^*(a)) = (\phi_1 \circ (g^*)^{\uparrow c})(a)$$

which lead us to ensure that the concepts $\langle (\phi_2 \circ g^*)^{\uparrow \downarrow}, (\phi_2 \circ g^*)^{\uparrow} \rangle$ and $\langle \phi_2 \circ (g^*)^{\uparrow c \downarrow}, \phi_1 \circ (g^*)^{\uparrow c} \rangle$ coincide. \square

Example 28. Given the concept $((g^*)^{\uparrow c \downarrow}, (g^*)^{\uparrow c}) \in \mathfrak{M}_L$ considered in Example 18, and the mappings in Example 3, by Proposition 27, the pair $(\phi_2 \circ (g^*)^{\uparrow c \downarrow}, \phi_1 \circ (g^*)^{\uparrow c})$ is a concept of the multi-adjoint concept lattice \mathfrak{M} .

For example, the mapping $\phi_1 \circ (g^*)^{\uparrow c}: A \rightarrow L_1$ is defined as: $(\phi_1 \circ (g^*)^{\uparrow c})(a_1) = b$ and $(\phi_1 \circ (g^*)^{\uparrow c})(a_2) = a$. \square

It is worth to take into account that the result above can be given analogously for any $f: A \rightarrow L_1$ as well.

As a consequence of the definition of L -connection and the above results, the following theorem is obtained.

Theorem 29. *The mappings $\Phi: (\mathfrak{M}_L, \preceq_L) \rightarrow (\mathfrak{M}, \preceq)$ and $\Psi: (\mathfrak{M}, \preceq) \rightarrow (\mathfrak{M}_L, \preceq_L)$ defined, for each $\langle g, f \rangle \in \mathfrak{M}$ and $\langle g^*, f^* \rangle \in \mathfrak{M}_L$, as follows*

$$\begin{aligned}\Phi(\langle g^*, f^* \rangle) &= \langle \phi_2 \circ g^*, \phi_1 \circ f^* \rangle \\ \Psi(\langle g, f \rangle) &= \langle \psi_2 \circ g, \psi_1 \circ f \rangle\end{aligned}$$

are well-defined and Φ and Ψ are (order-)isomorphism. Thus, \mathfrak{M}_L and \mathfrak{M} are isomorphic.

PROOF. From Propositions 24 and 27, Φ and Ψ are well-defined. Mapping Ψ is order-preserving since ψ_1, ψ_2 are order-preserving mappings and by Proposition 23.

To prove that Φ is order-preserving, let us to consider $\langle g_1^*, f_1^* \rangle$ and $\langle g_2^*, f_2^* \rangle \in \mathfrak{M}_L$, such that $\langle g_1^*, f_1^* \rangle \preceq_L \langle g_2^*, f_2^* \rangle$ and $\phi_2 \circ g_1^* \preceq_2 \phi_2 \circ g_2^*$ must be proved or, equivalently, $\phi_2 \circ g_1^* = \phi_2 \circ g_1^* \wedge_2 \phi_2 \circ g_2^*$.

As $g_1^* = (f_1^*)^{\downarrow c}$, $g_2^* = (f_2^*)^{\downarrow c}$, there exist mappings g_1, g_2 defined on $(L_2)^B$ such that $g_1^* = \psi_2(g_1)$, $g_2^* = \psi_2(g_2)$. Moreover, the equality $\langle g_1^*, f_1^* \rangle = \langle g_1^*, f_1^* \rangle \wedge \langle g_2^*, f_2^* \rangle$ is satisfied.

Therefore, $g_1^* = \psi_2 \circ \phi_2(g_1^* \wedge g_2^*) = \psi_2 \circ \phi_2(\psi_2(g_1) \wedge \psi_2(g_2)) = \psi_2(g_1 \wedge_2 g_2)$ and, applying ϕ_2 , we obtain the equality needed.

$$\phi_2(g_1^*) = \phi_2 \circ \psi_2(g_1 \wedge_2 g_2) = g_1 \wedge_2 g_2 = \phi_2 \circ \psi_2(g_1) \wedge_2 \phi_2 \circ \psi_2(g_2) = \phi_2(g_1^*) \wedge_2 \phi_2(g_2^*)$$

Furthermore, by Definition 1, we have that $\Phi \circ \Psi: \mathfrak{M} \rightarrow \mathfrak{M}$ is the identity mapping. The proof will be finished if $\Psi \circ \Phi: \mathfrak{M}_L \rightarrow \mathfrak{M}_L$ is also the identity mapping on \mathfrak{M}_L .

Consider $\langle g^*, f^* \rangle \in \mathfrak{M}_L$, in order to prove that $\Psi \circ \Phi(\langle g^*, f^* \rangle) = \langle g^*, f^* \rangle$ it is enough to prove that $\psi_1 \circ \phi_1 \circ f^* = f^*$; this is obtained from the following equalities, considering $a \in A$.

$$\begin{aligned}\psi_1 \circ \phi_1 \circ f^*(a) &= (\psi_1 \circ \phi_1)((g^*)^{\uparrow c}(a)) \\ &= (\psi_1 \circ \phi_1)(\psi_1(\inf\{R(a, b) \swarrow^b \phi_2(g^*(b)) \mid b \in B\})) \\ &= \psi_1((\phi_1 \circ \psi_1)(\inf\{R(a, b) \swarrow^b \phi_2(g^*(b)) \mid b \in B\})) \\ &= \psi_1(\inf\{R(a, b) \swarrow^b \phi_2(g^*(b)) \mid b \in B\}) \\ &= (g^*)^{\uparrow c}(a)\end{aligned}$$

□

Therefore, \mathfrak{M} and \mathfrak{M}_L are isomorphic posets and, as $(\mathfrak{M}_L, \preceq_L)$, (\mathfrak{M}, \preceq) are lattices, Φ and Ψ are lattice-isomorphisms.

As a consequence of the previous isomorphisms, in order to obtain the concept lattice (\mathfrak{M}, \preceq) , we firstly use an algorithm to build the concept lattice $(\mathfrak{M}_L, \preceq_L)$ and then, the mapping Φ is applied to obtain \mathfrak{M} .

There exist algorithms developed to obtain the concept lattices where the conjunctors have the same carrier for both arguments, as for instance, Lindig's algorithm [24], or its extension for graded attributes [7], or the efficient algorithm to compute the lattice of all fixpoints of a fuzzy closure operator [8]; however these algorithms cannot be applied to obtain the lattice $(\mathfrak{M}_L, \preceq_L)$ since \uparrow^c, \downarrow^c do not form a Galois connection, and neither $\uparrow^c \downarrow^c$ nor $\downarrow^c \uparrow^c$ are closure operators. Suitable modifications on these algorithms could be enough to obtain an efficient mechanism to obtain the lattice $(\mathfrak{M}_L, \preceq_L)$. As the complexity of the algorithm used depends on the size of L , we should find, whenever possible, the least lattice L such that L_1 and L_2 are L -connected, but this is beyond the scope of this work, and will be studied in the future.

5. Conclusions and future work

Sets of attributes and objects in fuzzy formal concept analysis are usually different and, hence, it might not make sense to evaluate them on the same carrier when interpreted in a fuzzy extension. In this context, the operators used to obtain the concept lattice could be defined by associating different lattices to attributes and objects, see [28].

There exist reasons which suggest the need to evaluate the set of attributes and objects in the same carrier and, in this direction, a new concept lattice has been introduced, in which objects and attributes are evaluated on the same lattice L , although using operators which evaluate objects and attributes in different carriers.

The relationship between the new concept lattice and the alternative one obtained directly considered different carriers to both set of attributes and objects, introduced in [28], has been studied. It is worth to recall that our framework based on L -connected lattices generalizes the well-known approach of concept lattices with hedges [11, 10, 19].

For future work, in order to define an efficient mechanism to obtain the lattice $(\mathfrak{M}_L, \preceq_L)$, modifications in the different algorithms introduced to compute the concept lattices in which the conjunctors have the same carrier for both arguments, will be studied. Moreover, we want to further develop

how the theory presented here can be applied to obtain t-concepts [17, 27] when, originally, the set of attributes and objects are evaluated in different lattices. Finally, another interesting topic to be studied in the short term is to obtain mechanisms to find the least lattice L such that L_1 and L_2 are L -connected.

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