On the Measure of Incoherent Information in Extended Multi-Adjoint Logic Programs

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***Abstract*—In this paper we continue analyzing the introduction of negation into the framework of residuated logic programming [8], [10]; specifically, we focus on extended programs, that is we consider programs with strong negation. The classical approach to extended logic programs consists in considering negated literals as new, independent, ones and, then apply the usual monotonic approach (based on the fix-point semantics and the $T_{\mathbb{P}}$ operator); if the least fix-point so obtained is inconsistent, then the approach fails and no meaning is attached to the program. This paper introduces several approaches to measure consistency (under the term *coherence*) into a multi-adjoint setting.**

I. Introduction and Preliminary Definitions

Inconsistency and conflictive information arise naturally in many situations; e.g. Medical Databases [2], Economics [7], Text Processing [5], etc. Although it is usually considered to be an undesirable feature of logic programs, the right approach is to learn to deal with it, since obtaining a inconsistency-free knowledge-base is not always possible. For instance, if you build a knowledge-base containing information from various newspaper reports about some political event, then the consistence of the knowledge-base is unlikely. As a result, tolerating inconsistency is advisable instead of rejecting it [1]. In fact, conflictive information can be useful by itself: consider a negotiation meeting where each part is searching for a hidden goal; then, the conflictive information in the negotiation could give an idea about the goal of each party.

This paper deals with measures of inconsistency in generalized logic programming. It is worth mentioning that most of the papers in this framework require at least three truth values $\{True, False, Inconsistent\}$ to deal with conflictive information. Yet the multi-valued and fuzzy logic seems a more flexible and advisable environment to develop inconsistent tolerance approaches.

Inconsistency can be seen as composed of two different levels [10]: "*lack of models*" (called instability) and "*contradictory models*" (called incoherence). The former occurs when a set of incompatible rules appears in the logic program, whereas the latter occurs when the existing models assign *contradictory* values to p and $\sim p$.

Our approach in this paper measures the conflictive information, in terms of incoherence, contained in an extended *multi-adjoint* logic program. Previous results have already been obtained by the authors in the residuated case [8]–[10], the novelty in this paper is to provide new measures to the more general and flexible framework of extended multi-adjoint logic programs, which allows for using a greater set of connectives and, especially important in this paper, a number of different negation operators.

In the rest of this section, we recall the preliminary syntactic and semantic definitions related to extended multi-adjoint logic programs.

Definition 1: A *multi-adjoint lattice* $\mathcal{L}$ is a tuple $(L, \leq, *_1, \leftarrow_1, *_2, \leftarrow_2, \dots, *_n, \leftarrow_n)$ such that for all $x, y, z \in L$:

1) $(L, \leq)$ is a complete bounded lattice, with top and bottom elements 1 and 0.
2) for all $i \in \{1, \dots, n\}$, it holds $1 *_i x = x *_i 1 = x$.
3) for all $i \in \{1, \dots, n\}$ the tuple $(*_i, \leftarrow_i)$ forms an adjoint pair, i.e. $z \leq (x \leftarrow_i y)$ iff $y *_i z \leq x$.

In multi-adjoint lattice frameworks, L represents the set of truth-values, operators $*_i$ are interpreted as conjunctions and operators $\leftarrow_i$ as implications. Hereafter, we will use $(L, \leq, *_i, \leftarrow_i)$ to denote multi-adjoint lattices.

As usual, a negation operator over $\mathcal{L}$ is any decreasing mapping $n\colon L \to L$ satisfying $n(0) = 1$ and $n(1) = 0$. In the rest of the paper we will consider a multi-adjoint lattice enriched with negation operators $\sim_j$, i.e $(L, \leq, *_i, \leftarrow_i, \sim_j)$. In order to introduce our logic programs, we will asume a set Π of propositional symbols. *Literals* are defined inductively as follows: any propositional symbol is a literal; if ℓ is a literal then $\sim_j \ell$ is also a literal for any negation $\sim_j \in \mathcal{L}$. Note that under this definition of literals we allow multiply negated propositional symbols; i.e. if $p \in \Pi$, then p and $\sim_1(\sim_2 p)$ are literals. Arbitrary literals will be denoted with the symbol ℓ (possible subscripted), and the set of all literals as Lit.

Definition 2: Given a multi-adjoint lattice with negations $(L, \leq, *_i, \leftarrow_i, \sim_j)$, an *extended*[1] *multi-adjoint logic program* $\mathbb{P}$ is a finite set of weighted rules of the form $\langle F; \vartheta \rangle$ satisfying the following conditions:

- F is a formula of the form $\ell \leftarrow_i \mathcal{B}$ where ℓ is a literal (called the *head* of F) and $\mathcal{B}$ (called the *body* of F) is built from literals $\ell_1, \dots, \ell_n$ and operators $*_i$.
- the weight ϑ is an element of the underlying multi-adjoint lattice L.

[1] Note that the use of *extended* refers to the fact that no default negation is allowed oin our programs, only strong negations.

Rules will be frequently denoted as $\langle \ell \leftarrow \mathcal{B}; \quad \vartheta \rangle$. We consider *facts* as rules with empty body, which are interpreted as a rule $\langle \ell \leftarrow 1; \quad \vartheta \rangle$.

The semantics for extended multi-adjoint logic programs is given as follows.

Definition 3: An *$\mathcal{L}$-interpretation* is a mapping $I\colon Lit \to L$; note that the domain of the interpretation is the set of literals, and it can be lifted to any rule by homomorphic extension.

We say that I *satisfies* a rule $\langle \ell \leftarrow \mathcal{B}; \quad \vartheta \rangle$ if and only if $I(\mathcal{B}) * \vartheta \leq I(\ell)$ or, equivalently, $\vartheta \leq I(\ \ell \leftarrow \mathcal{B})$. Finally, I is a *model* of $\mathbb{P}$ if it satisfies all rules (and facts) in $\mathbb{P}$.

Note that the order relation in the lattice $(L, \leq)$ can be extended over the set of all L-interpretations as follows: *Let I and J be two L-interpretations, then $I \leq J$ if and only if $I(\ell) \leq J(\ell)$ for all literal $\ell \in Lit$.*

For the sake of clarity, hereafter we write *interpretation* instead of $\mathcal{L}$-interpretation and *program* instead of extended multi-adjoint logic program.

II. Least model semantics and coherence

In logic programming (in both crisp and multi-valued frameworks), it is usual to consider two different negation operators: the *strong negation*[2] and *default negation*. The difference between both negations is purely semantical and is related with the way we infer the truth-value of a negated statement. Specifically, a default negated formula $\neg\phi$ is true if and only if ϕ is not true, whereas a strong negated statement $\sim\phi$ is true if and only if $\sim\phi$ can be inferred by the knowledge base. In other words, the truth-value of ϕ determines the truth-value of $\neg\phi$ but does not determine the value of $\sim\phi$; i.e. $\neg$ is compositional but $\sim$ is not. In this paper we are only concerned with programs including strong negations.

In the classical case, the semantics of extended programs is given by the least fix-point of the immediate consequence operator considering the negated literals as 'new' propositional symbols [4]. Hence, when the obtained fix-point turns out to be inconsistent, then the program has no meaning. In fuzzy logic, the semantics is obtained in a similar way by iterating the immediate consequence operator defined in [11] for multi-adjoint logic programs. The crux of the matter now is when one should reject the obtained model. The notion of consistence (or inconsistence) has to be generalized in order to answer this question.

There are many ideas underlying the notion of inconsistence (conflicting inference, inferring contradiction formulas, lack of models, etc) and, astonishingly, each of them entails a different generalization of inconsistency in fuzzy or multi-valued logics. We choose the notion of *coherence* as a convenient generalization of the notion of consistency. It is worth mentioning that *coherence* is closely related to the notions of N-contradictory fuzzy sets [14] and consistency in bilattices [3]. The term *coherence* was considered in order to not overlap with other definitions of fuzzy-consistency in the literature [13]. The notion of coherent interpretation is given below:

[2]Not to be confused with the notion of strong negation [14] on fuzzy sets, where strong negation is defined as any involutive operator.

Definition 4: Let $\mathcal{L}$ be a multi-adjoint lattice. An L-interpretation I is *coherent* if the inequality $I(\sim_i \ell) \leq \dot{\sim}_i I(\ell)$ holds for every literal ℓ and all strong negation $\sim_i \in \mathcal{L}$.

Note that *coherence* allows that two opposite literals, such as p and $\sim p$, live together ... under some requirements. Among all reasons supporting that *coherence* is a good generalization of consistency in fuzzy and multi-valued logic programming frameworks, we allude to just three. Firstly, it is easy to implement, because it only depends on a negation operator; secondly, it allows lack of knowledge (for instance, I such that $I(\ell) = 0$ for all $\ell \in Lit$ is always coherent); finally, our notion of coherence coincides with consistency in the classical framework.

As stated above, given an extended multi-adjoint logic program $\mathbb{P}$ we can obtain its least model by considering the negated literals as new, independent, propositional symbols and then, iterating the immediate consequence operator. So, the notion of coherence applies to programs as follows:

Definition 5: Let $\mathbb{P}$ be an extended residuated logic program, we say that $\mathbb{P}$ is *coherent* if its least model is coherent.

Although the definition of coherent program might look like as a hard restriction, the following property of coherent interpretations shows that a program is coherent if and only if it has, at least, one coherent model.

Proposition 1: Let I and J be two interpretations satisfying $I \leq J$. If J is coherent, then I is coherent as well.

Corollary 1: An extended multi-adjoint logic program is coherent if and only if it has one coherent model.

In order to continue with other mathematical properties of the notion of coherence, take into account that an interpretation I assigns a truth-degree to any negative literal $\sim_i \ell$ independently from the negation operator $\sim_i \in \mathcal{L}$. This way, if we have two different multi-adjoint lattices ($\mathcal{L}^1$ and $\mathcal{L}^2$), and each of them with two different negation operators to determine the coherence of $\sim_i \ell$ ($\sim_i^1$ and $\sim_i^2$), we can talk about the coherence of one interpretation w.r.t. any of these structures. The following result establishes that the lesser the negation operators in $\mathcal{L}$ the most restrictive the condition imposed by coherence.

Proposition 2: Let $\mathcal{L}^1$ and $\mathcal{L}^2$ be two multi-adjoint lattices such that $\sim_j^1 \leq \sim_j^2$ for all $\sim_j^1 \in \mathcal{L}^1$ and $\sim_j^2 \in \mathcal{L}^2$. Then any interpretation I that is coherent w.r.t. $\mathcal{L}^1$ is also coherent w.r.t. $\mathcal{L}^2$.

We can also define an ordering among extended residuated logic programs as follows: Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be two extended programs, then $\mathbb{P}_1 \subseteq \mathbb{P}_2$ if and only if for each rule $\langle r_i; \vartheta_1 \rangle$ in $\mathbb{P}_1$ there exists another rule[3] $\langle r_i; \vartheta_2 \rangle$ in $\mathbb{P}_2$ such that $\vartheta_1 \leq \vartheta_2$.

Proposition 3: Let $\mathbb{P}_1 \subseteq \mathbb{P}_2$ be two extended programs then the least model of $\mathbb{P}_1$ is smaller than the least model of $\mathbb{P}_2$.

Therefore we can say that a bigger program provides more information. Now, the following proposition holds easily.

[3]Note that the only difference between both rules is the assigned weight.

Somehow that holds because incoherence represents an excess of information [9].

Proposition 4: Let $\mathbb{P}_1 \subseteq \mathbb{P}_2$ be two extended programs. If $\mathbb{P}_2$ is coherent then $\mathbb{P}_1$ is coherent as well.

In the rest of the paper we use $M_{\mathbb{P}}$ to denote the least model of an extended residuated logic program $\mathbb{P}$ (either coherent or not).

On the use of various connectives

Once the semantics for extended logic programs has been presented, we can give the reasons why we consider different operators to interpret syntactic logic connectives (conjunctions, disjunctions, implications, strong negations, etc). The motivation in the use of different t-norms, aggregators, implications, etc. is widely studied in [12]: roughly speaking, we can simply say that it is a matter of greater flexibility. In this section, we only include motivation on the use of different strong negation operators.

One interesting feature of the notion of coherence is its link with the notion of N-contradiction and with computation with antonyms [15]. In this way, each $\sim_i p$ can be identified with an antonyms of p. Hence, allowing different strong negations in a extended program allows as well to deal with different antonyms. The following example shows how to incorporate antonyms in logic programming via strong negations. The example also show that a priori, there are neither relation between two different strong negated literals of p nor relation with p and a double strong negation $\sim_i(\sim_j p)$.

Example 1: Let us begin by providing the following semantical identifications of literals: $p \equiv$ *"the car is big"*; $\sim_1 p \equiv$ *"the car is small"*; and $\sim_2 p \equiv$ *"the car is tiny"*. Note that, obviously, p is an antonym of both $\sim p_1$ and $\sim p_2$. How does coherence represent this antonym relationship? Simply by determining upper bounds of the truth-values of $\sim_1 p$ and $\sim_2 p$ by fixing the truth-value of p. Moreover, the use of negation operators to determine these upper bounds imply that *"the greater the truth-value of p, the lesser the truth-values allowed for $\sim_1 p$ and $\sim_2 p$"*; and somehow, also vice versa.

With such meanings of $\sim_1 p$ and $\sim_2 p$, it seems convenient to establish a direct relationship between both negative literals; i.e. the greater the value of $\sim_1 p$, the greater the value of $\sim_2 p$; and vice versa. This relation can be done by incorporating the following two rules into the program:

$$\langle \sim_1 p \leftarrow \sim_2 p \quad ; \vartheta_1 \rangle$$

$$\langle \sim_2 p \leftarrow \sim_1 p \quad ; \vartheta_2 \rangle$$

Let us consider now the following semantical identification of $\sim_3 p$: *"the car is medium-sized"*. In this case we have another antonym of p ... but clearly in a lower degree than $\sim_1 p$. That feature requires the use of different negation operators for $\sim_1$ and $\sim_3$; i.e. to determine two different upper bounds of the truth-values of $\sim_1 p$ and $\sim_3 p$. Note also that in this case, *"medium-sized"* can be considered also an antonym of *"small"*. Hence, somehow we can identify $\sim_3 p$ with a literal with the form $\sim_4(\sim_1 p)$ by incorporating to the program the following two rules:

$$\langle \sim_4(\sim_1 p) \leftarrow \sim_3 p \quad ; 1 \rangle$$
$$\langle \sim_3 p \leftarrow \sim_4(\sim_1 p) \quad ; 1 \rangle$$

Therefore, between $\sim_1 p$ and $\sim_3 p$ there is not a direct relationship, but inverse. In conclusion, firstly we need different negations to represent different antonyms. Secondly, two strong negations of the same literal are not related a priori; neither directly (in this example $\sim_1 p$ and $\sim p_3$) nor inversely (in this example $\sim_1 p$ and $\sim p_2$). Thirdly, there is not necessarily a direct relation between a literal and its double negation (in this example between p and $\sim_4(\sim_1 p) \equiv \sim_3 p$). Finally note that, although we use fuzzy terminology throughout this example, it can be interpreted in a crisp environment. So, also in crisp logic programming the equality $\sim(\sim p) = p$ is not necessarily true; this feature exhibits a clear difference between strong negation and classical negation. □

III. Incoherence w.r.t. literals

Here, we deal with the atomic pieces of incoherence contained in each literal. Basically, the purpose of this section is to generalize the measure of incoherence defined on literals in [10]. In a multi-adjoint framework we have two ways to do that: on the one hand, we can measure the incoherence associated with each negated literal by taking into account only its negation; on the other hand, we can define a measure of incoherence of a literal by considering all its possible negations. Let us begin by the former.

A. Incoherence on each negated literals

We will measure the incoherence generated by one interpretation I on each pair $(\ell, \sim_i \ell)$. First of all we need to define when a pair $(\ell, \sim_i \ell)$ is considered to be coherent (respectively, incoherent) w.r.t. an interpretation I.

Definition 6: Let $\mathcal{L}$ be a multi-adjoint lattice and let I be an interpretation. We say that $(\ell, \sim_i \ell)$ is coherent w.r.t. I if and only if the inequality $I(\sim_i \ell) \leq \dot{\sim}_i I(\ell)$ holds. Otherwise the pair $(\ell, \sim_i \ell)$ is called incoherent.

Hereafter we will simply state $(\ell, \sim_i \ell)$ *is coherent (resp. incoherent)* without mentioning the interpretation, provided it is not ambiguous. Note that the definition above somehow determines a crisp measure of incoherence for the pair $(\ell, \sim_i \ell)$; 1 if it is incoherent and 0 otherwise. However, this crisp way to measure incoherence is unable to represent degrees of incoherence that, somehow, are inherent to the pairs $(\ell, \sim_i \ell)$. To represent such limitation, consider the following two interpretations with the same simple domain $\{p, \sim p\}$:

$$I_1(p) = 0.5 \qquad I_1(\sim p) = 0.6$$
$$I_2(p) = 1 \qquad I_2(\sim p) = 0.9$$

and consider the usual negation $\sim(x) = 1 - x$ as the operator used to determine the coherence. Certainly, the pair $(p, \sim p)$ is incoherent w.r.t. both I_1 and I_2. By using the crisp measure of incoherence above, the incoherence of $(p, \sim p)$ is 1 in both cases. However, the pair $(p, \sim p)$ seems to be *more* incoherent w.r.t. I_2 than w.r.t. I_1 since I_2 makes the pair $(p, \sim p)$ to break the coherence condition to a "bigger degree" than I_1.

To measure such degree of incoherence, we propose to assign a value to each element in the lattice corresponding to the inherent information it contains. Such assignment is done by associating an *information measure* (definition below) to the multi-adjoint lattice.

Definition 7: Let $(L, \leq)$ be a lattice, an information measure is an operator $m\colon L \to \mathbb{R}^+$ such that the following holds:

- $m(x) = 0$ if and only if $m(x) = 0$.
- m is monotonic.
- $m(\sup(x,y)) \geq m(x) + m(y) - m(\inf(x,y))$ for all $x, y \in L$.

The two first conditions required by the definition of information measure need little explanation: the only element in the lattice which provides no information is 0 and the closer the element to 1, the greater the information inherent in it. The third condition needs a more elaborated explanation. It represents that the information contained in the supremum of two elements should reflect the amount of information contained in each element separately; therefore, $m(\sup(x,y))$ should be greater than $m(x)+m(y)-m(\inf(x,y))$, since the latter part is counted twice when adding $m(x)$ and $m(y)$. Note that the third item imposes no restriction if the lattice is linear. Note finally that the notion of information measure is related with fuzzy measures [16].

From now on we will assume that our multi-adjoint lattices have an associated information measure m. To measure the degree of incoherence of a pair $(\ell, \sim_i \ell)$ w.r.t. I we focus on the minimal amount of information we have to remove from $I(\ell)$ and $I(\sim_i \ell)$ in order to recover the coherence of the pair $(\ell, \sim_i \ell)$. To do that, we define previously the set of coherent pairs w.r.t. one negation operator $\sim_i$ as:

$$\Delta^{\sim_i} = \{(x, y) \in L \times L \colon y \leq \dot{\sim}_i(x)\ \}$$

Basically, $\Delta^{\sim_i}$ determines the values that an interpretation can assign to a pair of literals $(\ell, \sim_i \ell)$ in order to keep coherence. Actually, this definition entails the following characterization of coherent interpretations.

Proposition 5: Let $\mathcal{L}$ be a multi-adjoint lattice with negations and let I be an L-interpretation. Then, I is coherent if and only if the pair $(I(\ell), I(\sim_i \ell)) \in \Delta^{\sim_i}$ for all $\ell \in Lit$ and all $\sim_i \in \mathcal{L}$.

Once the set of coherent pairs w.r.t. a negation operator has been presented, we define the *measure of incoherence of a pair* $(\ell, \sim_i \ell)$ *w.r.t. one interpretation* I (denoted by $\mathcal{I}^{\mathcal{L}}((\ell, \sim_i \ell); I)$) as follows:

$$\inf_{\substack{(x,y)\in\Delta^{\sim_i}\\(x,y)\leq(I(\ell),I(\sim_i\ell))}} \big\{m(I(\ell)) - m(x) + m(I(\sim_i \ell)) - m(y)\big\} \quad (1)$$

where the ordering within $\Delta^{\sim_i}$ is considered componentwise.

As we stated above, the idea underlying the definition of $\mathcal{I}^{\mathcal{L}}((\ell, \sim_i \ell); I)$ is to determine the minimum amount of information that has to be removed from $I(\ell)$ and $I(\sim \ell)$ in order to recover coherence. If we remove a certain amount of information α from $I(\ell)$, and a certain amount of information β from $I(\sim \ell)$, then we actually modify information for an amount of $\alpha + \beta$. Therefore, the measure $\mathcal{I}^{\mathcal{L}}((\ell, \sim_i \ell); I)$ determines, in some sense, the least amount of information that has to be removed in order to turn the pair $(\ell, \sim_i \ell)$ coherent w.r.t. I. Note that $\mathcal{I}^{\mathcal{L}}((\ell, \sim_i \ell); I) \geq 0$ since $m(I(\ell)) \geq m(x)$ and $m(I(\sim_i \ell)) \geq m(y)$ for all $(x, y) \leq (I(\ell), I(\sim_i \ell))$.

Example 2: For the interpretation I_1 and I_2 given above to motivate this measure, we obtain $\mathcal{I}^{\mathcal{L}}(I(p), I(\sim p), I_1) = 0.1$ and $\mathcal{I}^{\mathcal{L}}(I(p), I(\sim p), I_2) = 0.9$ by considering the information measure induced by the Euclidean norm. □

Remark 1: The definition of $\mathcal{I}^{\mathcal{L}}((\ell, \sim_i \ell); I)$ collapses to an extremely simple and intuitive form in the specific case of the unit interval ($L = [0, 1]$), the information measure induced by the Euclidean norm, and $\sim_i(x) = 1 - x$:

$$\mathcal{I}^{\mathcal{L}}((\ell, \sim_i \ell); I) = \begin{cases} 0 & \text{if } (\ell, \sim_i \ell) \text{ is coherent} \\ I(\sim_i \ell) - \dot{\sim}_i I(\ell) & \text{otherwise} \end{cases}$$

Below we present some properties of the measure $\mathcal{I}^{\mathcal{L}}((\ell, \sim_i \ell); I)$.

Proposition 6: If the pair $(\ell, \sim_i \ell)$ is coherent w.r.t. I then $\mathcal{I}^{\mathcal{L}}((\ell, \sim_i \ell); I) = 0$.

The converse is not true in general.

Example 3: Consider the lattice $[0, 1]$ with the information measure induced by the Euclidean norm and the following negation operator

$$\dot{\sim}(x) = \begin{cases} 0 & \text{if } x \geq 0.5 \\ 1 & \text{if } x < 0.5 \end{cases}$$

Then the pair $(p, \sim p)$ w.r.t. the interpretation $I(p) = 0.5$, $I(\sim p) = 0.5$ is incoherent and $\mathcal{I}^{\mathcal{L}}((p, \sim p); I) = 0$. The reason of the null measure is because we can recover the coherence of the pair $(p, \sim p)$ by removing a piece of information as small as we want from I. □

Although the equivalence between null measure of incoherence and coherence is not true in general, there exist frameworks in which both notions coincide:

Proposition 7: Assume that the multi-adjoint lattice is finite and the information measure used is injective. Then, $\mathcal{I}^{\mathcal{L}}((\ell, \sim_i \ell); I) = 0$ if and only if $(\ell, \sim_i \ell)$ is coherent w.r.t. I.

Proposition 8: On the unit interval $[0, 1]$ and under a continuous and injective information measure; if the operator associated to strong negation $\sim_i$ is continuous, then $\mathcal{I}^{\mathcal{L}}((\ell, \sim_i \ell); I) = 0$ iff $(\ell, \sim_i \ell)$ is coherent w.r.t. I.

The following result relates the ordering between L-interpretations and the measure of incoherence for pairs of literals: the greater an L-interpretation the more incoherent information assigns to literals.

Proposition 9: Let $I \leq J$ be two L-interpretations. Then $\mathcal{I}^{\mathcal{L}}((\ell, \sim_i \ell); I) \leq \mathcal{I}^{\mathcal{L}}((\ell, \sim_i \ell); J)$ for all pair of literals $(\ell, \sim_i \ell)$.

The following proposition shows that $\mathcal{I}^{\mathcal{L}}((\ell, \sim_i \ell); I)$ is bounded by the inherent information in $I(\ell)$ and $I(\sim_i \ell)$:

Proposition 10: Let I be an L-interpretation, then

$$\mathcal{I}^{\mathcal{L}}((\ell, \sim_i \ell); I) \leq \min\big\{m(I(\sim_i \ell)), m(I(\ell))\big\}$$

As a consequence of the proposition above, $m(1)$ is actually an upper bound for the value of each measure $\mathcal{I}^{\mathcal{L}}((\ell, \sim_i \ell); I)$.

We provide below an example to illustrate how to calculate the measure of incoherence $\mathcal{I}^{\mathcal{L}}$

Example 4: Consider on the unit interval $[0,1]$ the interpretation given by the following table:

x	p	$\sim_1 p$	q	$\sim_1 q$	$\sim_2 q$	r
$I(x)$	0.7	0.7	0.8	0.3	0.7	1

Let us interpret the strong negations by using the operators $\sim_1(x) = 1 - x^2$ and $\sim_2(x) = 1 - x$. If we use the Euclidean norm as the information measure of $[0,1]$, the incoherence measure of the pair of literals $(p, \sim_1 p)$ is equivalent to achieve the minimum of the mapping $f(x, y) = 0.7 - x + 0.7 - y = 1.4 - x - y$ in the compact set $\{(x, y) \in [0, 0.7] \times [0, 0.7] \colon y \leq 1 - x^2\}$. Specifically $\mathcal{I}^{\mathcal{L}}((p, \sim_1 p); I) = 0.15$. Similarly, the value $\mathcal{I}^{\mathcal{L}}((q, \sim_2 q); I) = 0.5$ is the minimum of the mapping $g(x, y) = 0.8 - x + 0.7 - y$ in the compact $\{(x, y) \in [0, 0.8] \times [0, 0.7] \colon y \leq 1 - x\}$. The rest of pairs of opposite literals are coherent, so in that case $\mathcal{I}^{\mathcal{L}}((\ell, \sim_i \ell); I) = 0$. □

B. General measures on literals

The measure of incoherence given in the previous section is defined on pairs of literals with the form $(\ell, \sim_i \ell)$. Thus, $\mathcal{I}^{\mathcal{L}}$ defines a degree of incoherence for each literal and each strong negation $\sim_i$. The aim of this section is to define a general degree of incoherence just on literals without fixing a strong negation; i.e. by considering all possible negations of the literal. As above, first we define the notion of coherent (resp. incoherent) literal w.r.t. one interpretation.

Definition 8: Let $\mathcal{L}$ be a multi-adjoint lattice and let I be an interpretation. We say that a literal ℓ is coherent w.r.t. I if and only if the inequality $I(\sim_i \ell) \leq \dot{\sim}_i I(\ell)$ holds for all strong negation $\sim_i \in \mathcal{L}$. Otherwise the literal ℓ is called incoherent.

As in the case of coherent/incoherent pairs, hereafter we will simply state ℓ *is coherent/incoherent* without mentioning the interpretation, whenever it is not ambiguous. We can also define a crisp measure of incoherence for literals by using the definition above, but as in the case of coherent/incoherent pairs of literals, this measure is not able to deal with inherent degrees of incoherence in literals. So, we use the same idea underlying $\mathcal{I}^{\mathcal{L}}$: to determine the least amount of information we have to remove from I in order to recover the coherence of ℓ. Hence, given a multi-adjoint lattice $\mathcal{L} = (L, \leq)$ with negations $\{\sim_1, \sim_2, \dots, \sim_n\}$, we define the set of coherent tuples w.r.t. $\mathcal{L}$ as

$$\Delta_{\mathcal{L}} = \{(x, y_1, \dots, y_n) \in L^{n+1} \colon y_i \leq \dot{\sim}_i(x)\ \}$$

As $\Delta^{\sim_1}$, the tuples in $\Delta_{\mathcal{L}}$ represent the allowed values on every coherent L-interpretation to one literal ℓ and all its possible negations $\sim_1 \ell, \dots \sim_n \ell$. Moreover, coherent interpretations can be characterized by using $\Delta_{\mathcal{L}}$ as follows:

Proposition 11: Let $\mathcal{L}$ be a multi-adjoint lattice with negations $\{\sim_1, \sim_2, \dots, \sim_n\}$ and let I be an L-interpretation. Then, I is coherent if and only if for all $\ell \in Lit$ the tuple $(I(\ell), I(\sim_1 \ell), \dots, I(\sim_n \ell)) \in \Delta_{\mathcal{L}}$.

Now, we define a measure of incoherence for literals in a similar way to $\mathcal{I}^{\mathcal{L}}$ but by considering the set $\Delta_{\mathcal{L}}$ instead of $\Delta^{\sim_i}$. Formally, we define the measure of incoherence $\mathcal{I}^{\mathcal{G}}(\ell; I)$ by:

$$\inf_{\substack{(x, y_i) \in \Delta_{\mathcal{L}} \\ x \leq I(\ell) \\ y_i\ \leq I(\sim_i \ell)}} \Big\{ m(I(\ell)) - m(x) + \Big(\sum_i^n m(I(\sim_i \ell)) - m(y_i) \Big) \Big\} \quad (2)$$

The subsequent result shows an intuitive connection between the measures $\mathcal{I}^{\mathcal{L}}$ and $\mathcal{I}^{\mathcal{G}}$. Specifically, the least amount of information necessary to be removed from the tuple $(I(\ell), I(\sim_1 \ell), \dots, I(\sim_n \ell))$ in order to recover the coherence of $\ell, \sim_1 \ell, \dots, \sim_n \ell$ is

- bigger or equal than the amount of information necessary to be removed just from $(I(\ell), I(\sim_i \ell))$ in order to recover the coherence only of the pair $(\ell, \sim_i \ell)$.
- but lesser or equal than the sum of all amount of information necessary to be removed from each $(I(\ell), I(\sim_i \ell))$ in order to recover (separately) the coherence of pair $(\ell, \sim_i \ell)$.

In other words, we need to remove a lesser amount of information from I if we deal directly with the incoherence of all negated literals $\{\sim_i \ell\}_i$ of ℓ than if we deal independently with each incoherent pair $(\ell, \sim_i \ell)$.

Proposition 12: Let $\mathcal{L}$ be a multi-adjoint lattice and let I be an interpretation. Then for all $\ell \in Lit$ and $\sim_i \in \mathcal{L}$ we have that:

$$\sum_{\sim_j \in \mathcal{L}} \mathcal{I}^{\mathcal{L}}((\ell, \sim_j \ell); I) \geq \mathcal{I}^{\mathcal{G}}(\ell; I) \geq \mathcal{I}^{\mathcal{L}}((\ell, \sim_i \ell); I)$$

The proposition above entails two interesting corollaries.

Corollary 2: Let $\mathcal{L}$ be a multi-adjoint lattice, let I be an interpretation and let ℓ be a literal. If $\mathcal{I}^{\mathcal{G}}(\ell; I) = 0$ then $\mathcal{I}^{\mathcal{L}}((\ell, \sim_i \ell); I) = 0$ for all pair $(\ell, \sim_i \ell)$.

Corollary 3: Let $\mathcal{L}$ be a multi-adjoint lattice, let I be an interpretation and let ℓ be a literal. If $\mathcal{I}^{\mathcal{L}}((\ell, \sim_i \ell); I) = 0$ for all $\sim_i \in \mathcal{L}$ then $\mathcal{I}^{\mathcal{G}}(\ell; I) = 0$.

We introduce below one example on how to compute the measure of incoherence $\mathcal{I}^{\mathcal{G}}(\ell; I)$.

Example 5: Consider the $[0,1]$-interpretation given by the following table:

x	p	$\sim_1 p$	$\sim_2 p$
$I(x)$	0.8	0.64	0.7

Let us consider as well that the negation operators associated to $\sim_2$ and $\sim_3$ are $\dot{\sim}_1(x) = 1 - x^2$ and $\dot{\sim}_2(x) = 1 - x$ respectively. The values of $\mathcal{I}^{\mathcal{L}}((p, \sim_1 p); I) = 0.4$ and $\mathcal{I}^{\mathcal{L}}((p, \sim_2 p); I) = 0.25$ can be calculated by using a similar approach as that in Example 4. On the other hand, the calculation of the value $\mathcal{I}^{\mathcal{G}}(p; I)$ is equivalent to determine the solution of the following non-linear optimization problem[4]:

$$\begin{aligned} \min: & \ 2.14 - x - y - z \\ \text{subject to}: & \ y \leq 1 - x^2 \\ & \ z \leq 1 - x \\ & \ 0 \leq x \leq 0.8 \\ & \ 0 \leq y \leq 0.64 \\ & \ 0 \leq z \leq 0.7 \end{aligned}$$

where we obtain $\mathcal{I}^{\mathcal{G}}(p; I) = 0.5$. Note that, as expected, the chain of inequalities in Proposition 12 holds, with the values

$$0.4 + 0.25 = 0.65 \geq 0.5 \geq 0.4$$

□

The following results on $\mathcal{I}^{\mathcal{G}}$ reproduce the previous study done on $\mathcal{I}^{\mathcal{L}}$ above. Actually, the following propositions are the respective results linked with Propositions 6, 7, 8 and 9 given on $\mathcal{I}^{\mathcal{L}}$. Let us begin by showing that coherence implies a null measure of incoherence.

Proposition 13: If the literal ℓ is coherent w.r.t. I then $\mathcal{I}^{\mathcal{G}}(\ell; I) = 0$.

The following two results establish frameworks where coherence and null measure of incoherence are equivalent.

Proposition 14: Assume that the multi-adjoint lattice is finite and the information measure used is injective. Then, $\mathcal{I}^{\mathcal{G}}(\ell; I) = 0$, if and only if ℓ is coherent w.r.t. I.

Proposition 15: On the unit interval $[0, 1]$ and under a continuous and injective information measure; if all operator associated to strong negations is continuous, then $\mathcal{I}^{\mathcal{G}}(\ell; I) = 0$ iff ℓ is coherent w.r.t. I.

In $\mathcal{I}^{\mathcal{G}}$, as in the case of the measure $\mathcal{I}^{\mathcal{L}}$, the greater an L-interpretation the more incoherent information contains.

Proposition 16: Let $I \leq J$ be two L-interpretations. Then $\mathcal{I}^{\mathcal{G}}(\ell; I) \leq \mathcal{I}^{\mathcal{G}}(\ell; J)$ for all literal ℓ.

C. Degrees of concentration and dispersion of incoherence

To end this section, we present two new measures related with incoherence just by comparing the measures $\mathcal{I}^{\mathcal{L}}$ and $\mathcal{I}^{\mathcal{G}}$. Note that the former represents a local degree of contradiction by considering just a pair of opposite literals, whereas the latter represents a global degree of contradiction by considering all opposite literal of a fixed literal. Therefore it is interesting to study the information that both measures can provide by comparison. For instance we can measure how "local" is the incoherence of $\mathcal{I}^{\mathcal{L}}((\ell, \sim_i \ell); I)$ with respect to the global degree $\mathcal{I}^{\mathcal{G}}(\ell; I)$. That is, we define the degree of concentration of coherence in $(\ell, \sim_i \ell)$ w.r.t. an interpretation I as:

$$\mathcal{I}^C((\ell, \sim_i \ell); I) = \begin{cases} 0 & \text{if } \mathcal{I}^{\mathcal{G}}(\ell; I) = 0 \\ \frac{\mathcal{I}^{\mathcal{L}}(\sim_i \ell; I)}{\mathcal{I}^{\mathcal{G}}(\ell; I)} & \text{Otherwise} \end{cases} \tag{3}$$

On the other hand we can be interested in measuring how "disperse" is the global incoherence of $\mathcal{I}^{\mathcal{G}}(\ell; I)$ among the local incoherences $\mathcal{I}^{\mathcal{L}}((\ell, \sim_i \ell); I)$. Hence, we define the dispersion of the incoherence of the literal ℓ w.r.t. an interpretation I as:

$$\mathcal{I}^D(\ell; I) = \begin{cases} 0 & \text{if } \mathcal{I}^{\mathcal{G}}(\ell; I) = 0 \\ \frac{\sum_{\sim_i \in \mathcal{L}} \mathcal{I}^{\mathcal{L}}(\sim_i \ell; I)}{\mathcal{I}^{\mathcal{G}}(\ell; I)} & \text{Otherwise} \end{cases} \tag{4}$$

The following result states that zero concentration is equivalent to local coherence, whereas zero dispersion is equivalent to global coherence.

Proposition 17: Let $\mathcal{L}$ be a multi-adjoint lattice, let I be an interpretation and let ℓ be a literal. Then:

- $\mathcal{I}^C((\ell, \sim_i \ell); I) = 0$ if and only if $\mathcal{I}^{\mathcal{L}}((\ell, \sim_i \ell); I) = 0$
- $\mathcal{I}^D(\ell; I) = 0$ if and only if $\mathcal{I}^{\mathcal{G}}(\ell; I) = 0$

Example 6: Reconsider Example 5. The measure of concentration and dispersion of coherence for the case of p are:

$$\mathcal{I}^C((p, \sim_1 p); I) = \frac{0.4}{0.5} = 0.8 \quad \mathcal{I}^C((p, \sim_2 p); I) = \frac{0.25}{0.5} = 0.5$$

$$\mathcal{I}^D(p; I) = \frac{0.65}{0.5} = 1.3$$

□

IV. Incoherence on Extended Programs

As the main goal of the paper is to define measures of incoherence on extended logic programs, in this section, we will provide some extensions of the measures $\mathcal{I}^{\mathcal{L}}$ and $\mathcal{I}^{\mathcal{G}}$ already defined on literals in Section III. The idea underlying in the extension consists in identifying each program $\mathbb{P}$ with its least model $M_{\mathbb{P}}$.

In principle, there are three ways to extend such measures in an extended program $\mathbb{P}$: either estimating the average number of incoherent literals (resp. pair of literals), or estimating the maximal size of incoherence, or estimating the average size of incoherence in the least model of $\mathbb{P}$. For the former, we denote the number of incoherent literals w.r.t. $M_{\mathbb{P}}$ (Definition 8) as $\mathcal{NI}(\mathbb{P})$ and the number of incoherent pairs of opposite literals w.r.t. $M_{\mathbb{P}}$ (Definition 6) as $\mathcal{NIP}(\mathbb{P})$. So we can consider the measures of incoherence $\mathcal{I}_1^{\mathcal{L}}(\mathbb{P})$ and $\mathcal{I}_1^{\mathcal{G}}(\mathbb{P})$ defined on an extended programs $\mathbb{P}$ as:

$$\mathcal{I}_1^{\mathcal{L}}(\mathbb{P}) = \frac{\mathcal{NIP}(\mathbb{P})}{|Lit_{\mathbb{P}}| - |\Pi_{\mathbb{P}}|} \tag{5}$$

$$\mathcal{I}_1^{\mathcal{G}}(\mathbb{P}) = \frac{\mathcal{NI}(\mathbb{P})}{|Lit_{\mathbb{P}}|} \tag{6}$$

The formula for $\mathcal{I}_1^{\mathcal{G}}$ does not need explanation. For $\mathcal{I}_1^{\mathcal{L}}$, on the other hand, the value $|Lit_{\mathbb{P}}| - |\Pi_{\mathbb{P}}|$ is just the number of negative literals occurring in $\mathbb{P}$. It is clear that the number of literals is always greater or equal than the number propositional variables occurring in $\mathbb{P}$, and the measure $\mathcal{I}_1^{\mathcal{L}}(\mathbb{P})$ is undefined when $|Lit_{\mathbb{P}}| = |\Pi_{\mathbb{P}}|$, but this happens just when the strong negation does not appear in $\mathbb{P}$ and therefore, the measure of incoherence makes no sense. Anyway, we can extend the domain of $\mathcal{I}_1^{\mathcal{L}}$ to every extended program by defining $\mathcal{I}_1^{\mathcal{L}}(\mathbb{P}) = 0$ if $\mathbb{P}$ does not contain strong negations.

Note finally that the values of $\mathcal{I}_1^{\mathcal{L}}$ and $\mathcal{I}_1^{\mathcal{G}}$ belong to the unit interval $[0, 1]$. If $\mathcal{I}_1^{\mathcal{L}}(\mathbb{P}) = 0$ (resp. $\mathcal{I}_1^{\mathcal{G}}(\mathbb{P}) = 0$) then there are

[4]This kind of non-linear problem can be resolved by the well-known *Karush-Kuhn-Tucker* method [6].

no incoherent literals in $\mathbb{P}$, that is, $\mathbb{P}$ is a coherent program. However, if $\mathcal{I}_1^{\mathcal{L}}(\mathbb{P}) = 1$ (resp. $\mathcal{I}_1^{\mathcal{G}}(\mathbb{P}) = 1$) then every pair of opposite literals (resp. every literal) is incoherent in $\mathbb{P}$.

Obviously, the measures $\mathcal{I}_1^{\mathcal{L}}$ and $\mathcal{I}_1^{\mathcal{G}}$ are unable to deal with the inherent degree of incoherence measured by $\mathcal{I}^{\mathcal{L}}$ and $\mathcal{I}^{\mathcal{G}}$ on literals. Therefore it is necessary to introduce new measures of incoherence. The measures $\mathcal{I}_2^{\mathcal{L}}$ and $\mathcal{I}_2^{\mathcal{G}}$ focus on estimating the maximal size of incoherence in $\mathbb{P}$. So, given an extended program $\mathbb{P}$, we consider:

$$\mathcal{I}_2^{\mathcal{L}}(\mathbb{P}) = \max_{\sim_i \ell \in Lit_{\mathbb{P}}} \{\mathcal{I}((\ell, \sim_i \ell); M_{\mathbb{P}})\} \tag{7}$$

and

$$\mathcal{I}_2^{\mathcal{G}}(\mathbb{P}) = \max_{\ell \in Lit_{\mathbb{P}}} \{\mathcal{I}^{\mathcal{G}}(\ell; M_{\mathbb{P}})\} \tag{8}$$

The following relationship between $\mathcal{I}_2^{\mathcal{L}}(\mathbb{P})$ and $\mathcal{I}_2^{\mathcal{G}}(\mathbb{P})$ is a consequence of Proposition 12.

Proposition 18: Let $\mathbb{P}$ be an extended program. Then $\mathcal{I}_2^{\mathcal{L}}(\mathbb{P}) \leq \mathcal{I}_2^{\mathcal{G}}(\mathbb{P})$

On the other hand, other measures $\mathcal{I}_3^{\mathcal{L}}$ and $\mathcal{I}_3^{\mathcal{G}}$ can be defined by focusing on estimating the average size of incoherence in $\mathbb{P}$. In this case, given an extended program $\mathbb{P}$, we can consider:

$$\mathcal{I}_3^{\mathcal{L}}(\mathbb{P}) = \frac{\sum_{\sim_i \ell \in Lit_{\mathbb{P}}} \mathcal{I}((\ell, \sim_i \ell); M_{\mathbb{P}})}{\sum_{\sim_i \ell \in Lit_{\mathbb{P}}} \mathcal{I}((\ell, \sim_i \ell); I_{\top})} \tag{9}$$

and

$$\mathcal{I}_3^{\mathcal{G}}(\mathbb{P}) = \frac{\sum_{\ell \in Lit_{\mathbb{P}}} \mathcal{I}^{\mathcal{G}}(\ell; M_{\mathbb{P}})}{\sum_{\ell \in Lit_{\mathbb{P}}} \mathcal{I}^{\mathcal{G}}(\ell; I_{\top})} \tag{10}$$

The quotients in the definition of $\mathcal{I}_3^{\mathcal{L}}$ and $\mathcal{I}_3^{\mathcal{G}}$ represent the maximal degree of incoherence between extended programs; since Propositions 9 and 16 entail that the interpretation $I_{\top}$ (defined by $I_{\top}(\ell) = 1$ for all $\ell \in Lit$) assigns to every literal the maximal degree of incoherence. Thus, for all extended program $\mathbb{P}$ the measures $\mathcal{I}_3(\mathbb{P})$ and $\mathcal{I}_3^{\mathcal{G}}(\mathbb{P})$ belong to the unit interval $[0, 1]$. However, due to the syntactic structure of an extended program, it could be suitable to consider different quotients; the following example illustrates this fact.

Example 7: Consider the following multi-adjoint lattice with negations $\mathcal{L} = ([0,1], \leq, *_P, \leftarrow_P, *_Ł, \leftarrow_Ł, *_G, \leftarrow_G, \sim_1(x) = 1 - x^2, \sim_2(x) = 1 - x, \sim_3(x) = 1 - \sqrt{x})$, and the extended program $\mathbb{P}$ defined by:[5]

$\langle \sim_1 p \leftarrow_P q *_G \sim_3 s \quad ; 0.8\rangle$ $\qquad$ $\langle \sim_2 p \leftarrow_Ł \sim_3 s \quad ; 0.7\rangle$
$\langle p \leftarrow_G t \quad ; 0.8\rangle$ $\qquad$ $\langle \sim_3 s \leftarrow_Ł \quad ; 1\rangle$
$\langle q \leftarrow_P \quad ; 0.8\rangle$ $\qquad$ $\langle t \leftarrow_G \quad ; 0.9\rangle$

The least model of $\mathbb{P}$ is given by the following table:

x	p	$\sim_1 p$	$\sim_2 p$	q	t	$\sim_3 s$
$M_{\mathbb{P}}(x)$	0.8	0.64	0.7	0.8	0.9	1

The only incoherent literal in $\mathbb{P}$ is p and the only incoherent pairs of opposite literals are $(p, \sim_1 p)$ and $(p, \sim_2 p)$. This implies that the measures of incoherence $\mathcal{I}^{\mathcal{L}}$ and $\mathcal{I}^{\mathcal{G}}$ are null for the rest of literals and pairs of opposite literals. Thus we have:

$$\mathcal{I}_1^{\mathcal{L}}(\mathbb{P}) = \frac{2}{3} \approx 0.66 \qquad \mathcal{I}_1^{\mathcal{G}}(\mathbb{P}) = \frac{1}{6} \approx 0.16$$

Note that in Example 5 we already computed the measures $\mathcal{I}^{\mathcal{L}}(\sim_1 p; M_{\mathbb{P}}) = 0.4$, $\mathcal{I}^{\mathcal{L}}(\sim_2 p; M_{\mathbb{P}}) = 0.25$ and $\mathcal{I}^{\mathcal{G}}(p; M_{\mathbb{P}}) = 0.5$. Therefore, the measures $\mathcal{I}_2(\mathbb{P}) = 0.4$ and $\mathcal{I}_2^{\mathcal{G}}(\mathbb{P}) = 0.5$ are easily obtained. To compute the values of the measures $\mathcal{I}_3^{\mathcal{L}}$ and $\mathcal{I}_3^{\mathcal{G}}$, we have to take into account that $\mathcal{I}(\sim_1 \ell; I_{\top}) = 0.75$ and that $\mathcal{I}(\sim_2 \ell; I_{\top}) = \mathcal{I}(\sim_3 \ell; I_{\top}) = \mathcal{I}^{\mathcal{G}}(p; \top) = 1$ (such values are obtained in a similar way as in Example 5). Hence:

$$\mathcal{I}_3^{\mathcal{L}}(\mathbb{P}) = \frac{0.65}{2.75} \approx 0.23 \qquad \mathcal{I}_3^{\mathcal{G}}(\mathbb{P}) = \frac{0.5}{6} \approx 0.08$$

The big difference between the values $\mathcal{I}_1^{\mathcal{L}}(\mathbb{P})$ and $\mathcal{I}_3^{\mathcal{L}}(\mathbb{P})$ with respect to $\mathcal{I}_1^{\mathcal{G}}(\mathbb{P})$ and $\mathcal{I}_3^{\mathcal{G}}(\mathbb{P})$ is due to the quotients considered in such measures. The measures based on $\mathcal{I}^{\mathcal{L}}$ use pairs of opposite literals, whereas the measures based on $\mathcal{I}^{\mathcal{G}}$ consider just literals. Thus, somehow, $\mathcal{I}_1^{\mathcal{L}}(\mathbb{P})$ and $\mathcal{I}_3^{\mathcal{L}}(\mathbb{P})$ consider only the atomic pairs in which incoherence can be measured by ignoring the non-negated literals in the program. For instance, in this example $\mathcal{I}_1^{\mathcal{L}}(\mathbb{P})$ and $\mathcal{I}_3^{\mathcal{L}}(\mathbb{P})$ ignore, among others, all the pairs of opposite literals built from the propositional symbols q and t. However, $\mathcal{I}_1^{\mathcal{G}}(\mathbb{P})$ and $\mathcal{I}_3^{\mathcal{G}}(\mathbb{P})$ consider all literals in the program. Obviously, this is not a bad feature, but sometimes we could be interested just in literals which, by the syntactic structure of $\mathbb{P}$, could contain incoherence. For instance, in this example there are no negations of q and t in $\mathbb{P}$, so these propositional symbols cannot be incoherent. Hence, one could consider appropriate to remove these literals from the quotient of $\mathcal{I}_1^{\mathcal{G}}(\mathbb{P})$ and $\mathcal{I}_3^{\mathcal{G}}(\mathbb{P})$ to estimate the average of the size of incoherence. □

As a consequence of the example above, it seems interesting to be able to substitute the quotients in $\mathcal{I}_3^{\mathcal{L}}(\mathbb{P})$ and $\mathcal{I}_3^{\mathcal{G}}(\mathbb{P})$ by another value to represent different aspects of incoherence, for instance, if we were interested in estimating the average of the size of incoherence just with respect to the literals "susceptible" of being incoherent. Aiming at this goal, in a general way, leads to the definition of weighted measure of incoherence. Thus, given an extended program $\mathbb{P}$ and a set of weights (with the form $\{\vartheta_{(\ell,\sim_i\ell)}\}$, resp. $\{\vartheta_\ell\}$, with $\ell \in Lit_{\mathbb{P}}$) we consider:

$$\mathcal{I}_4^{\mathcal{L}}(\mathbb{P}; \{\vartheta_{(\ell,\sim_i\ell)}\}) = \sum_{\sim_i \ell \in Lit_{\mathbb{P}}} \vartheta_{(\ell,\sim_i\ell)} \cdot \mathcal{I}^{\mathcal{L}}((\ell, \sim_i \ell); M_{\mathbb{P}}) \tag{11}$$

and

$$\mathcal{I}_4^{\mathcal{G}}(\mathbb{P}; \{\vartheta_\ell\}) = \sum_{\ell \in Lit_{\mathbb{P}}} \vartheta_\ell \cdot \mathcal{I}^{\mathcal{G}}(\ell; M_{\mathbb{P}}) \tag{12}$$

As one could expect, these new measures of incoherence generalize $\mathcal{I}_3^{\mathcal{L}}$ and $\mathcal{I}_3^{\mathcal{G}}$, as stated in the proposition below:

Proposition 19: Let $\mathbb{P}$ be an extended program defined on a multi-adjoint lattice $\mathcal{L}$. Then, if we consider the set of weights $\{\vartheta_{(\ell,\sim_i\ell)}\}$ (reps. $\{\overline{\vartheta_\ell}\}$) defined by

$$\vartheta_{(\ell,\sim_i\ell)} = \frac{1}{\sum\limits_{\sim_i \ell \in Lit_{\mathbb{P}}} \mathcal{I}(\sim_i \ell; I_{\top})} \text{ and } \overline{\vartheta_\ell} = \frac{1}{\sum\limits_{\ell \in Lit_{\mathbb{P}}} \mathcal{I}^{\mathcal{G}}(\ell; I_{\top})}$$

[5]The subscripts of the operators indicate the usual product, Łukasiewicz and Gödel connectives.

for all $\ell \in Lit_{\mathbb{P}}$ we obtain the equalities:

$$\mathcal{I}_3^{\mathcal{L}}(\mathbb{P}) = \mathcal{I}_4^{\mathcal{L}}(\mathbb{P}; \{\vartheta_{(\ell,\sim_i\ell)}\}) \text{ and } \mathcal{I}_3^{\mathcal{G}}(\mathbb{P}) = \mathcal{I}_4^{\mathcal{G}}(\mathbb{P}; \{\overline{\vartheta_\ell}\}).$$

In the following example we apply the weighted measures of incoherence $\mathcal{I}_4^{\mathcal{L}}$ and $\mathcal{I}_4^{\mathcal{G}}$ to the program described in Example 7. The aim of the example is to obtain a value which represents the average size of incoherence between the negative literals appearing in $\mathbb{P}$.

Example 8: Let us consider the following subsets of literals of an extended program $\mathbb{P}$:

$$\Omega(\mathbb{P}) = \{\ell \in Lit_{\mathbb{P}} \mid \sim_i\ell \in Lit_{\mathbb{P}}, \text{for some } \sim_i\}$$

In other words, $\Omega(\mathbb{P})$ is the set of literals of $\mathbb{P}$ which also appear negated in $\mathbb{P}$. Note that only literals in $\Omega(\mathbb{P})$ can be incoherent since if $\ell \notin \Omega(\mathbb{P})$ then either $M_{\mathbb{P}}(\ell) = 0$ or $M_{\mathbb{P}}(\sim_i\ell) = 0$ for all $\sim_i \in \mathcal{L}$. We define the weights $\vartheta_{(\ell,\sim_i\ell)}$ and $\overline{\vartheta_\ell}$ as:

$$\vartheta_{(\ell,\sim_i\ell)} = \begin{cases} \left(\displaystyle\sum_{\substack{\ell \in \Omega(\mathbb{P}) \\ \sim_i\ell \in Lit_{\mathbb{P}}}} \mathcal{I}^{\mathcal{L}}((\ell,\sim_i\ell); I_\top) \right)^{-1} & \text{if } \ell \in \Omega(\mathbb{P}) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\overline{\vartheta_\ell} = \begin{cases} \left(\displaystyle\sum_{\ell \in \Omega(\mathbb{P})} \mathcal{I}^{\mathcal{G}}(\ell; I_\top) \right)^{-1} & \text{if } \ell \in \Omega(\mathbb{P}) \\ 0 & \text{otherwise} \end{cases}$$

Note that in the specific case of the program $\mathbb{P}$ given in Example 7, $\Omega(\mathbb{P})$ is the singleton $\{p\}$. So the weights above are $\vartheta_{(p,\sim_1 p)} = \vartheta_{(p,\sim_2 p)} = \frac{1}{0.75+1} = \frac{1}{1.75}$, $\overline{\vartheta_p} = 1$, and 0 otherwise. Hence

$$\mathcal{I}_4^{\mathcal{L}}(\mathbb{P}; \{\vartheta_{(\ell,\sim_i\ell)}\}) = \frac{0.65}{1.75} \approx 0.37$$

and

$$\mathcal{I}_4^{\mathcal{G}}(\mathbb{P}; \{\overline{\vartheta_\ell}\}_{\ell \in Lit_{\mathbb{P}}}) = 0.5$$

The difference between the values $\mathcal{I}_3^{\mathcal{G}}(\mathbb{P}) = 0.08$ and $\mathcal{I}_4^{\mathcal{G}}(\mathbb{P}; \{\vartheta_\ell\}_{\ell \in Lit_{\mathbb{P}}}) = 0.5$ informs us that, although there is not much incoherence in the whole program ($\mathcal{I}_3^{\mathcal{G}}(\mathbb{P}) = 0.08$), the part of the program which can generate incoherence actually produces a lot ($\mathcal{I}_4^{\mathcal{G}}(\mathbb{P}; \{\vartheta_\ell\}_{\ell \in Lit_{\mathbb{P}}}) = 0.5$). □

Note that the use of weights in the measures $\mathcal{I}_4^{\mathcal{L}}$ and $\mathcal{I}_4^{\mathcal{G}}$ allows us to establish a degree of importance between the atomic incoherence of literals. For instance, in a control system of a nuclear power plant, it is not so important an incoherence related to the electric lightning system as an incoherence related to the nuclear fusion in the reactor.

Last but not least, it is worth to mention that Propositions 6, 7, 8, 13, 14 and 15 given on $\mathcal{I}^{\mathcal{L}}$ and $\mathcal{I}^{\mathcal{G}}$ can be proved in the framework of extended programs. So, we have that if $\mathbb{P}$ is a coherent program, then $\mathcal{I}_i^{\mathcal{L}}(\mathbb{P}) = \mathcal{I}_i^{\mathcal{G}}(\mathbb{P}) = 0$ for all $i = 1, 2, 3$ and $\mathcal{I}_4^{\mathcal{L}}(\mathbb{P}, \{\vartheta_\ell\}) = \mathcal{I}_4^{\mathcal{G}}(\mathbb{P}, \{\vartheta_\ell\}) = 0$ for all set of weights $\{\vartheta_\ell\}$. And the converse is true if we are working in one of the frameworks described by Propositions 7 and 8.

V. Conclusions and future work

We have introduced different measures of incoherence; firstly at atomic level, a measure on pairs of opposite literals ($\mathcal{I}^{\mathcal{L}}$) and a measure on literals ($\mathcal{I}^{\mathcal{G}}$); secondly, two different measures, one of dispersion and another of concentration, have been also defined by comparing these two atomic measures. Finally, other various measures of incoherence on extended multi-adjoint logic programs in the previous section by using the atomic measures of incoherence $\mathcal{I}^{\mathcal{L}}$ and $\mathcal{I}^{\mathcal{G}}$.

Concerning future work, the particular definition of measures $\mathcal{I}_4^{\mathcal{L}}$ and $\mathcal{I}_4^{\mathcal{L}}$ in terms of weights in order to represent different aspects of incoherence, naturally leads to the possible integration of OWA operators within this research line.

The initial study of the relation of strong negation with antonyms in Section II motivates at least the use of chains of two strong negations. Yet the admission of longer chains of strong negations in literals encourages us to start a thorough study of its meaning and, last but not least, the interaction of the strong negations with default negation concerning consistency in a multi-valued or fuzzy environment.

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