CONGRUENCE RELATIONS ON MULTILATTICES

P. Cordero, G. Gutiérrez, J. Martínez, M. Ojeda-Aciego, I. de las Peñas

Dept. Matemática Aplicada. Univ. de Málaga. Spain
E-mail: pcordero@uma.es

We focus on a possible generalisation of the theory of congruences on a lattice to a more general framework. In this paper, we prove that the set of congruences on an m-distributive multilattice forms a complete lattice and, moreover, show that the classical relationship between homomorphisms and congruences can be adequately adapted to work with multilattices.

Keywords: Multilattice, congruence, L-fuzzy sets

1. Introduction

The problem of providing suitable fuzzifications of crisp concepts is an important topic which attracts the attention of a number of researchers. Since the inception of fuzzy sets and fuzzy logic, there have been approaches to consider underlying sets of truth-values more general than the unit interval; for instance, consider the L-fuzzy sets introduced by Goguen,[1] where L is a complete lattice.

This paper is part of a research line aimed at investigating L-fuzzy sets where L has the structure of a multilattice. The concepts of ordered and algebraic multilattice were introduced by Benado[2] in 1954. A multilattice is an algebraic structure in which the restrictions imposed on a (complete) lattice, namely, the "existence of least upper bounds and greatest lower bounds" are relaxed to the "existence of minimal upper bounds and maximal lower bounds".

Much more recently, Cordero *et al.*[3,4] proposed an alternative algebraic definition of multilattice which is more closely related to that of lattice, allowing for natural definitions of related structures such that multisemilattices and, in addition, is better suited for applications. For instance, Medina *et al.*[5] developed a general approach to fuzzy logic programming based on a multilattice as underlying set of truth-values for the logic.

Several papers have been published on the lattice of fuzzy congruences on different algebraic structures,[6–10] and in this paper we initiate research in this direction. Specifically, we focus on the theory of congruences on a multilattice, as this is a necessary step prior to considering the multilattice-based generalization of the concept of L-fuzzy congruence.

The structure of this paper follows, as much as possible, the development presented in Gratzer's General Lattice Theory.[11] After some preliminary results, we prove that the set of congruences on a multilattice which satisfies a sort of distributivity, that we have called m-distributivity, forms a complete lattice. Finally, we show that the classical relationship between homomorphisms and congruences can be adequately adapted to work with multilattices.

2. Preliminary definitions

We need some previous concepts that allow us to introduce the multilattice structure:

Given $(M, \leq)$ a partially ordered set (henceforth poset) and $B \subseteq M$, a *multi-supremum of* B is a minimal element of the set of upper bounds of B and $\text{Multi-sup}(B)$ denote the set of multi-suprema of B. Dually, we define the *multi-infima*.

Definition 2.1. A poset, $(M, \leq)$, is an ordered multilattice if and only if it satisfies that, for all $a, b, x \in M$ with $a \leq x$ and $b \leq x$, there exists $z \in \text{Multi-sup}(\{a, b\})$ such that $z \leq x$ and its dual.[a]

Similarly to lattice theory, if we define $a \vee b = \text{Multi-sup}\{a, b\}$ and define $a \wedge b = \text{Multi-inf}\{a, b\}$, it is possible to define multilattices algebraically and, conversely, if we define $a \leq b$ if and only if $a \vee b = b$ it is possible to obtain the ordered version of multilattice. Both definitions of multilattice are proved to be equivalent (see [3, Theorem 2.11]).

In the category of lattices, factor objects are determined by congruence relations, that is, equivalence relations which are compatible with the operations. As a result, it is worth to investigate congruences in the framework of multilattices.

Firstly, we will introduce a notation which will be useful hereafter. Let $\mathcal{R}$ be a binary relation in M and $X, Y \subseteq M$ then $X \widehat{\mathcal{R}} Y$ denotes that, for all $x \in X$, there exists $y \in Y$ such that $x\mathcal{R}y$ and for all $y \in Y$ there exists $x \in X$ such that $x\mathcal{R}y$.

Definition 2.2. Let $(M, \vee, \wedge)$ be a multilattice, a **congruence** on M is

[a] Note that the definition is consistent with the existence of two incomparable elements *without* any multisupremum. In other words, $\text{Multi-sup}(\{a, b\})$ can be empty.

an equivalence relation $\equiv$ which for all $a, b, c \in M$ satisfies that if $a \equiv b$, then $a \vee c \,\widehat{\equiv}\, b \vee c$ and $a \wedge c \,\widehat{\equiv}\, b \wedge c$.

3. On the lattice of congruences

In the rest of the paper, unless stated otherwise, we will always assume that our relations are defined on a multilattice $(M, \wedge, \vee)$. The following results are consequences from the definition:

Lemma 3.1. *Let $\equiv$ be a congruence relation and $[a]$ be the equivalence class of an element a.*

(1) If $b \in [a]$ then $\varnothing \neq a \vee b \subseteq [a]$ and $\varnothing \neq a \wedge b \subseteq [a]$
(2) If there exists $z \in a \wedge b$ and $w \in a \vee b$ such that $z \equiv w$, then $a \equiv b$

Lemma 3.2. *Let $\equiv$ be a congruence relation. If $a \leq b$ with $a \equiv b$ then:*

(1) For all $z \in a \wedge t$ we have that $\varnothing \neq (b \wedge t) \cap z{\uparrow} \subseteq [z]$.
(2) For all $w \in b \vee t$ we have that $\varnothing \neq (a \vee t) \cap w{\downarrow} \subseteq [w]$.

where $z{\uparrow} = \{x \mid x \geq z\}$ and $w{\downarrow} = \{x \mid x \leq w\}$.

The following proposition will be useful in the characterisation of congruences on a multilattice. Specifically, note that any congruence relation in a multilattice satisfies the two hypotheses in the statement.

Proposition 3.1. *Let $\mathcal{R}$ be a relation in a multilattice $(M, \vee, \wedge)$ satisfying the following conditions for all $x, y \in M$*

C1 *If there exist $z \in x \wedge y$ and $w \in x \vee y$ such that $z\mathcal{R}w$, then $x\mathcal{R}y$*
C2 *If $x \leq y$ with $x\mathcal{R}y$ then*

(a) $z \in x \wedge t$ implies that there exists $w \in y \wedge t$ such that $z\mathcal{R}w$.
(b) $w \in y \vee t$ implies that there exists $z \in x \vee t$ such that $z\mathcal{R}w$.

Then, if $a \leq x, y \leq b$ and $a\mathcal{R}b$ then $x\mathcal{R}y$.

Proof. Since $(M, \vee, \wedge)$ is a multilattice, if $a \leq x, y \leq b$ then there exist $z \in x \wedge y$ and $w \in x \vee y$ such that $a \leq z \leq w \leq b$. So $a \leq b$, $a\mathcal{R}b$ and $a \in a \wedge w$, by C2.a, there exists $w \in b \wedge w$ with $a\mathcal{R}w$. Now, as $a \leq w$, $a\mathcal{R}w$ and $w \in w \vee z$, by C2.b, there exists $z \in a \vee z$ with $z\mathcal{R}w$. Finally, by C1, we have that $x\mathcal{R}y$. $\square$

Lemma 3.3. *Let $\equiv$ be a congruence relation.*

(1) If $y \in x \vee z$ and $x \equiv y$, then $x \vee z \subseteq [x]$ and $\varnothing \neq x \wedge z \subseteq [z]$
(2) If $y \in x \wedge z$ and $x \equiv y$, then $x \wedge z \subseteq [x]$ and $\varnothing \neq x \vee z \subseteq [z]$

Theorem 3.1. *Let $(M, \vee, \wedge)$ be a multilattice and $\mathcal{R}$ be a binary relation. Then $\mathcal{R}$ is a congruence relation if and only if the following conditions hold:*

(1) $\mathcal{R}$ is reflexive
(2) $x\mathcal{R}y$ if and only if there exist $z \in x \wedge y$ and $w \in x \vee y$ with $z\mathcal{R}w$
(3) If $x \leq y \leq z$ with $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$
(4) If $x \leq y$ with $x\mathcal{R}y$, then $x \wedge t\,\widehat{\mathcal{R}}\,y \wedge t$ and $x \vee t\,\widehat{\mathcal{R}}\,y \vee t$.

Proof. If $\mathcal{R}$ is a congruence relation, then all the conditions are satisfied using the previous results. Conversely, let us suppose that all the conditions are satisfied. Firstly we will prove that $\mathcal{R}$ is an equivalence relation. Symmetry is a straightforward consequence of (2). For the transitivity, let us suppose that $x\mathcal{R}y$ and $y\mathcal{R}z$. Then by (2) there exist $u \in x \wedge y$, $w \in x \vee y$, $u' \in y \wedge z$ and $w' \in y \vee z$ such that $u\mathcal{R}w$ and $u'\mathcal{R}w'$. Since $u \leq w$ and $u\mathcal{R}w$, by (4), $w' = u \vee w'\widehat{\mathcal{R}}w \vee w'$, thus,[b] there exists $q \in w \vee w'$ such that $w'\mathcal{R}q$. Analogously, $u \wedge u'\,\widehat{\mathcal{R}}\,w \wedge u' = u'$ so, there exists $p \in u' \wedge u$ such that $p\mathcal{R}u'$. Since $p \leq u' \leq y \leq w' \leq q$ and $p\mathcal{R}u'\mathcal{R}w'\mathcal{R}q$, by (3), $p\mathcal{R}q$. Finally, since $p \leq x, z \leq q$ and by Proposition 3.1, $x\mathcal{R}z$.

Now let us prove the compatibility with the operations. If $a\mathcal{R}b$, by (2) there exist $z \in a \wedge b$ and $w \in a \vee b$ such that $z\mathcal{R}w$ and so $a\mathcal{R}w$. Then using (4) we have that since $a \leq w$ then $a \vee t\,\widehat{\mathcal{R}}\,w \vee t$ and since $b \leq w$ then $b \vee t\,\widehat{\mathcal{R}}\,w \vee t$. Then we have that $a \vee t\,\widehat{\mathcal{R}}\,b \vee t$. □

It is well-known that, for every set A, the set of equivalence relations on A, $Eq(A)$, with the inclusion ordering (in the powerset of $A \times A$) is a complete lattice in which the infimum is the meet and the supremum is the transitive closure of the join. A suitable generalization of distributivity will be proved to be a sufficient condition for the set of congruences being a complete lattice.

Definition 3.1. A multilattice $(M, \vee, \wedge)$ is said to be **m-distributive** if the following conditions hold, for all $a, b \in M$ with $a \leq b$ and $t \in M$:

(1) $b \wedge t \subseteq (a \wedge t) \vee (b \wedge t)$ (2) $a \vee t \subseteq (a \vee t) \wedge (b \vee t)$

Theorem 3.2. *The set of the congruences in an m-distributive multilattice M, $Con(M)$, is a sublattice of $Eq(M)$ and, moreover is a complete lattice wrt the inclusion ordering.*

Proof. Let $\{\equiv_i\}_{i \in \Lambda}$ be a set of congruences in M, consider $\equiv_\cap$ to be the intersection and $\equiv_{tc}$ be the transitive closure of union.

[b]We are abusing the notation here, in that singletons are not written between braces.

Since $\equiv_\cap$ and $\equiv_{tc}$ are equivalence relations, they satisfy the conditions (1) and (3) of Theorem 3.1. On the other hand, condition (2) is a consequence of Lemma 3.1. Thus, we have just to check condition (4) in order to show that both $\equiv_\cap$ and $\equiv_{tc}$ are congruences.

Let us consider $x \leq y$ with $x \equiv_\cap y$. Lemma 3.2 ensures that, if $z \in x \wedge t$, then there exists $w \in y \wedge t$ with $z \equiv_\cap w$. Conversely, if $w \in y \wedge t$, since M is m-distributive, there exist $z \in x \wedge t$ and $v \in y \wedge t$ such that $w \in z \vee v$. In particular, $z \leq w$, which implies that $w \equiv_i z$, for all $i \in \Lambda$, by Lemma 3.2.

The proof for $\equiv_{tc}$ follows by a routine calculation. □

4. Homomorphisms and congruences

The notion of homomorphism is extended to the theory of multilattices as follows: $h\colon M \to M'$ is a homomorphism if $h(a \vee b) \subseteq h(a) \vee h(b)$ and $h(a \wedge b) \subseteq h(a) \wedge h(b)$.

Proposition 4.1. *Let $(M, \vee, \wedge)$ be a multilattice and $\equiv$ a congruence relation, then $M/_\equiv$ is a multilattice with*

$$[a] \vee [b] = \{[x] \mid x \in a \vee b\} \qquad \text{and} \qquad [a] \wedge [b] = \{[x] \mid x \in a \wedge b\}$$

Moreover, the mapping $h : M \to M/_\equiv$ such that $h(x) = [x]$ is a surjective homomorphism.

Theorem 4.1. *Let $h : M \to M'$ be a homomorphism between multilattices. The relation in M given by $a \equiv b \Leftrightarrow h(a) = h(b)$ is a congruence iff:*

(1) $h(a) = h(b)$ implies $a \vee b \neq \varnothing$ and $a \wedge b \neq \varnothing$
(2) $h(a \vee b) = h(a)$ implies $a \wedge b \neq \varnothing$
(3) $h(a \wedge b) = h(a)$ implies $a \vee b \neq \varnothing$

Proof. We prove just the converse implication, for which we will use the characterisation given in Theorem 3.1.

It is obvious that $\equiv$ is an equivalence relation. Now, let us consider $x, y \in M$ where $x \leq y$ and $h(x) = h(y)$, and $z \in x \wedge t$. As $z \leq y$ and $z \leq t$ there must be $w \in y \wedge t$ with $z \leq w$; consequently, $h(z) \leq h(w)$. On the other hand, by properties of homomorphism, we have $h(w) \in h(y \wedge t) \subseteq h(y) \wedge h(t) = h(x) \wedge h(t)$ and $h(z) \in h(x \wedge t) \subseteq h(x) \wedge h(t)$. As a result, $h(z) = h(w)$, that is, $z \equiv w$.

Let us consider $x, y, t \in M$ with $x \leq y$, $h(x) = h(y)$ and $w \in y \wedge t$. Firstly, $x \leq y$ and $w \leq y$ so there must exists $y' \in x \vee w$ with $x \leq y' \leq y$. As h is a homomorphism we have that $h(x) \leq h(y') \leq h(y) = h(x)$, that is, $h(y') = h(x)$. As $h(y') = h(x) \in h(x) \vee h(w)$ so $h(x) \vee h(w) = h(x)$. In a nutshell, $h(x \vee w) = h(x)$.

Furthermore, $h(x \wedge w) \subseteq h(x) \wedge h(w) = h(w)$, thus $h(x \wedge w) = h(w)$. By condition (2), $x \wedge w \neq \varnothing$, so we can take $x' \in x \wedge w$ and, by definition of multilattice, there exists $z \in x \wedge t$ such that $x' \leq z$. Notice that $h(x') \leq h(z)$ and $h(z), h(w) \in h(x) \wedge h(t)$; hence, we obtain that $h(w) = h(z)$. $\square$

It is remarkable that in most of the applications of multilattices it is the case that $\text{Multi-sup}(\{a,b\}) \neq \varnothing \neq \text{Multi-inf}(\{a,b\})$ and, as a result, every homomorphism defines a congruence.

5. Conclusions and future work

We have started the investigation of congruences on a multilattice, and shown that the set of congruences of an m-distributive multilattice is a complete latice and a sublattice of the set of its equivalence relations. Moreover, the well-known relation between congruences and homomorphisms has been shown to be preserved when considered in the framework of multilattices.

As future work, we are planning to investigate the multilattice-based generalization of the concept of L-fuzzy congruence, following the line of the several papers published on the lattice of fuzzy congruences on different algebraic structures.[6–10]

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