Galois connections between a fuzzy preordered structure and a general fuzzy structure

I.P. Cabrera, P. Cordero, F. García-Pardo, M. Ojeda-Aciego, B. De Baets

***Abstract*—We continue the study of (isotone) Galois connections, also called adjunctions, in the framework of fuzzy preordered structures, which generalize fuzzy preposets by considering underlying fuzzy equivalence relations. Specifically, we present necessary and sufficient conditions so that, given a mapping $f\colon \mathbb{A} \to B$ from a fuzzy preordered structure $\mathbb{A} = \langle A, \approx_A, \rho_A\rangle$ into a fuzzy structure $\langle B, \approx_B\rangle$, it is possible to construct a fuzzy relation ρ_B that induces a suitable fuzzy preorder structure on B and such that there exists a mapping $g\colon B \to \mathbb{A}$ such that the pair (f, g) constitutes an Galois connection.**

***Index Terms*—Galois connection, Preorder, Fuzzy sets**

I. INTRODUCTION

Galois connections (both in isotone and in antitone forms) can be found in different areas, and it is common to find papers dealing with them either from a practical or a theoretical point of view. In the literature, one can find numerous papers on theoretical developments on Galois connections [1], [2], [9], [25], [27] and also on applications thereof [19], [20], [29], [32], [35], [36], [38], [44].

Concerning the generalization to the fuzzy case, to the best of our knowledge, the first approach was due to Bělohlávek [1]. Later, a number of authors have introduced different approaches to so-called fuzzy (isotone or antitone) Galois connections; see [6], [20], [21], [25], [27], [30], [45]. It is remarkable that the mappings forming the Galois connection in all the above-mentioned approaches are crisp rather than fuzzy. In our opinion the term 'fuzzy Galois connection' should be reserved for the case in which the involved mappings are actually fuzzy mappings, and that is why we prefer to stick to the term 'Galois connection' rather than 'fuzzy Galois connection', notwithstanding the fact that we are working in the context of fuzzy structures.

In previous works, some of the present authors have studied the problem of constructing a right adjoint (or residual mapping) associated to a given mapping $f\colon \mathbb{A} \to B$ where $\mathbb{A}$ is endowed with some order-like structure and B is unstructured: in [24], we consider $\mathbb{A}$ to be a crisp partially (pre)ordered set $\langle A, \leq_A\rangle$; later, in [10], we considered $\mathbb{A}$ to be a fuzzy preposet $\langle A, \rho_A\rangle$.

In this paper, we consider the case in which there are two underlying fuzzy equivalence relations in both the domain and the codomain of the mapping f, more specifically, f is a *morphism* between the *fuzzy structures* $\langle A, \approx_A\rangle$ and $\langle B, \approx_B\rangle$ where, in addition, $\langle A, \approx_A\rangle$ is a fuzzy pre-ordered structure. Firstly, we have to characterize when it is possible to endow B with the adequate structure (namely, enrich it to a fuzzy pre-ordered structure) and, then, construct a mapping g from B to A compatible with the fuzzy equivalence relations such that the pair (f, g) forms a Galois connection.

Although all the results will be stated in terms of the existence and construction of right adjoints (or residual mappings), they can be straightforwardly modified for the existence and construction of left adjoints (or residuated mappings). On the other hand, it is worth remarking that the construction developed in this paper can be extended to the different types of Galois connections (see [22]).

Galois connections (both in a crisp and in a fuzzy setting) have found applications in areas such as (fuzzy) Mathematical Morphology [14]–[16], in which the (fuzzy) erosion and (fuzzy) dilation operations are known to form a Galois connection [7], [26], [39], [40]; another important source of applications of Galois connections is within the field of Formal Concept Analysis, in which the concept-forming operators form either an antitone or isotone Galois connection (depending on the specific definition); in this research direction, one still can find recent papers on the theoretical background of the discipline [3]–[5], [11], [31], [37], [42] and a number of applications [13], [33], [34].

The structure of the paper is as follows. In Section II, some preliminary notions on Galois connections between fuzzy preordered structures used in the rest of the paper are introduced. Then, in Section III we study the canonical decomposition of Galois connections in our framework, followed by an analysis of conditions for the existence of the right adjoint in Sections IV and V. As a consequence of the canonical decomposition, we propose a two-step procedure for verifying the existence of the right adjoint in a constructive manner; this is studied in detail in Section VI. Finally, in Section VII, we state the conclusions and prospects for future work.

II. GALOIS CONNECTIONS BETWEEN FUZZY PREORDERED STRUCTURES

The most common underlying structure for considering fuzzy generalizations of Galois connections is that of a complete residuated lattice $\mathbb{L} = (L, \leq, \top, \bot, \otimes, \to)$. As usual, supremum and infimum will be denoted by $\vee$ and $\wedge$, respectively. An $\mathbb{L}$-*fuzzy set* X on a universe U is a mapping $X\colon U \to L$ from U to L, where $X(u)$ denotes the degree to which u belongs to X. Given two $\mathbb{L}$-fuzzy sets X and Y, X is

---

I.P. Cabrera, P. Cordero, F. García-Pardo and M. Ojeda-Aciego are with the Universidad de Málaga. Departamento de Matemática Aplicada. Blv. Louis Pasteur 35, 29071 Málaga, Spain, and B. De Baets is with the research unit KERMIT at the Department of Mathematical Modelling, Statistics and Bioinformatics. Ghent University, Coupure links 653, 9000 Gent, Belgium

said to *be included in* Y, denoted as $X \subseteq Y$, if $X(u) \le Y(u)$ for all $u \in U$.

A mapping $R\colon U \times U \to L$ is a (binary) $\mathbb{L}$*-fuzzy relation on* U. An $\mathbb{L}$-fuzzy relation R is said to be:

(i) *Reflexive* if $R(a,a) = \top$ for all $a \in U$.
(ii) $\otimes$*-Transitive* if $R(a,b) \otimes R(b,c) \le R(a,c)$ for all $a, b, c \in U$.
(iii) *Symmetric* if $R(a,b) = R(b,a)$ for all $a, b \in U$.

From now on, when no confusion arises, we will omit the prefixes "$\mathbb{L}$-" and "$\otimes$-".

Definition 1: A fuzzy relation $\approx$ on A is said to be a:

(i) *Fuzzy equivalence relation* if it is reflexive, symmetric and transitive.
(ii) *Fuzzy equality relation* if it is a fuzzy equivalence relation such that $\approx(a,b) = \top$ implies $a = b$, for all $a, b \in A$.

We will use the infix notation for a fuzzy equivalence relation, that is: for a fuzzy equivalence relation $\approx\colon A \times A \to L$, we write $a_1 \approx a_2$ to refer to $\approx(a_1, a_2)$.

Definition 2: For a fuzzy equivalence relation $\approx\colon A \times A \to L$, the equivalence class of an element $a \in A$ is the fuzzy set $[a]_\approx\colon A \to L$ defined by $[a]_\approx(u) = (a \approx u)$ for all $u \in A$.

Remark 1: Note that $[x]_\approx = [y]_\approx$ if and only if $(x \approx y) = \top$. Indeed, if $[x]_\approx = [y]_\approx$, then $(x \approx y) = [x]_\approx(y) = [y]_\approx(y) = \top$, by reflexivity; conversely, if $(x \approx y) = \top$, then $[x]_\approx(u) = (x \approx u) = (y \approx x) \otimes (x \approx u) \le (y \approx u) = [y]_\approx(u)$, for all $u \in A$ by transitivity; the converse inequality follows in the same way.

Definition 3:

(i) A *fuzzy structure* $\mathcal{A} = \langle A, \approx_A \rangle$ is a set A endowed with a fuzzy equivalence relation $\approx_A$.
(ii) A *morphism* between two fuzzy structures $\mathcal{A}$ and $\mathcal{B}$ is a mapping $f\colon A \to B$ such that for all $a_1, a_2 \in A$ the following inequality holds: $(a_1 \approx_A a_2) \le (f(a_1) \approx_B f(a_2))$. In this case, we write $f\colon \mathcal{A} \to \mathcal{B}$, and we say that f is *compatible* with $\approx_A$ and $\approx_B$.

It is worth mentioning that fuzzy structures and their morphisms form a category and, in fact, in this categorical framework, our fuzzy structures are called *global* $\mathbb{L}$*-valued sets* associated to a GL-monoid $\mathbb{L}$ [28]. Furthermore, Demirci [18] proved it to be a full subcategory of the so-called $\mathbb{L}$-valued sets (a generalized form of our notion of fuzzy structure just introduced) which, in addition, coincides with the category of $\mathbb{L}^d$-pseudometric spaces, where $\mathbb{L}^d$ is the dual GL-monoid associated with $\mathbb{L}$, showing an essential duality between global $\mathbb{L}$-valued sets (fuzzy structures) and $\mathbb{V}$-pseudometric spaces, where $\mathbb{V}$ denotes a dual GL-monoid.

Definition 4: A morphism between two fuzzy structures $\mathcal{A}$ and $\mathcal{B}$ is said to be

(i) $\approx$*-injective* if $(f(a_1) \approx_B f(a_2)) \le (a_1 \approx_A a_2)$, for all $a_1, a_2 \in A$ (or, equivalently, $(f(a_1) \approx_B f(a_2)) = (a_1 \approx_A a_2)$, for all $a_1, a_2 \in A$.)
(ii) $\approx$*-surjective* if for all $b \in B$ there exists $a \in A$ such that $(f(a) \approx_B b) = \top$.
(iii) a $\approx$*-isomorphism* if it is $\approx$-injective and $\approx$-surjective. In such case, for all $b_1, b_2 \in B$, there exist $a_1, a_2 \in A$ such that $(b_1 \approx_B b_2) = (f(a_1) \approx_B f(a_2)) = (a_1 \approx_A a_2)$.

Remark 2: Consider a morphism $f\colon \langle A, \approx_A \rangle \to \langle B, \approx_B \rangle$.

(i) If f is surjective, then it is $\approx$-surjective (see Example 1 for a counterexample for the converse implication).
In addition, if $\approx_B$ is a fuzzy equality, then f is $\approx$-surjective if and only if f is surjective.
(ii) The $\approx$-injectivity and the injectivity of f are independent (see Examples 2 and 3).
Furthermore, if $\approx_A$ is a fuzzy equality and f is $\approx$-injective, then it is injective. However, the converse implication is false in general, as shown in Example 3.

Some examples are worked out below in order to illustrate the previous remarks. All of them are based on the standard residuated lattice structure generated by the product t-norm on the real unit interval, that is $\mathbb{L} = ([0,1], \sup, \inf, 1, 0, \cdot, \to)$.

Example 1: Consider two fuzzy structures $\mathcal{A} = \langle \{o, p\}, \approx_A \rangle$ and $\mathcal{B} = \langle \{o, p, q\}, \approx_B \rangle$, where $\approx_A$ and $\approx_B$ are the fuzzy equivalence relations given by the tables below:

$\approx_A$	o	p
o	1	0.9
p	0.9	1

$\approx_B$	o	p	q
o	1	0.9	0.9
p	0.9	1	1
q	0.9	1	1

The inclusion mapping $i\colon \mathcal{A} \to \mathcal{B}$ is obviously a morphism which, in addition, is also $\approx$-surjective, since $(o \approx_B i(o)) = 1$, $(p \approx_B i(p)) = 1$ and $(q \approx_B i(p)) = 1$. However, it is not surjective.

Example 2: Consider two fuzzy structures $\mathcal{A} = \langle \{o, p, q, r\}, \approx_A \rangle$ and $\mathcal{B} = \langle \{o, p, q\}, \approx_B \rangle$, where $\approx_A$ and $\approx_B$ are the fuzzy equality relations given by the tables below:

$\approx_A$	o	p	q	r
o	1	0.5	0.7	1
p	0.5	1	0.5	0.5
q	0.7	0.5	1	0.7
r	1	0.5	0.7	1

$\approx_B$	o	p	q
o	1	0.5	0.7
p	0.5	1	0.5
q	0.7	0.5	1

The mapping $f\colon \mathcal{A} \to \mathcal{B}$ defined by $f(o) = o$, $f(p) = p$, $f(q) = q$ and $f(r) = o$ is a $\approx$-injective morphism; however, it is not injective.

Example 3: Consider two fuzzy structures $\mathcal{A} = \langle \{a, b, c\}, \approx_A \rangle$ and $\mathcal{B} = \langle \{o, p, q\}, \approx_B \rangle$, where $\approx_A$ and $\approx_B$ are the fuzzy equality relations given by the tables below:

$\approx_A$	a	b	c
a	1	0.8	0.7
b	0.8	1	0.6
c	0.7	0.6	1

$\approx_B$	o	p	q
o	1	0.9	0.8
p	0.9	1	0.8
q	0.8	0.8	1

and consider the mapping $f\colon \mathcal{A} \to \mathcal{B}$ with $f(a) = o$, $f(b) = p$ and $f(c) = q$. It is easy to check that f is a morphism, and an injective but not $\approx$-injective mapping, since $(a \approx_A b) = 0.8 < 0.9 = (f(a) \approx_B f(b))$.

Concerning our underlying ordered structure, in [10] we worked with the notion of *fuzzy preposet* defined below:

Definition 5: A *fuzzy preposet* is a pair $\mathbb{A} = \langle A, \rho_A \rangle$ in which ρ_A is a reflexive and transitive fuzzy relation on A.

The additional consideration of an underlying fuzzy equivalence relation suggests considering the following notions:

Definition 6 ([8]): Let $\approx_A$ be a fuzzy equivalence relation on A. A fuzzy relation $\rho_A\colon A \times A \to L$ is said to be

(i) $\approx_A$-*reflexive* if $(a_1 \approx_A a_2) \leq \rho_A(a_1, a_2)$ for all $a_1, a_2 \in A$.

(ii) $\otimes$-$\approx_A$-*antisymmetric* if $\rho_A(a_1, a_2) \otimes \rho_A(a_2, a_1) \leq (a_1 \approx_A a_2)$ for all $a_1, a_2 \in A$.

Definition 7: Given a fuzzy structure $\mathcal{A} = \langle A, \approx_A \rangle$, the pair $\mathbb{A} = \langle \mathcal{A}, \rho_A \rangle$ will be called a $\otimes$-$\approx_A$- *fuzzy preordered structure* or simply *fuzzy preordered structure* (when there is no risk of confusion), if ρ_A is a fuzzy relation that is $\approx_A$-reflexive, $\otimes$-$\approx_A$-antisymmetric and $\otimes$-transitive.

If the underlying fuzzy structure is not clear from the context, we will sometimes write a fuzzy preordered structure as a triplet $\mathbb{A} = \langle A, \approx_A, \rho_A \rangle$.

Remark 3: Note that although in the above definition the fuzzy relation ρ_A is required to be $\otimes$-$\approx_A$-antisymmetric, this condition serves to establish a natural correspondence between the fuzzy equivalence relation $\approx_A$ and the fuzzy relation ρ_A, and should by no means be seen as playing the same role as the usual antisymmetry condition satisfied by an order relation (in such a case, we would at least have to restrict to a fuzzy *equality* relation $\approx_A$). For the same reason, we consider the name *fuzzy order relation* not suitable for a fuzzy relation that is reflexive, $\otimes$-$\approx_A$-antisymmetric and $\otimes$-transitive, with $\approx_A$ a fuzzy equivalence relation, and advocate that such fuzzy relations should rather be seen as particularly interesting *fuzzy pre-order relations*.

A reasonable approach to introduce the notion of Galois connection between fuzzy preordered structures $\mathbb{A}$ and $\mathbb{B}$ would be the following

Definition 8: Let $\mathbb{A}$ and $\mathbb{B}$ be two fuzzy preordered structures. Given two morphisms $f\colon \mathcal{A} \to \mathcal{B}$ and $g\colon \mathcal{B} \to \mathcal{A}$, the pair (f, g) is said to be a *Galois connection* between $\mathbb{A}$ and $\mathbb{B}$ (briefly, $(f, g)\colon \mathbb{A} \leftrightarrows \mathbb{B}$) if the following conditions hold for all $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$:

(G1) $(a_1 \approx_A a_2) \otimes \rho_A(a_2, g(b)) \leq \rho_B(f(a_1), b)$;

(G2) $(b_1 \approx_B b_2) \otimes \rho_B(f(a), b_1) \leq \rho_A(a, g(b_2))$;

The mapping f is said to be the left adjoint of g and, reciprocally, g is said to be the right adjoint of f.

Proposition 1 shows that the previous definition is strongly related to the definition given in [10] which straightforward generalizes the usual notion of Galois connection between posets (namely, $a \leq g(b)$ if and only if $f(a) \leq b$) in the same line of [1], [45].

For convenience, the definition used in [10] is recalled below:

Definition 9: Let $\mathbb{A} = \langle A, \rho_A \rangle$ and $\mathbb{B} = \langle B, \rho_B \rangle$ be fuzzy preposets. A pair of mappings $f\colon A \to B$ and $g\colon B \to A$ forms a *Galois connection* between $\mathbb{A}$ and $\mathbb{B}$, denoted $(f, g) : \mathbb{A} \leftrightarrows \mathbb{B}$ if, for all $a \in A$ and $b \in B$, the equality $\rho_A(a, g(b)) = \rho_B(f(a), b)$ holds.

Proposition 1: Consider two fuzzy preordered structures $\mathbb{A} = \langle \mathcal{A}, \rho_A \rangle$ and $\mathbb{B} = \langle \mathcal{B}, \rho_B \rangle$, and two mappings $f\colon A \to B$ and $g\colon B \to A$. It holds that the pair (f, g) is a Galois connection between $\mathbb{A}$ and $\mathbb{B}$ if and only if both mappings are morphisms and $\rho_A(a, g(b)) = \rho_B(f(a), b)$ for all $a \in A$ and $b \in B$.

As a consequence of the previous theorem we obtain the following result linking Galois connections between fuzzy preordered structures and Galois connections between fuzzy preposets.

Corollary 1: If a pair (f, g) is a Galois connection between two fuzzy preordered structures $\langle A, \approx_A, \rho_A \rangle$ and $\langle B, \approx_B, \rho_B \rangle$, then (f, g) is also a Galois connection between the two fuzzy preposets $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$.

Conversely, if a pair (f, g) is a Galois connection between two fuzzy preposets $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$, then (f, g) is also a Galois connection between the two fuzzy preordered structures $\langle A, =, \rho_A \rangle$ and $\langle B, =, \rho_B \rangle$, where $=$ denotes the standard (crisp) equality.

Definition 10: Let $\mathbb{A} = \langle \mathcal{A}, \rho_A \rangle$ be a fuzzy preordered structure. The upset and the downset of an element $a \in A$ are defined as the fuzzy sets $a^\uparrow, a^\downarrow\colon A \to L$ where

$$a^\downarrow(u) = \rho_A(u, a) \quad \text{and} \quad a^\uparrow(u) = \rho_A(a, u) \quad \text{for all } u \in A\,.$$

An element $m \in A$ is called a *quasi-maximum* of a fuzzy set $X\colon A \to L$ if

(i) $X(m) = \top$ and

(ii) $X \subseteq m^\downarrow$, i.e., $X(u) \leq \rho_A(u, m)$ for all $u \in A$.

The definition of *quasi-minimum* is similar.

Observe that, given two quasi-maxima x_1, x_2 of a fuzzy set X in a fuzzy preordered structure, we obtain $\rho_A(x_1, x_2) = \top = \rho_A(x_2, x_1)$ and by $\otimes$-$\approx_A$-antisymmetry, also $(x_1 \approx_A x_2) = \top$. This fact justifies both the use of the adjective *preordered* (even when a form of antisymmetry holds), the use of the prefix *quasi-* and, hence, the notation $\operatorname{qmax}_{\mathbb{A}}(X)$ (resp., $\operatorname{qmin}_{\mathbb{A}}(X)$) to refer to the crisp set of quasi-maxima (resp. quasi-minima).

Example 4: Consider the Łukasiewicz residuated lattice and the fuzzy preordered structure $\langle A, \approx_A, \rho_A \rangle$ where $A = \{a_1, a_2, a_3, a_4\}$, and $\approx_A$ and ρ_A are the fuzzy relations given by the tables below:

$\approx_A$	a_1	a_2	a_3	a_4
a_1	1	0.4	0	0.4
a_2	0.4	1	0.2	1
a_3	0	0.2	1	0.2
a_4	0.4	1	0.2	1

ρ_A	a_1	a_2	a_3	a_4
a_1	1	1	1	1
a_2	0.4	1	0.4	1
a_3	0	0.3	1	0.3
a_4	0.4	1	0.4	1

Then, for instance, we have

$$\begin{aligned} \operatorname{qmax}_{\mathbb{A}}(\{(a_1,1),(a_2,1),(a_3,0.7),(a_4,1)\}) &= \varnothing \\ \operatorname{qmax}_{\mathbb{A}}(\{(a_1,1),(a_2,1),(a_3,0.2),(a_4,1)\}) &= \{a_2, a_4\} \\ \operatorname{qmax}_{\mathbb{A}}(\{(a_1,1),(a_2,0.9),(a_3,0.2),(a_4,1)\}) &= \{a_4\}. \end{aligned}$$

Definition 11: Let $\mathbb{A} = \langle \mathcal{A}, \rho_A \rangle$ be a fuzzy preordered structure. A mapping $f\colon A \to A$ is said to be *inflationary* if $\rho_A(a, f(a)) = \top$ for all $a \in A$. Similarly, a mapping f is said to be *deflationary* if $\rho_A(f(a), a) = \top$ for all $a \in A$.

By Proposition 1, we can easily adapt the existing equivalences between different alternative definitions of a Galois connection. In the theorem below, as it could be expected, the general structure of all the definitions is preserved, but those concerning the actual definition of Galois connection and inverse image have to be modified: in the former case, by using the notions of quasi-maximum and quasi-minimum and, in the latter case, for a mapping $f\colon A \to B$ and a fuzzy subset Y of B, the fuzzy set $f^{-1}(Y)$ is defined as $f^{-1}(Y)(a) = Y(f(a))$, for all $a \in A$.

Proposition 2 ([22]): Consider two fuzzy preordered structures $\mathbb{A} = \langle \mathcal{A}, \rho_A \rangle$, $\mathbb{B} = \langle \mathcal{B}, \rho_B \rangle$, and two morphisms $f\colon \mathcal{A} \to \mathcal{B}$ and $g\colon \mathcal{B} \to \mathcal{A}$. The following statements are equivalent:

1) $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$.
2) f and g are isotone, $g \circ f$ is inflationary, and $f \circ g$ is deflationary.
3) $f(a)^{\uparrow} = g^{-1}(a^{\uparrow})$ for all $a \in A$.
4) $g(b)^{\downarrow} = f^{-1}(b^{\downarrow})$ for all $b \in B$.
5) f is isotone and $g(b) \in \operatorname{qmax}_{\mathbb{A}}\big(f^{-1}(b^{\downarrow})\big)$ for all $b \in B$.
6) g is isotone and $f(a) \in \operatorname{qmin}_{B}\big(g^{-1}(a^{\uparrow})\big)$ for all $a \in A$.

Theorem 1: Consider two fuzzy preordered structures $\mathbb{A}$ and $\mathbb{B}$. If the pair (f, g) is a Galois connection between $\mathbb{A}$ and $\mathbb{B}$, then $\big(fgf(a) \approx_B f(a)\big) = \top$ and $\big(gfg(b) \approx_A g(b)\big) = \top$, for all $a \in A$ and $b \in B$.

Corollary 2: Consider two fuzzy preordered structures $\mathbb{A}$ and $\mathbb{B}$. If the pair (f, g) is a Galois connection between $\mathbb{A}$ and $\mathbb{B}$ then, for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$, the following equalities hold:

1) $\big(f(a_1) \approx_B f(a_2)\big) = \big(gf(a_1) \approx_A gf(a_2)\big)$.
2) $\big(g(b_1) \approx_A g(b_2)\big) = \big(fg(b_1) \approx_B fg(b_2)\big)$.

III. THE CANONICAL DECOMPOSITION

In this section, we show that the canonical decomposition of a mapping $f\colon A \to B$ can be used in order to find its right adjoint (whenever it exists) by building the right adjoints to the canonical projection and the canonical embedding.

Recall that a given mapping $f\colon A \to B$ can be canonically decomposed as the composition $i_f \circ \varphi_f$ where $\varphi_f\colon A \to f(A)$ is the canonical projection defined by $\varphi_f(a) = f(a)$ and $i_f\colon f(A) \to B$ is the canonical embedding defined by $i_f(b) = b$; by construction, φ_f is surjective and i_f is injective. Moreover, if $f\colon \langle A, \approx_A \rangle \to \langle B, \approx_B \rangle$ is a morphism, then φ_f is a surjective (and hence $\approx$-surjective due to Remark 2) morphism and i_f is both injective and $\approx$-injective morphism.

Theorem 2: Consider two fuzzy preordered structures $\mathbb{A}$ and $\mathbb{B}$, and two morphisms $f\colon \mathcal{A} \to \mathcal{B}$ and $g\colon \mathcal{B} \to \mathcal{A}$. We have that $(f, g)\colon \mathbb{A} \leftrightharpoons \mathbb{B}$ if and only if there exist $(\varphi_f, \psi)\colon \mathbb{A} \leftrightharpoons f(\mathbb{A})$ and $(i_f, h)\colon f(\mathbb{A}) \leftrightharpoons \mathbb{B}$, where $f(\mathbb{A}) = \langle f(A), \approx_B, \rho_B \rangle$, such that for all $a \in A$ and $b \in B$ it holds that

$$\big(i_f \varphi_f(a) \approx_B f(a)\big) = \top \text{ and } \big(\psi h(b) \approx_A g(b)\big) = \top\,. \tag{1}$$

In the statement of the previous theorem, a given Galois connection $(f, g)\colon \mathbb{A} \leftrightharpoons \mathbb{B}$ has been decomposed through $f(\mathbb{A})$ into two Galois connections $(\varphi_f, \psi)\colon \mathbb{A} \leftrightharpoons f(\mathbb{A})$ and $(i_f, h)\colon f(\mathbb{A}) \leftrightharpoons \mathbb{B}$, which we will call the *canonical decomposition* of (f, g) in which φ_f is the *canonical projection* and i_f is the *canonical embedding*.

The following result introduces some properties of the corresponding morphisms involved in the decomposition.

Theorem 3: Given a Galois connection $(f, g)\colon \mathbb{A} \leftrightharpoons \mathbb{B}$ and its corresponding canonical decomposition $(\varphi_f, \psi)\colon \mathbb{A} \leftrightharpoons f(\mathbb{A})$ and $(i_f, h)\colon f(\mathbb{A}) \leftrightharpoons \mathbb{B}$, the following conditions hold:

(i) φ_f and h are $\approx$-surjective mappings.
(ii) ψ and i_f are $\approx$-injective mappings.

The preceding result enables to divide the analysis of the existence of a right adjoint into two parts, based on the canonical projection and the canonical embedding. The key properties of these mappings are $\approx$-surjectivity and $\approx$-injectivity; in Section V, the existence of a right adjoint will be studied in these frameworks.

IV. NECESSARY CONDITIONS FOR THE EXISTENCE OF A RIGHT ADJOINT

In order to provide necessary conditions for the existence of a right adjoint, the following notions are needed.

Definition 12: Let $\mathcal{A}$ and $\mathcal{B}$ be two fuzzy structures and let $f\colon \mathcal{A} \to \mathcal{B}$ be a morphism. The *fuzzy kernel relation* $\equiv_f\colon A \times A \to L$ associated with f is defined as follows, for $a_1, a_2 \in A$,

$$(a_1 \equiv_f a_2) = \big(f(a_1) \approx_B f(a_2)\big).$$

The fuzzy kernel relation trivially is a fuzzy equivalence relation, and the equivalence class of an element $a \in A$ is the fuzzy set $[a]_f\colon A \to L$ defined by $[a]_f(u) = (f(a) \approx_B f(u))$ for all $u \in A$.

The following definitions rephrase the notion of Hoare ordering [43, pag. 166] including weak (W) and strong (S) versions between crisp subsets (that is, $C \sqsubseteq_H D$ iff for all $c \in C$ there exists $d \in D$ such that $c \leq d$), and the subsequent lemma proves that all of them coincide and can be computed in an extremely easy manner when comparing sets of quasi-maxima.

Definition 13: Let $\mathbb{A}$ be a fuzzy preordered structure. For crisp subsets C and D of A, we define the following fuzzy relations

(i) $(C \sqsubseteq_W D) = \displaystyle\bigvee_{c \in C} \bigvee_{d \in D} \rho_A(c, d)$;
(ii) $(C \sqsubseteq_H D) = \displaystyle\bigwedge_{c \in C} \bigvee_{d \in D} \rho_A(c, d)$;
(iii) $(C \sqsubseteq_S D) = \displaystyle\bigwedge_{c \in C} \bigwedge_{d \in D} \rho_A(c, d)$.

Lemma 1: Consider a fuzzy preordered structure $\mathbb{A}$, and crisp subsets X, Y of A such that $\operatorname{qmax}_{\mathbb{A}}(X) \neq \varnothing \neq \operatorname{qmax}_{\mathbb{A}}(Y)$. It holds that

$$\begin{aligned}\big(\operatorname{qmax}_{\mathbb{A}}(X) \sqsubseteq_W \operatorname{qmax}_{\mathbb{A}}(Y)\big) &= \big(\operatorname{qmax}_{\mathbb{A}}(X) \sqsubseteq_H \operatorname{qmax}_{\mathbb{A}}(Y)\big)\\ &= \big(\operatorname{qmax}_{\mathbb{A}}(X) \sqsubseteq_S \operatorname{qmax}_{\mathbb{A}}(Y)\big)\\ &= \rho_A(x, y)\end{aligned}$$

for any $x \in \operatorname{qmax}_{\mathbb{A}}(X)$ and $y \in \operatorname{qmax}_{\mathbb{A}}(Y)$.

Recall that given a fuzzy preordered structure $\mathbb{A}$ and two crisp subsets $X, Y \subseteq A$, for all $x_1, x_2 \in \operatorname{qmax}_{\mathbb{A}}(X)$ and $y_1, y_2 \in \operatorname{qmax}_{\mathbb{A}}(Y)$ we have $(x_1 \approx_A x_2) = \top = (y_1 \approx_A y_2)$. Therefore, we can write

$$\begin{aligned}(x_1 \approx_A y_1) &= (x_2 \approx_A x_1) \otimes (x_1 \approx_A y_1) \otimes (y_1 \approx_A y_2)\\ &\leq (x_2 \approx_A y_2)\\ &= (x_1 \approx_A x_2) \otimes (x_2 \approx_A y_2) \otimes (y_2 \approx_A y_1) \leq (x_1 \approx_A y_1)\end{aligned}$$

and obtain that $(x_1 \approx_A y_1) = (x_2 \approx_A y_2)$.

Definition 14: Consider a fuzzy preordered structure $\mathbb{A} = \langle A, \approx_A, \rho_A \rangle$, and crisp subsets X, Y of A. The fuzzy relations $\approx_A$ and ρ_A can be extended to the sets of quasi-maxima as follows:

$$\big(\mathrm{qmax}_{\mathbb{A}}(X) \approx_A \mathrm{qmax}_{\mathbb{A}}(Y)\big) \overset{\text{def}}{=} (x \approx_A y)$$
$$\rho_A\big(\mathrm{qmax}_{\mathbb{A}}(X), \mathrm{qmax}_{\mathbb{A}}(Y)\big) \overset{\text{def}}{=} \rho_A(x, y)$$

for any $x \in \mathrm{qmax}_{\mathbb{A}}(X), y \in \mathrm{qmax}_{\mathbb{A}}(Y)$.

Note that the above definition makes sense, since, by Lemma 1 and the preceding result, it does not depend of the specific choice of the elements x and y.

The preceding definitions allow us to state necessary conditions on f in order to have a right adjoint in a more compact form which essentially follows the scheme already obtained in [10] and [24].

Theorem 4 (Necessary conditions): Consider two fuzzy preordered structures $\mathbb{A}$ and $\mathbb{B}$, together with two morphisms $f \colon \mathcal{A} \to \mathcal{B}$ and $g \colon \mathcal{B} \to \mathcal{A}$. If (f, g) is a Galois connection between $\mathbb{A}$ and $\mathbb{B}$, then

1) $\mathrm{qmax}_{\mathbb{A}}([a]_f)$ is not empty for all $a \in A$.
2) $\rho_A(a_1, a_2) \leq \rho_A\big(\mathrm{qmax}_{\mathbb{A}}([a_1]_f), \mathrm{qmax}_{\mathbb{A}}([a_2]_f)\big)$, for all $a_1, a_2 \in A$.
3) $(a_1 \equiv_f a_2) \leq \big(\mathrm{qmax}_{\mathbb{A}}([a_1]_f) \approx_A \mathrm{qmax}_{\mathbb{A}}([a_2]_f)\big)$, for all $a_1, a_2 \in A$.

It is worth remarking that, due to Corollary 2, the third condition actually becomes an equality, that is,

$$(a_1 \equiv_f a_2) = \big(\mathrm{qmax}_{\mathbb{A}}([a_1]_f) \approx_A \mathrm{qmax}_{\mathbb{A}}([a_2]_f)\big) .$$

for all $a_1, a_2 \in A$.

V. EXISTENCE OF A RIGHT ADJOINT OF $\approx$-SURJECTIVE OR $\approx$-INJECTIVE MORPHISMS

We show now that the necessary conditions in Theorem 4 are sufficient in the case of a $\approx$-surjective mapping. Afterwards, we also identify necessary and sufficient conditions in the case of a $\approx$-injective mapping.

Theorem 5 (Sufficient conditions): Consider a fuzzy preordered structure $\mathbb{A}$, a fuzzy structure $\mathcal{B} = \langle B, \approx_B \rangle$, and a $\approx$-surjective morphism $f \colon \mathcal{A} \to \mathcal{B}$. If the following conditions hold

1) $\mathrm{qmax}_{\mathbb{A}}([a]_f)$ is not empty for all $a \in A$;
2) $\rho_A(a_1, a_2) \leq \rho_A\big(\mathrm{qmax}_{\mathbb{A}}([a_1]_f), \mathrm{qmax}_{\mathbb{A}}([a_2]_f)\big)$, for all $a_1, a_2 \in A$;
3) $(a_1 \equiv_f a_2) \leq \big(\mathrm{qmax}_{\mathbb{A}}([a_1]_f) \approx_A \mathrm{qmax}_{\mathbb{A}}([a_2]_f)\big)$, for all $a_1, a_2 \in A$;

then there exists a $\approx_B$-reflexive, $\otimes$-$\approx_B$-antisymmetric and $\otimes$-transitive fuzzy relation ρ_B on B and a morphism $g \colon \mathcal{B} \to \mathcal{A}$ such that (f, g) is a Galois connection between the fuzzy preordered structures $\mathbb{A}$ and $\mathbb{B} = \langle \mathcal{B}, \rho_B \rangle$.

The problem of finding a right adjoint of a $\approx$-injective morphism can be reduced to the case of embeddings. For this aim, we introduce the notion of contraction, which allows to characterize this problem, as stated in Theorem 6 below.

Definition 15: Let $\mathcal{B} = \langle B, \approx_B \rangle$ be a fuzzy structure, and consider a crisp subset $X \subseteq B$. A mapping $h \colon B \to X$ is said to be a *contraction* if it is a morphism $h \colon \mathcal{B} \to \langle X, \approx_B \rangle$ and $h(x) = x$ for all $x \in X$.

Theorem 6: Consider two fuzzy preordered structures $\mathbb{A} = \langle \mathcal{A}, \rho_A \rangle$ and $\mathbb{B} = \langle \mathcal{B}, \rho_B \rangle$. For a $\approx$-injective morphism $f \colon \mathcal{A} \to \mathcal{B}$, the following statements are equivalent:

1) There exists a morphism $g \colon \mathcal{B} \to \mathcal{A}$ such that $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$.
2) There exist a contraction $h \colon \langle B, \approx_B \rangle \to \langle f(A), \approx_B \rangle$ and a fuzzy relation $\rho_{f(A)}$ defined as $\rho_{f(A)}(f(a_1), f(a_2)) = \rho_A(a_1, a_2)$ such that the pair (i, h) is a Galois connection between $\langle f(A), \approx_B, \rho_{f(A)} \rangle$ and $\langle B, \approx_B, \rho_B \rangle$, where $i \colon f(A) \to B$ denotes the canonical embedding.

The previous theorem allows to reduce the problem of finding a right adjoint to the case of embedding morphisms. That is, given a subset $X \neq \varnothing$ of a fuzzy structure $\mathcal{B} = \langle B, \approx_B \rangle$ together with a $\approx_B$-reflexive, $\otimes$-$\approx_B$-antisymmetric and $\otimes$-transitive fuzzy relation ρ_X on $\langle X, \approx_B \rangle$, we study necessary and sufficient conditions guaranteeing the existence of a fuzzy relation ρ_B with the required properties and a contraction $h \colon B \to X$ such that $(i, h) \colon \langle X, \approx_B, \rho_X \rangle \leftrightharpoons \langle B, \approx_B, \rho_B \rangle$.

In order to analyze the existence of an appropriate extension ρ_B of a given ρ_X, we consider the notion of h-reflexive closure of ρ_X introduced below.

Definition 16: Given a fuzzy structure $\mathcal{B} = \langle B, \approx_B \rangle$, a nonempty crisp subset $X \subseteq B$, a $\approx_B$-reflexive, $\otimes$-$\approx_B$-antisymmetric and $\otimes$-transitive fuzzy relation ρ_X on $\langle X, \approx_B \rangle$, and a contraction $h \colon B \to X$, the *h-reflexive closure of ρ_X* is the fuzzy relation $\mu_h \colon B \times B \to L$ defined as follows

$$\mu_h(b_1, b_2) = \begin{cases} \rho_X(b_1, h(b_2)) & \text{, if } b_1 \in X \,, \\ b_1 \approx_B b_2 & \text{, if } b_1 \notin X \,. \end{cases}$$

The term h-reflexive closure makes sense since μ_h is $\approx_B$-reflexive, as will be shown below, and, moreover, any suitable fuzzy relation ρ_B which extends $\mathcal{B} = \langle B, \approx_B \rangle$ to a fuzzy preordered structure for which there exists a contraction h such that $(i, h) \colon \langle X, \approx_B, \rho_X \rangle \leftrightharpoons \langle B, \approx_B, \rho_B \rangle$ should satisfy $\mu_h \leq \rho_B$.

Lemma 2: The fuzzy relation μ_h is $\approx_B$-reflexive.

Although μ_h is $\approx_B$-reflexive, it might fail to be $\otimes$-transitive, as shown in Example 5. Therefore, the transitive closure of μ_h, denoted μ_h^t, should be contained in ρ_B as well.

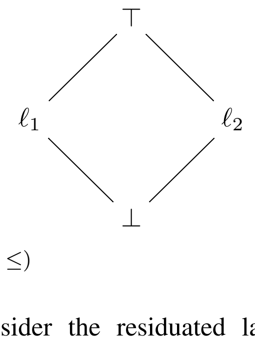

Fig. 1. The lattice $(L, \leq)$

Example 5: Consider the residuated lattice $\mathbb{L} = (L, \leq, \top, \bot, \otimes, \to)$ where $(L, \leq)$ is depicted in Figure 1 and the product $\otimes$ is the meet operation.

Consider $B = \{x_1, x_2, b\}$, the subset $X = \{x_1, x_2\}$ and the two $\mathbb{L}$-fuzzy relations below:

$\approx_B$	x_1	x_2	b
x_1	$\top$	ℓ_2	ℓ_1
x_2	ℓ_2	$\top$	ℓ_2
b	ℓ_1	ℓ_2	$\top$

ρ_X	x_1	x_2
x_1	$\top$	ℓ_2
x_2	ℓ_2	$\top$

For the contraction $h\colon B \to X$, where $h(x_1) = h(b) = x_1$ and $h(x_2) = x_2$, the h-reflexive closure of ρ_X is given in the following table:

μ_h	x_1	x_2	b
x_1	$\top$	ℓ_2	$\top$
x_2	ℓ_2	$\top$	ℓ_2
b	ℓ_1	ℓ_2	$\top$

Note that μ_h is not $\otimes$-transitive, since $\mu_h(b, x_2) \otimes \mu_h(x_2, x_1)$ and $\mu_h(b, x_1)$ are not comparable.

Concerning $\otimes$-$\approx_B$-antisymmetry, if a fuzzy relation ρ_B is $\otimes$-$\approx_B$-antisymmetric, then any other relation μ such that $\mu \le \rho_B$ is also $\otimes$-$\approx_B$-antisymmetric. If there were a Galois connection $(i, h)\colon \langle X, \approx_B, \rho_X\rangle \leftrightarrows \langle B, \approx_B, \rho_B\rangle$ for a contraction h and a suitable fuzzy relation ρ_B, we would have $\mu_h^t \le \rho_B$, and then μ_h^t would be $\otimes$-$\approx_B$-antisymmetric and, therefore, that is a necessary condition for μ_h^t.

Lemma 3: Let $X \neq \varnothing$ be a subset of B such that $\langle X, \approx_B, \rho_X\rangle$ is a fuzzy preordered structure and let $h\colon B \to X$ be a contraction. The h-reflexive closure of ρ_X (i.e. μ_h) satisfies the following properties:

1) $\mu_h(b_1, b_2) \le \mu_h^2(b_1, b_2)$ for all $b_1, b_2 \in B$.
2) $\mu_h^2(x, b) = \mu_h(x, b)$ for all $x \in X$ and $b \in B$.
3) $\mu_h^2(b_1, b_2) = \mu_h^3(b_1, b_2)$ for all $b_1, b_2 \in B$.
4) μ_h^2 is the transitive closure of μ_h.

Theorem 7: Consider a nonempty subset X of a fuzzy structure $\mathcal{B} = \langle B, \approx_B\rangle$ together with a $\approx_B$-reflexive, $\otimes$-$\approx_B$-antisymmetric and $\otimes$-transitive fuzzy relation ρ_X on $\mathcal{X} = \langle X, \approx_B\rangle$. For a contraction $h\colon B \to X$ and its h-reflexive closure μ_h, the following statements are equivalent:

1) There exists a $\approx_B$-reflexive, $\otimes$-$\approx_B$-antisymmetric and $\otimes$-transitive fuzzy relation ρ_B on $\mathcal{B}$ such that the pair (i, h) is a Galois connection between $\langle \mathcal{X}, \rho_X\rangle$ and $\langle \mathcal{B}, \rho_B\rangle$.
2) μ_h^2 is $\otimes$-$\approx_B$-antisymmetric.

According to Theorem 7 and Proposition 3 (below), the necessary and sufficient condition for the existence of a fuzzy preorder structure on B and the right adjoint of the embedding $i\colon X \to B$, for a subset X of B, is the existence of a contraction $h\colon B \to X$ such that μ_h^2 is $\otimes$-$\approx_B$-antisymmetric.

In the rest of the section, we will identify suitable conditions that guarantee this kind of antisymmetry.

Proposition 3: Consider a nonempty subset X of a fuzzy structure $\mathcal{B} = \langle B, \approx_B\rangle$ together with a $\approx_B$-reflexive, $\otimes$-$\approx_B$-antisymmetric and $\otimes$-transitive fuzzy relation ρ_X on X. For a contraction $h\colon B \to X$ and its h-reflexive closure μ_h, it holds that μ_h^2 is $\otimes$-$\approx_B$-antisymmetric if and only if the following conditions are satisfied:

1) $\rho_X(x, h(b)) \le \bigwedge\limits_{y \in X} \Big(\big((b \approx_B y) \otimes \rho_X(y, x)\big) \to (b \approx_B x)\Big)$ for all $x \in X$ and $b \notin X$.
2) $(b_1 \approx_B x) \otimes \rho_X(x, h(b_2)) \otimes (b_2 \approx_B y) \otimes \rho_X(y, h(b_1)) \le (b_1 \approx_B b_2)$, for all $x, y \in X$ and $b_1, b_2 \in B \smallsetminus X$.

Combining the previous results, we obtain the following conclusive theorem.

Theorem 8: Consider a fuzzy preordered structure $\mathbb{A} = \langle \mathcal{A}, \rho_A\rangle$, a fuzzy structure $\mathcal{B}$ and a $\approx$-injective morphism $f\colon \mathcal{A} \to \mathcal{B}$. There exist a morphism $g\colon \mathcal{B} \to \mathcal{A}$ and a $\approx_B$-reflexive, $\otimes$-$\approx_B$-antisymmetric and $\otimes$-transitive fuzzy relation ρ_B such that $(f, g) : \langle \mathcal{A}, \rho_A\rangle \leftrightarrows \langle \mathcal{B}, \rho_B\rangle$ if and only if there exists a contraction $h\colon \langle B, \approx_B\rangle \to \langle f(A), \approx_B\rangle$ such that

1) $\rho_{f(A)}(x, h(b)) \le \bigwedge_{y \in f(A)} \Big(\big((b \approx_B y) \otimes \rho_{f(A)}(y, x)\big) \to (b \approx_B x)\Big)$ for all $x \in f(A)$ and $b \in B \smallsetminus f(A)$.
2) $\rho_{f(A)}(x, h(b_2)) \otimes \rho_{f(A)}(y, h(b_1)) \le \big((b_1 \approx_B x) \otimes (b_2 \approx_B y)\big) \to (b_1 \approx_B b_2)$, for all $x, y \in f(A)$ and $b_1, b_2 \in B \smallsetminus f(A)$.

where the fuzzy relation $\rho_{f(A)}$ is defined as $\rho_{f(A)}(f(a_1), f(a_2)) = \rho_A(a_1, a_2)$.

VI. CONSTRUCTING THE RIGHT ADJOINTS

The results of the previous sections lead to the following procedure for checking the existence and constructing a right adjoint of a given morphism f:

(i) Firstly, consider the canonical projection φ_f and the canonical embedding i_f of f, so that we have $f = i_f \circ \varphi_f$.
(ii) Since φ_f is surjective and, thus, $\approx$-surjective, we can verify the sufficient conditions of Theorem 5 and, in case of fulfillment, construct a right adjoint ψ of φ_f. In case the conditions do not hold, by Theorem 4, there does not exist a right adjoint of φ_f and then the procedure ends since, by Theorem 2 , there does not exist a right adjoint of f either.
(iii) If the previous strategy was successful, since i_f is injective and $\approx$-injective, we proceed by verifying the necessary and sufficient conditions of Theorem 8. In case of fulfillment, we construct a right adjoint h of i_f and construct the right adjoint of f as the composition $\psi \circ h$. Obviously, in case the conditions do not hold, the procedure ends again unsuccessfully.

The preceding procedure can be more formally stated as Algorithm 1.

In the following examples we consecutively illustrate several cases of application of this procedure: an example in which the existence of a right adjoint of φ_f fails, then an example in which φ_f has a right adjoint but i_f does not and, finally, and example in which both parts of the canonical decomposition have a right adjoint.

Example 6: Consider the underlying truth-values set $\mathbb{L}$ to be the real unit interval with its residuated lattice structure induced by the Łukasiewicz t-norm.

Consider the following fuzzy preordered structure $\mathbb{A} = \langle A, \approx_A, \rho_A\rangle$ where $A = \{a_1, a_2, a_3\}$ and the fuzzy relations

Algorithm 1: Building Galois Connection

```
Data: A finite fuzzy preordered structure $\langle A, \approx_A, \rho_A \rangle$, a
      finite fuzzy structure $\langle B, \approx_B \rangle$ and a morphism
      $f\colon \langle A, \approx_A \rangle \to \langle B, \approx_B \rangle$.
Result: A morphism $g\colon \langle B, \approx_B \rangle \to \langle A, \approx_A \rangle$ and a
        $\approx_B$-reflexive, $\otimes$-$\approx_B$-antisymmetric and
        $\otimes$-transitive fuzzy relation $\rho_B$ such that
        $(f, g) : \langle A, \approx_A, \rho_A \rangle \leftrightharpoons \langle B, \approx_B, \rho_B \rangle$ if they exist,
        or the message "It is not possible to build a
        Galois connection" otherwise.
Compute the relation $\equiv_f$ on $A$ defined by
  $(a_1 \equiv_f a_2) := (f(a_1) \approx_B f(a_2))$
foreach $a \in A$ do
    Compute $\mathrm{qmax}_{\mathbb{A}}([a]_f)$ where $[a]_f$ is the equivalence
      class of $a$ w.r.t. $\equiv_f$
    if $\mathrm{qmax}_{\mathbb{A}}([a]_f) = \varnothing$ then return "It is not possible
      to build a Galois connection"
    else Let $b = f(a)$ and consider an arbitrary element
      $\psi(b)$ from $\mathrm{qmax}_{\mathbb{A}}([a]_f)$
foreach $a_1, a_2 \in A$ do
    if $\rho_A(a_1, a_2) \not\leq \rho_A(\psi f(a_1), \psi f(a_2))$ or
      $(a_1 \equiv_f a_2) \not\leq (\psi f(a_1) \approx_A \psi f(a_2))$ then
        return "It is not possible to build a Galois
          connection"
Define $\rho_{f(A)}$ as $\rho_{f(A)}(b_1, b_2) := \rho_A(\psi(b_1), \psi(b_2))$ for
  each $b_1, b_2 \in f(A)$
foreach contraction $h\colon B \to f(A)$ do
    Define $\mu_h$ in $B$ as:
        $\mu_h(b_1, b_2) := \rho_{f(A)}(b_1, h(b_2))$ if $b_1 \in f(A)$ and
        $\mu_h(b_1, b_2) := (b_1 \approx_B b_2)$ otherwise
    Compute $\rho_B := \mu_h^2$ and $g := \psi \circ h$
    if $\rho_B$ is $\otimes$-$\approx_B$-antisymmetric then return $g$ and $\rho_B$
return "It is not possible to build a Galois connection"
```

$\approx_A$ and ρ_A given below:

$\approx_A$	a_1	a_2	a_3
a_1	1	0.5	0
a_2	0.5	1	0.5
a_3	0	0.5	1

ρ_A	a_1	a_2	a_3
a_1	1	1	1
a_2	0.5	1	1
a_3	0	0.5	1

Now, consider $B = \{b_1, b_2, b_3\}$ and the fuzzy equivalence relation $\approx_B$ given below:

$\approx_B$	b_1	b_2	b_3
b_1	1	0.7	0.8
b_2	0.7	1	0.7
b_3	0.8	0.7	1

Finally, consider the morphism $f\colon \mathcal{A} \to \mathcal{B}$ defined by $f(a_1) = f(a_2) = b_1$ and $f(a_3) = b_2$.

We proceed to the construction of a right adjoint of f as described above by considering the canonical decomposition of f as $i_f \circ \varphi_f$; so, let us check the conditions of Theorem 5 for the morphism $\varphi_f\colon \mathcal{A} \to \langle f(A), \approx_B \rangle$.

The equivalence classes w.r.t. the fuzzy kernel relation are the following: $[a_1]_{\varphi_f} = [a_2]_{\varphi_f} = \{(a_1, 1), (a_2, 1), (a_3, 0.7)\}$ and $[a_3]_{\varphi_f} = \{(a_1, 0.7), (a_2, 0.7), (a_3, 1)\}$. It is straightforward to check that $\mathrm{qmax}_{\mathbb{A}}([a_1]_{\varphi_f})$ is empty and, hence, there is not a right adjoint of φ_f. Therefore, we conclude that there does not exist a right adjoint of f.

Example 7: Consider the same residuated lattice $\mathbb{L}$ and the fuzzy preordered structure $\mathbb{A}$ given in the previous example, but consider the (different) fuzzy equivalence relation on B defined by

$\approx_B$	b_1	b_2	b_3
b_1	1	0.5	0.8
b_2	0.5	1	0.7
b_3	0.8	0.7	1

It is not difficult to check that the mapping f in the previous example is also a morphism between the fuzzy structures $\mathcal{A}$ and $\langle B, \approx_B \rangle$. Once again, we consider the canonical decomposition of f as $i_f \circ \varphi_f$.

In this case, the morphism $\varphi_f\colon \mathcal{A} \to \langle f(A), \approx_B \rangle$ fulfills the conditions of Theorem 5: the equivalence classes w.r.t. the fuzzy kernel relation are the following: $[a_1]_{\varphi_f} = [a_2]_{\varphi_f} = \{(a_1, 1), (a_2, 1), (a_3, 0.5)\}$ and $[a_3]_{\varphi_f} = \{(a_1, 0.5), (a_2, 0.5), (a_3, 1)\}$. Observe that $\mathrm{qmax}_{\mathbb{A}}([a]_{\varphi_f})$ is not empty for all $a \in A$ since $\mathrm{qmax}_{\mathbb{A}}([a_1]_{\varphi_f}) = \mathrm{qmax}_{\mathbb{A}}([a_2]_{\varphi_f}) = \{a_2\}$ and $\mathrm{qmax}_{\mathbb{A}}([a_3]_{\varphi_f}) = \{a_3\}$.

Furthermore, condition 2 of Theorem 5 holds since

$$\begin{aligned}\rho_A(a_1, a_3) = 1 &\leq \rho_A\big(\mathrm{qmax}_{\mathbb{A}}([a_1]_{\varphi_f}), \mathrm{qmax}_{\mathbb{A}}([a_3]_{\varphi_f})\big) \\ &= \rho_A(a_2, a_3) = 1\end{aligned}$$

and the remaining cases are straightforward.

Condition 3 of Theorem 5 is also fulfilled since

$$\begin{aligned}(a_1 \equiv_{\varphi_f} a_3) &= \big(\varphi_f(a_1) \approx_B \varphi_f(a_3)\big) = 0.5 \\ &\leq \big(\mathrm{qmax}_{\mathbb{A}}([a_1]_{\varphi_f}) \approx_A \mathrm{qmax}_{\mathbb{A}}([a_3]_{\varphi_f})\big) \\ &= (a_2 \approx_A a_3) = 0.5\end{aligned}$$

and the remaining cases are similar.

Thus, according to Theorem 5, it is possible to define $\rho_{f(A)}$ as follows:

$\rho_{f(A)}$	b_1	b_2
b_1	1	1
b_2	0.5	1

and the right adjoint of φ_f as $\psi(b_1) = a_2$ and $\psi(b_2) = a_3$.

Now, we will use Theorem 8 for studying the existence of a right adjoint of the canonical embedding i_f. In this case, no contraction can be defined from B to $f(A)$, therefore there does not exist a right adjoint of the canonical embedding and as a consequence, a right adjoint of the initial mapping f does not exist either.

Example 8: Continuing with the same residuated lattice $\mathbb{L}$ and the fuzzy preordered structure $\mathbb{A}$ given in the previous example, consider a third fuzzy equivalence relation in B defined by

$\approx_B$	b_1	b_2	b_3
b_1	1	0.5	0
b_2	0.5	1	0.5
b_3	0	0.5	1

The mapping f given in the previous example is still a morphism between the fuzzy structures $\mathcal{A}$ and $\langle B, \approx_B \rangle$.

Following the first steps of the previous example, we obtain the same right adjoint of the canonical projection. For the canonical embedding, we will consider the mapping $h\colon B \to f(A)$ given by $h(b_1) = b_1, h(b_2) = b_2$ and $h(b_3) = b_1$, which is a contraction since

$$(b_2 \approx_B b_3) = 0.5 \leq \big(h(b_2) \approx_B h(b_3)\big) = (b_2 \approx_B b_1) = 0.5\,.$$

For condition 1 of Theorem 8 two cases have to be considered: to begin with, given $b_3 \in B \smallsetminus f(A)$ and $b_2 \in f(A)$ we have $\rho_{f(A)}(b_2, h(b_3)) = \rho_{f(A)}(b_2, b_1) = 0.5$ and

$$\begin{aligned}\big((b_3 \approx_B b_1) \otimes \rho_{f(A)}(b_1, b_2) \to (b_3 \approx_B b_2)\big)&\\ \wedge \big((b_3 \approx_B b_2) \otimes \rho_{f(A)}(b_2, b_2) \to (b_3 \approx_B b_2)\big)&\\ = \big((0 \otimes 1) \to 0.5\big) \wedge \big((0.5 \otimes 1) \to 0.5\big) = 1\,.&\end{aligned}$$

For the other possible case, $b_3 \in B \smallsetminus f(A)$ and $b_1 \in f(A)$, we proceed analogously.

As condition 2 of Theorem 8 holds trivially, we obtain that the canonical embedding has a right adjoint as well, for which the fuzzy relation ρ_B is given by μ_h^2, with μ_h the h-reflexive closure of $\rho_{f(A)}$. More specifically, μ_h and ρ_B are given by the following tables:

μ_h	b_1	b_2	b_3
b_1	1	1	1
b_2	0.5	1	0.5
b_3	0	0.5	1

μ_h^2	b_1	b_2	b_3
b_1	1	1	1
b_2	0.5	1	0.5
b_3	0	1	1

The right adjoint of the canonical embedding i_f is the contraction h given above.

Finally, the right adjoint of the initial morphism f is the composition of ψ and h which is given by $g(b_1) = \psi(h(b_1)) = a_2, g(b_2) = \psi(h(b_2)) = a_3$ and $g(b_3) = \psi(h(b_3)) = \psi(b_1) = a_2$.

ACKNOWLEDGMENT

I.P. Cabrera, P. Cordero, F. García-Pardo and M. Ojeda-Aciego are partially supported by the Spanish Ministry of Science projects TIN2014-59471-P, and TIN2015-70266-C2-1-P, both co-funded by the European Regional Development Fund (ERDF).

VII. CONCLUSIONS AND FUTURE WORK

Given a mapping $f\colon \mathbb{A} \to B$ from a fuzzy preordered structure $\mathbb{A}$ into a fuzzy structure $\langle B, \approx_B\rangle$, we have characterized when it is possible to construct a fuzzy relation ρ_B that induces a suitable fuzzy preorder structure on B and such that there exists a mapping $g\colon B \to \mathbb{A}$ such that the pair (f, g) constitutes a Galois connection. In the case of existence of right adjoint, it is worth remarking that the right adjoint need not be unique since, actually, its construction is given with several of degrees of freedom, in particular for extending the fuzzy ordering from the image of f to the entire codomain. Although a convenient extension has been given, our results do not imply that every right adjoint can be constructed in this way, and there may exist other constructions that are adequate as well. This is a first topic for future work.

This paper continues the line of [23] where we consider a mapping $f\colon \langle A, \rho_A\rangle \to B$ (and ρ_A is a fuzzy relation satisfying reflexivity, $\otimes$-transitivity and the weakest form of antisymmetry, namely, $\rho_A(a, b) = \rho_A(b, a) = \top$ implies $a = b$, for all $a, b \in A$); a further step was given in [10] for the same case $f\colon \langle A, \rho_A\rangle \to B$, in which antisymmetry was dropped. Both cases above can be seen as fuzzy preordered structures, in the sense of this paper, just by considering the so-called symmetric kernel relation (the conjunction of $\rho_A(a, b)$ and $\rho_A(a, b)$); the relationship between these and other kinds of structures can be found in [46]. Summarizing, the problem in [10] can be seen as constructing a right adjoint of a mapping $f\colon \langle A, \approx_A, \rho_A\rangle \to B$ which involves the construction of both $\approx_B$ and ρ_B, whereas in this paper our problem is to find a right adjoint to a mapping $f\colon \langle A, \approx_A, \rho_A\rangle \to \langle B, \approx_B\rangle$ in which the fuzzy equivalence $\approx_B$ is already given and has to be preserved; therefore, the main result in [10] is not exactly a particular case. We have considered a fuzzy mapping as a morphism $\langle A, \approx_A\rangle \to \langle B, \approx_B\rangle$ between fuzzy structures, adopting the approach of [17], while our long-term goal is to study fuzzy Galois connections constituted of truly fuzzy mappings.

As stated in the introduction, Galois connections have found applications in areas such as formal concept analysis, where the intent and extent operators form a Galois connection, and in mathematical morphology, where the erosion and the dilation operations are often required to form a Galois connection as well (one of the approaches not requiring this can be seen in [41]). The results presented in this work pave the way to build specific settings of mathematical morphology parameterized by a fixed candidate to be an erosion (or dilation) operator; and the same approach would also apply to the development of new settings of formal concept analysis. In general, the construction of new Galois connections is of interest in fields in which there are two approaches to certain reality and one has more information about one of them, since the existence of a Galois connection allows to retrieve the unknown information in the other approach. In this respect, as future work, we will explore the application of the obtained results in the area of compression of data (images, signals, etc.) in which the existence of the right adjoint of a given compressing mapping might allow to recover as much information as possible.

Last but not least, it is worth to study the two following extensions of the present work: on the one hand, we could consider an even more general notion of fuzzy mapping, for instance that proposed in [12]; on the other hand, we could consider $\mathbb{L}$-valued sets as a suitable generalization of our fuzzy structures.

REFERENCES

[1] R. Bělohlávek. Fuzzy Galois connections. *Mathematical Logic Quarterly*, 45(4):497–504, 1999.

[2] R. Bělohlávek. Lattices of fixed points of fuzzy Galois connections. *Mathematical Logic Quarterly*, 47(1):111–116, 2001.

[3] R. Bělohlávek, B. De Baets and J. Konecny, Granularity of attributes in formal concept analysis, *Information Sciences*, 260:149–170, 2014.

[4] R. Bělohlávek, B. De Baets, J. Outrata and V. Vychodil. Characterizing trees in concept lattices, *Int. J. of Uncertainty, Fuzziness and Knowledge-Based Systems*, 16:1–15, 2008.

[5] R. Bělohlávek, B. De Baets, J. Outrata and V. Vychodil, Inducing decision trees via concept lattices, *Int. J. of General Systems*, 38:455–467, 2009.

[6] R. Bělohlávek and P. Osička. Triadic fuzzy Galois connections as ordinary connections. *Fuzzy Sets and Systems*, 249:83–99, 2014.

[7] I. Bloch. Fuzzy sets for image processing and understanding, *Fuzzy Sets and Systems*, 281:280–291, 2015.

[8] U. Bodenhofer, B. De Baets and J. Fodor. A compendium of fuzzy weak orders: Representations and constructions. *Fuzzy Sets and Systems*, 158(8):811–829, 2007.

[9] F. Börner. Basics of Galois connections. *Lect. Notes in Computer Science*, 5250:38–67, 2008.

[10] I.P. Cabrera, P. Cordero, F. García-Pardo, M. Ojeda-Aciego, and B. De Baets. On the construction of adjunctions between a fuzzy preposet and an unstructured set. *Fuzzy Sets and Systems*, 320:81–92, 2017

[11] G. Ciobanu and C. Vaideanu. Similarity relations in fuzzy attribute-oriented concept lattices. *Fuzzy Sets and Systems*, 275:88–109, 2015.

[12] M. Ćirić, J. Ignjatović and S. Bogdanović. Uniform fuzzy relations and fuzzy functions. *Fuzzy Sets and Systems*, 160:1054–1081, 2009.

[13] P. Cordero, M. Enciso, A. Mora, M. Ojeda-Aciego, and C. Rossi. Knowledge discovery in social networks by using a logic-based treatment of implications. *Knowledge-Based Systems*, 87:16–25, 2015.

[14] B. De Baets, E. Kerre and M. Gupta. The fundamentals of fuzzy mathematical morphology. Part 1: Basic concepts, *Int. J. of General Systems*, 23:155–171, 1994.

[15] B. De Baets. Fuzzy morphology: a logical approach. In *Uncertainty Analysis in Engineering and Sciences: Fuzzy Logic, Statistics and Neural Network Approach* (B. Ayyub and M. Gupta, eds.), pp. 53–67, Kluwer Academic Publishers, 1997.

[16] B. De Baets. Generalized idempotence in fuzzy mathematical morphology. In *Fuzzy Techniques in Image Processing* (E. Kerre and M. Nachtegael, eds.), Studies in Fuzziness and Soft Computing 52:58–75, Physica-Verlag, 2000.

[17] M. Demirci. A theory of vague lattices based on many-valued equivalence relations—I: general representation results. *Fuzzy Sets and Systems*, 151:437–472, 2005.

[18] M. Demirci. The order-theoretic duality and relations between partial metrics and local equalities. *Fuzzy Sets and Systems*, 192:45–57, 2012.

[19] K. Denecke, M. Erné, and S. L. Wismath. *Galois Connections and Applications*, Kluwer Academic Publishers, 2004.

[20] Y. Djouadi and H. Prade. Interval-valued fuzzy Galois connections: Algebraic requirements and concept lattice construction. *Fundamenta Informaticae*, 99(2):169–186, 2010.

[21] A. Frascella. Fuzzy Galois connections under weak conditions. *Fuzzy Sets and Systems*, 172(1):33–50, 2011.

[22] F. García-Pardo, I.P. Cabrera, P. Cordero, and M. Ojeda-Aciego. On Galois connections and soft computing. *Lect. Notes in Computer Science*, 7903:224–235, 2013.

[23] F. García-Pardo, I.P. Cabrera, P. Cordero, and M. Ojeda-Aciego. On the construction of fuzzy Galois connections. In *Proc. of the XVII Spanish Conf. on Fuzzy Logic and Technology*, 99–102, 2014.

[24] F. García-Pardo, I.P. Cabrera, P. Cordero, M. Ojeda-Aciego and F.J. Rodríguez. On the definition of suitable orderings to generate adjunctions over an unstructured codomain. *Information Sciences*, 286:173-187, 2014.

[25] G. Georgescu and A. Popescu. Non-commutative fuzzy Galois connections. *Soft Computing*, 7(7):458–467, 2003.

[26] M. González-Hidalgo, S. Massanet, A. Mir and D. Ruiz-Aguilera. A fuzzy morphological hit-or-miss transform for grey-level images: A new approach. *Fuzzy Sets and Systems*, 286:30–65, 2016.

[27] J. Gutiérrez-García, I. Mardones-Pérez, M. A. de Prada-Vicente, and D. Zhang. Fuzzy Galois connections categorically. *Mathematical Logic Quarterly*, 56(2):131–147, 2010.

[28] U. Höhle. Classification of subsheaves over GL-algebras. In S.R. Buss, P. Hájek and P. Pudlak (eds) Logic Colloquium'98, *Lecture Notes in Logic* 13, 2000.

[29] J. Järvinen. Pawlak's information systems in terms of Galois connections and functional dependencies. *Fundamenta Informaticae*, 75:315–330, 2007.

[30] J. Konecny. Isotone fuzzy Galois connections with hedges. *Information Sciences*, 181(10):1804–1817, 2011.

[31] J. Konecny and M. Krupka. Block relations in formal fuzzy concept analysis. *Intl. J. of Approximate Reasoning*, 73:27–55, 2016.

[32] S. Kuznetsov. Galois connections in data analysis: Contributions from the Soviet era and modern Russian research. *Lect. Notes in Computer Science*, 3626:196–225, 2005.

[33] S.-T. Li and F.-C. Tsai. A fuzzy conceptualization model for text mining with application in opinion polarity classification. *Knowledge-Based Systems*, 39:23–33, 2013.

[34] T. Martin and A. Majidian. Finding fuzzy concepts for creative knowledge discovery. *Intl. J. of Intelligent Systems*, 28(1):93–114, 2013.

[35] A. Melton, D. A. Schmidt and G. E. Strecker. Galois connections and computer science applications. *Lect. Notes in Computer Science*, 240:299–312, 1986.

[36] S.-C. Mu and J. Oliveira. Programming from Galois connections. *J. of Logic and Algebraic Programming*, 81(6):680–704, 2012.

[37] J. Pócs. Note on generating fuzzy concept lattices via Galois connections. *Information Sciences*, 185(1):128–136, 2012.

[38] J. Propp. A Galois connection in the social network. *Mathematics Magazine*, 85(1):34–36, 2012.

[39] Y. Shi, M. Nachtegael, D. Ruan and E. Kerre. Fuzzy adjunctions and fuzzy morphological operations based on implications, *Intl. J. of Intelligent Systems*, 24(12):1280–1296, 2009.

[40] P. Sussner. Lattice fuzzy transforms from the perspective of mathematical morphology, *Fuzzy Sets and Systems*, 288:115–128, 2016.

[41] P. Sussner and M.E. Valle. Classification of fuzzy mathematical morphologies based on concepts of inclusion measure and duality, *Journal of Mathematical Imaging and Vision*, 32:139–159, 2008.

[42] F. J. Valverde-Albacete, C. Peláez-Moreno and C. del Campo. Activating Generalized Fuzzy Implications from Galois Connections. In *Enric Trillas: A Passion for Fuzzy Sets*, Studies in Fuzziness and Soft Computing 322:201–212, 2015.

[43] S. Vickers. *Topology via logic*. Cambridge University Press, 1996.

[44] M. Wolski. Galois connections and data analysis. *Fundamenta Informaticae*, 60:401–415, 2004.

[45] W. Yao and L.-X. Lu. Fuzzy Galois connections on fuzzy posets. *Mathematical Logic Quarterly*, 55(1):105–112, 2009.

[46] W. Yao. Quantitative domains via fuzzy sets. Part I: Continuity of fuzzy directed complete posets. *Fuzzy Sets and Systems*, 161:973–987, 2010.

APPENDIX

PROOFS OF THE RESULTS

Proof of Proposition 1: Assume that f and g are morphisms and the equality $\rho_A(a, g(b)) = \rho_B(f(a), b)$ holds for all $a \in A$ and $b \in B$.

Let $a_1, a_2 \in A$ and $b \in B$. Since f is a morphism, it holds

$$(a_1 \approx_A a_2) \otimes \rho_A(a_2, g(b)) \leq (f(a_1) \approx_B f(a_2)) \otimes \rho_A(a_2, g(b)).$$

By the hypothesis, we obtain that

$$\begin{aligned}(f(a_1) \approx_B f(a_2)) &\otimes \rho_A(a_2, g(b)) \\ &= (f(a_1) \approx_B f(a_2)) \otimes \rho_B(f(a_2), b).\end{aligned}$$

As ρ_B is $\approx_B$-reflexive and transitive, we have that

$$\begin{aligned}(f(a_1) \approx_B f(a_2)) &\otimes \rho_B(f(a_2), b) \\ &\leq \rho_B(f(a_1), f(a_2)) \otimes \rho_B(f(a_2), b) \leq \rho_B(f(a_1), b).\end{aligned}$$

Therefore, $(a_1 \approx_A a_2) \otimes \rho_A(a_2, g(b)) \leq \rho_B(f(a_1), b)$ for all $a_1, a_2 \in A$ and $b \in B$ and Condition (G1) holds. Condition (G2) follows similarly.

Conversely, assume now that (f, g) is a Galois connection, then conditions (G1) and (G2) hold and f and g are morphisms. Applying condition (G1), for $a \in A$ and $b \in B$, we have that $(a \approx_A a) \otimes \rho_A(a, g(b)) \leq \rho_B(f(a), b)$. As $\approx_A$ is reflexive, we obtain that $\rho_A(a, g(b)) \leq \rho_B(f(a), b)$ for all $a \in A$ and $b \in B$. The inequality $\rho_B(f(a), b) \leq \rho_A(a, g(b))$ follows similarly. ■

Proof of Theorem 1: Since f is isotone and $g \circ f$ inflationary, $\top = \rho_A(a, gf(a)) \leq \rho_B(f(a), fgf(a))$, thus, $\rho_B(f(a), fgf(a)) = \top$. Moreover, $\rho_B(fgf(a), f(a)) = \rho_A(gf(a), gf(a)) = \top$. Therefore, by $\otimes$-$\approx_B$-antisymmetry, we obtain that $\big(fgf(a) \approx_B f(a)\big) = \top$. ■

Proof of Corollary 2: We will only prove the first item, since the second is similar.

Since g is a morphism, given $a_1, a_2 \in A$, we have $\big(f(a_1) \approx_B f(a_2)\big) \leq \big(gf(a_1) \approx_A gf(a_2)\big)$. Moreover, since f is a morphism, we also have

$$\big(gf(a_1) \approx_A gf(a_2)\big) \leq \big(fgf(a_1) \approx_B fgf(a_2)\big).$$

Now, by Theorem 1, we have that $\big(f(a) \approx_B fgf(a)\big) = \top$, for all $a \in A$. Finally, the $\otimes$-transitivity of $\approx_B$ leads to

$$\big(fgf(a_1) \approx_B fgf(a_2)\big) \leq \big(f(a_1) \approx_B f(a_2)\big).$$

■

Proof of Theorem 2: Assume $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$, and consider the mappings $\varphi_f \colon A \to f(A)$ and $\psi \colon f(A) \to A$ to be, respectively, the corresponding restriction of f and g to $f(A)$, that is $\varphi_f(a) = f(a)$ for all $a \in A$, and $\psi(x) = g(x)$ for all $x \in f(A)$. It is straightforward to see that ψ is a morphism and the pair (φ_f, ψ) is a Galois connection between $\mathbb{A}$ and $f(\mathbb{A})$ because, for each $a \in A$ and $x \in f(A)$, we have that

$$\rho_A(a, \psi(x)) = \rho_A(a, g(x)) = \rho_B(f(a), x) = \rho_B(\varphi_f(a), x).$$

Consider now the embedding $i_f \colon f(A) \to B$ such that $i_f(x) = x$ for all $x \in f(A)$ and $h \colon B \to f(A)$ such that $h(y) = f(g(y))$ for all $y \in B$. Since f and g are morphisms, the composition h is a morphism as well.

We will now prove that $\rho_B(x, h(y)) = \rho_B(i_f(x), y)$ for all $x \in f(A)$ and $y \in B$. Since $f \circ g$ is deflationary, we have $\rho_B(fg(y), y) = \top$ and, then

$$\begin{aligned}\rho_B(x, h(y)) &= \rho_B(x, fg(y)) \\ &= \rho_B(x, fg(y)) \otimes \rho_B(fg(y), y) \leq \rho_B(i_f(x), y).\end{aligned}$$

On the other hand, since $x \in f(A)$, there exists $a \in A$ such that $f(a) = x = i_f(x)$ and, using the fact that the pair (f, g) is a Galois connection, we obtain

$$\begin{aligned}\rho_B(i_f(x), y) &= \rho_B(f(a), y) \\ &= \rho_A(a, g(y)) \leq \rho_B(f(a), fg(y)) = \rho_B(x, h(y)).\end{aligned}$$

Having proved that both (φ_f, ψ) and (i_f, h) are Galois connections, we will prove the identities in (1):

(i) Using the reflexivity of $\approx_B$, for all $a \in A$, we have that

$$\top = \big(f(a) \approx_B f(a)\big) = \big(i_f\varphi_f(a) \approx_B f(a)\big).$$

(ii) By Theorem 1, for all $b \in B$, we have that

$$\big(\psi h(b) \approx_B g(b)\big) = \big(gfg(b) \approx_B g(b)\big) = \top.$$

Conversely, assume $(\varphi_f, \psi) \colon \mathbb{A} \leftrightharpoons f(\mathbb{A})$ and $(i_f, h) \colon f(\mathbb{A}) \leftrightharpoons \mathbb{B}$, and consider the pair of morphisms $(i_f \circ \varphi_f, \psi \circ h)$, which obviously forms a Galois connection between $\mathbb{A}$ and $\mathbb{B}$ since

$$\rho_A(a, \psi h(b)) = \rho_B(\varphi_f(a), h(b)) = \rho_B(i_f\varphi_f(a), b)\,.$$

Finally, the identities in (1) imply that, for all $a \in A$ and $b \in B$, $\rho_A(a, g(b)) = \rho_A(a, \psi h(b))$ and $\rho_B(i_f\varphi_f(a), b) = \rho_B(f(a), b)$. Therefore, we obtain that $(f, g) \colon \mathbb{A} \leftrightharpoons \mathbb{B}$.

■

Proof of Theorem 3: The properties of φ_f and i_f were stated before.

(i) The mapping h is $\approx$-surjective because, for all $x \in f(A)$, there exists $a \in A$ with $f(a) = x$ and, by Theorem 1, $\big(h(x) \approx_B x\big) = \big(hf(a) \approx_B f(a)\big) = \big(fgf(a) \approx_B f(a)\big) = \top$.

(ii) Let us prove that ψ is $\approx$-injective. Consider $x_1, x_2 \in f(A)$ and $a_1, a_2 \in A$ such that $f(a_1) = x_1$ and $f(a_2) = x_2$. Since (f, g) is a Galois connection, by Corollary 2, we obtain $\big(\psi(x_1) \approx_A \psi(x_2)\big) = \big(gf(a_1) \approx_A gf(a_2)\big) = \big(f(a_1) \approx_B f(a_2)\big)$ and, hence, ψ is $\approx$-injective.

■

Proof of Lemma 1: Recalling that $\rho_A(x, \bar{x}) = \top = \rho_A(\bar{x}, x)$ for all $x, \bar{x} \in \operatorname{qmax}_{\mathbb{A}}(X)$, and using the transitivity of ρ_A, we have that

$$\rho_A(x, y) = \rho_A(\bar{x}, x) \otimes \rho_A(x, y) \otimes \rho_A(y, \bar{y}) \leq \rho_A(\bar{x}, \bar{y})\,.$$

Similarly, $\rho_A(\bar{x}, \bar{y}) \leq \rho_A(x, y)$. Therefore, $\rho_A(x, y) = \rho_A(\bar{x}, \bar{y})$, for any $x, \bar{x} \in \operatorname{qmax}_{\mathbb{A}}(X)$ and $y, \bar{y} \in \operatorname{qmax}_{\mathbb{A}}(Y)$, and it turns out that all the elements computed in the definitions of the Hoare ordering collapse to the same value. ■

Proof of Theorem 4:

1) We will show that $g(f(a)) \in \operatorname{qmax}_{\mathbb{A}}([a]_f)$.
By Theorem 1, we have $\big(f(a) \approx_B fgf(a)\big) = \top$. On the other hand, using $\approx_B$-reflexivity and that (f, g) is a Galois connection, for all $u \in A$, it follows that

$$\begin{aligned}[a]_f(u) = \big(f(u) \approx_B f(a)\big) &\leq \rho_B\big(f(u), f(a)\big) \\ &= \rho_A\big(u, g(f(a))\big) = g(f(a))^{\downarrow}(u)\,.\end{aligned}$$

2) By Proposition 2, f and g are isotone maps, thus

$$\rho_A(a_1, a_2) \leq \rho_A\big(g(f(a_1)), g(f(a_2))\big)$$

for all $a_1, a_2 \in A$. We have just shown that $g(f(a)) \in \operatorname{qmax}_{\mathbb{A}}([a]_f)$ for all $a \in A$, thus, from Lemma 1, we obtain that $\rho_A(a_1, a_2) \leq \rho_A\big(\operatorname{qmax}_{\mathbb{A}}([a_1]_f), \operatorname{qmax}_{\mathbb{A}}([a_2]_f)\big)$ for all $a_1, a_2 \in A$.

3) Since g is a morphism, it holds that

$$\begin{aligned}(a_1 \equiv_f a_2) &= \big(f(a_1) \approx_B f(a_2)\big) \\ &\leq \big(g(f(a_1)) \approx_A g(f(a_2))\big).\end{aligned}$$

Finally, by Condition 1, $g(f(a_i)) \in \operatorname{qmax}_{\mathbb{A}}([a_i]_f)$ for $i \in \{1, 2\}$.

■

Proof of Theorem 5: We define the fuzzy relation $\rho_B \colon B \times B \to L$ as follows

$$\begin{aligned}\rho_B(b_1, b_2) &= \big(\operatorname{qmax}_{\mathbb{A}}([a_1]_f) \sqsubseteq_H \operatorname{qmax}_{\mathbb{A}}([a_2]_f)\big) \\ &= \rho_A\big(\operatorname{qmax}_{\mathbb{A}}([a_1]_f), \operatorname{qmax}_{\mathbb{A}}([a_2]_f)\big)\end{aligned}$$

where $a_i \in A$ satisfies $(f(a_i) \approx_B b_i) = \top$ for each $i \in \{1, 2\}$.

First, given $b \in B$, since f is a $\approx$-surjective mapping, there exists $a \in A$ such that $(f(a) \approx_B b) = \top$, so the previous construction makes sense. Furthermore, the definition of ρ_B

does not depend on the choice of a_i because, if $(f(\bar{a}_i) \approx_B b_i) = \top$, then $(\bar{a}_i \equiv_f a_i) = (f(\bar{a}_i) \approx_B f(a_i)) = \top$ and thus, by Remark 1, $[\bar{a}_i]_f = [a_i]_f$.

By Lemma 1, there always exists $c_i \in \mathrm{qmax}_{\mathbb{A}}([a_i]_f) \neq \varnothing$, for $i \in \{1,2\}$, such that $\rho_B(b_1,b_2) = \rho_A(c_1,c_2)$. We will use this equality in order to prove that ρ_B is $\approx_B$-reflexive, $\otimes$-$\approx_B$-antisymmetric and $\otimes$-transitive.

- $\approx_B$-*reflexivity*: By definition of quasi-maximum, for $i \in \{1,2\}$, we have $(f(a_i) \approx_B f(c_i)) = \top$ and $(f(a_i) \approx_B f(x)) \leq \rho_A(x,c_i)$, for all $x \in A$. Thus, since f is $\approx$-surjective, we obtain
$$\begin{aligned}
&(b_1 \approx_B b_2) = \\
&\qquad (f(a_1) \approx_B b_1) \otimes (b_1 \approx_B b_2) \otimes (b_2 \approx_B f(a_2)) \\
&\quad \leq (f(a_1) \approx_B f(a_2)) \\
&\quad = (f(a_2) \approx_B f(a_1)) \otimes (f(a_1) \approx_B f(c_1)) \\
&\quad \leq (f(a_2) \approx_B f(c_1)) \\
&\quad \leq \rho_A(c_1,c_2) = \rho_B(b_1,b_2)\,.
\end{aligned}$$
- $\otimes$-$\approx_B$-*antisymmetry*: By definition of ρ_B and the $\otimes$-$\approx_A$-antisymmetry property of ρ_A, we have
$$\begin{aligned}
\rho_B(b_1,b_2) \otimes \rho_B(b_2,b_1) &= \rho_A(c_1,c_2) \otimes \rho_A(c_2,c_1) \\
&\leq (c_1 \approx_A c_2)\,.
\end{aligned}$$
Since f is a morphism and using the fact that $(f(a_i) \approx_B f(c_i)) = \top$, we obtain
$$\begin{aligned}
&(c_1 \approx_A c_2) \leq (f(c_1) \approx_B f(c_2)) \\
&= (f(a_1) \approx_B f(c_1)) \otimes (f(c_1) \approx_B f(c_2)) \otimes (f(c_2) \approx_B f(a_2)) \\
&\leq (f(a_1) \approx_B f(a_2)) = (b_1 \approx_B b_2)\,.
\end{aligned}$$
- $\otimes$-*transitivity*: From the transitivity of ρ_A, it is straightforward that ρ_B is transitive.

In order to define $g\colon B \to A$, there are a number of suitable possibilities all of which can be obtained as follows: given $b \in B$, we choose $g(b)$ as an element $x_b \in \mathrm{qmax}_{\mathbb{A}}([a]_f)$ where $a \in A$ verifies $(f(a) \approx_B b) = \top$. The existence of x_b is guaranteed by the fact that f is $\approx_B$-surjective and Condition 1, namely, $\mathrm{qmax}_{\mathbb{A}}([a]_f)$ is not empty. Similarly as for ρ_B, it is easy to prove that $g(b)$ does not depend on the choice of a.

By Condition 3, given b_1, b_2, and for all a_i such that $(f(a_i) \approx_B b_i) = \top$, we have that
$$\begin{aligned}
(b_1 \approx_B b_2) &= (f(a_1) \approx_B b_1) \otimes (b_1 \approx_B b_2) \otimes (b_2 \approx_B f(a_2)) \\
&\leq (f(a_1) \approx_B f(a_2)) = (a_1 \equiv_f a_2) \\
&\leq (g(b_1) \approx_A g(b_2))\,,
\end{aligned}$$
hence, g is a morphism.

Now, due to Proposition 1, it suffices to prove that $\rho_A(a,g(b)) = \rho_B(f(a),b)$, for all $a \in A$ and $b \in B$. Recall that, by Lemma 1, we have $\rho_B(f(a),b) = \rho_A(u,v)$ for all $u \in \mathrm{qmax}_{\mathbb{A}}([a]_f)$ and $v \in \mathrm{qmax}_{\mathbb{A}}([z]_f)$ where $(f(z) \approx_B b) = \top$. By definition, $g(b) \in \mathrm{qmax}_{\mathbb{A}}([z]_f)$, hence $\rho_B(f(a),b) = \rho_A(u,g(b))$. Thus, we just have to prove that
$$\rho_A(u,g(b)) = \rho_A(a,g(b))$$
for all $u \in \mathrm{qmax}_{\mathbb{A}}([a]_f)$.

Given $u \in \mathrm{qmax}_{\mathbb{A}}([a]_f)$, we have $(f(a) \approx_B f(u)) = \top$ and $(f(a) \approx_B f(x)) \leq \rho_A(x,u)$, for all $x \in A$. In particular, $(f(a) \approx_B f(a)) \leq \rho_A(a,u)$, and then, since $\approx_A$ is reflexive, we obtain $\rho_A(a,u) = \top$. Therefore,
$$\rho_A(u,g(b)) = \rho_A(a,u) \otimes \rho_A(u,g(b)) \leq \rho_A(a,g(b))\,.$$
On the other hand, for any $x \in A$ with $(f(x) \approx_B b) = \top$, we have that $g(b) \in \mathrm{qmax}_{\mathbb{A}}([x]_f)$, and thus $[g(b)]_f = [x]_f$. Applying Condition 2, we obtain
$$\begin{aligned}
\rho_A(a,g(b)) &\leq \rho_A\big(\mathrm{qmax}_{\mathbb{A}}([a]_f), \mathrm{qmax}_{\mathbb{A}}([g(b)]_f)\big) \\
&= \rho_A\big(\mathrm{qmax}_{\mathbb{A}}([a]_f), \mathrm{qmax}_{\mathbb{A}}([x]_f)\big) = \rho_B(f(a),b)\,.
\end{aligned}$$
■

Proof of Theorem 6: Assume that $(f,g) : \mathbb{A} \leftrightharpoons \mathbb{B}$. Consider the embedding $i\colon f(A) \to B$ and the mapping $h\colon B \to f(A)$ defined as follows:
$$h(x) = \begin{cases} x & \text{, if } x \in f(A), \\ f(g(x)) & \text{, if } x \notin f(A)\,. \end{cases}$$
Let us prove that $(b_1 \approx_B b_2) \leq \big(h(b_1) \approx_B h(b_2)\big)$ for all $b_1, b_2 \in B$:

(i) If $b_1, b_2 \in f(A)$, then $(b_1 \approx_B b_2) = \big(h(b_1) \approx_B h(b_2)\big)$.

(ii) If $b_1 = f(a_1) \in f(A)$ and $b_2 \notin f(A)$, then using the fact that g is a morphism, f is $\approx$-injective, and Theorem 1, we have that
$$\begin{aligned}
&(b_1 \approx_B b_2) = (f(a_1) \approx_B b_2) \\
&\quad \leq (g(f(a_1)) \approx_A g(b_2)) \\
&\quad = (f(g(f(a_1))) \approx_B f(g(b_2))) \\
&\quad = (f(a_1) \approx_B f(g(f(a_1)))) \otimes (f(g(f(a_1))) \approx_B f(g(b_2))) \\
&\quad \leq (f(a_1) \approx_B f(g(b_2))) \\
&\quad = (b_1 \approx_B h(b_2)) = (h(b_1) \approx_B h(b_2)).
\end{aligned}$$

(iii) If $b_1, b_2 \notin f(A)$ then, using the fact that g and f are morphisms, we have $(b_1 \approx_B b_2) \leq (g(b_1) \approx_A g(b_2)) \leq (f(g(b_1)) \approx_B f(g(b_2))) = (h(b_1) \approx_B h(b_2))$.

Therefore, the mapping h is a morphism. On the other hand, it is straightforward that the embedding is a morphism.

Let us now show the properties of $\rho_{f(A)}$. The relation is $\approx_B$-reflexive by the $\approx$-injectivity of f and the $\approx_A$-reflexivity of ρ_A:
$$\begin{aligned}
(f(a_1) \approx_B f(a_2)) &= (a_1 \approx_A a_2) \\
&\leq \rho_A(a_1,a_2) = \rho_{f(A)}(f(a_1),f(a_2))\,.
\end{aligned}$$
The $\otimes$-$\approx_B$-antisymmetry of $\rho_{f(A)}$ is a consequence of the $\otimes$-$\approx_A$-antisymmetry of ρ_A and the $\approx$-injectivity of f:
$$\begin{aligned}
\rho_{f(A)}(f(a_1),f(a_2)) \otimes \rho_{f(A)}(f(a_2),f(a_1)) &\leq (a_1 \approx_A a_2) \\
&= (f(a_1) \approx_B f(a_2)).
\end{aligned}$$
Furthermore, the transitivity of $\rho_{f(A)}$ is a direct consequence of the definition of $\rho_{f(A)}$ and the transitivity of ρ_A.

Next, we will prove that $\rho_{f(A)}(b_1,h(b_2)) = \rho_B(i(b_1),b_2)$ for all $b_1 = f(a_1) \in f(A)$ and $b_2 \in B$. First, we prove
$$\rho_{f(A)}(b_1,h(b_2)) = \rho_{f(A)}(f(a_1),f(g(b_2)))\,.$$

If $b_2 \notin f(A)$, it is straightforward from the definition of h. In case $b_2 \in f(A)$, there exists a_2 such that $h(b_2) = b_2 = f(a_2)$ and by Corollary 2, $\rho_{f(A)}(h(b_2), f(g(b_2))) = \rho_{f(A)}(f(g(b_2)), h(b_2)) = \top$. Therefore, by transitivity of $\rho_{f(A)}$, it follows that

$$\begin{aligned}\rho_{f(A)}(b_1,h(b_2)) &= \rho_{f(A)}(b_1, h(b_2)) \otimes \rho_{f(A)}(h(b_2), f(g(b_2)))\\ &\le \rho_{f(A)}(f(a_1), f(g(b_2)))\\ &= \rho_{f(A)}(f(a_1), f(g(b_2))) \otimes \rho_{f(A)}(f(g(b_2)), h(b_2))\\ &\le \rho_{f(A)}(b_1, h(b_2))\,.\end{aligned}$$

Finally, since (f,g) is a Galois connection between $\mathbb{A}$ and $\mathbb{B}$, we have

$$\begin{aligned}\rho_{f(A)}(f(a_1),f(g(b_2))) &= \rho_A(a_1, g(b_2))\\ &= \rho_B(f(a_1), b_2) = \rho_B(b_1, b_2) = \rho_B(i(b_1), b_2)\,.\end{aligned}$$

Conversely, assume that there exists a contraction $h\colon \langle B,\approx_B\rangle \to \langle f(A),\approx_B\rangle$ satisfying $(i,h)\colon \langle f(A),\approx_B, \rho_{f(A)}\rangle \leftrightharpoons \langle B,\approx_B,\rho_B\rangle$.

By the axiom of choice, a number of suitable definitions of $g\colon B \to A$ exist such that $g(b)$ is an element of $f^{-1}(h(b))$, for each $b \in B$. That is, $f(g(b)) = h(b)$ for each $b \in B$.

Since h is a contraction and f is $\approx$-injective, we have that g is a morphism because, for all $b_1, b_2 \in B$,

$$\begin{aligned}(b_1 \approx_B b_2) \le (h(b_1) \approx_B h(b_2)) &= (f(g(b_1)) \approx_B f(g(b_2)))\\ &= (g(b_1) \approx_A g(b_2))\,.\end{aligned}$$

To conclude, again by the $\approx$-injectivity of f and using the fact that (i,h) is a Galois connection between $\langle f(A),\approx_B, \rho_{f(A)}\rangle$ and $\langle B,\approx_B,\rho_B\rangle$, we have

$$\begin{aligned}\rho_A(a, g(b)) = \rho_{f(A)}(f(a), f(g(b))) &= \rho_{f(A)}(f(a), h(b))\\ = \rho_B(i(f(a)), b)) &= \rho_B(f(a), b)\end{aligned}$$

for all $a \in A$ and $b \in B$. ■

Proof of Lemma 2: Given $b_1, b_2 \in B$, if $b_1 \in X$, then

$$\begin{aligned}(b_1 \approx_B b_2) &\le (h(b_1) \approx_B h(b_2))\\ &\le \rho_X(h(b_1), h(b_2)) = \rho_X(b_1, h(b_2)) = \mu_h(b_1, b_2)\end{aligned}$$

and if $b_1 \notin X$, we directly have $(b_1 \approx_B b_2) = \mu_h(b_1, b_2)$. ■

Proof of Lemma 3:

1. If $b_1 \in X$, then for all $b_2 \in B$

$$\begin{aligned}\mu_h(b_1, b_2) &= \rho_X(b_1, h(b_2))\\ &= \rho_X(b_1, h(b_2)) \otimes \rho_X(h(b_2), h(b_2))\\ &\overset{(*)}{=} \mu_h(b_1, h(b_2)) \otimes \mu_h(h(b_2), b_2)\\ &\le \bigvee_{x\in B} (\mu_h(b_1, x) \otimes \mu_h(x, b_2)) = \mu_h^2(b_1, b_2)\end{aligned}$$

where $(*)$ follows since $h(b_2) \in X$ and h is a contraction (and, hence, idempotent).

If $b_1 \notin X$, then

$$\begin{aligned}\mu_h(b_1, b_2) &= (b_1 \approx_B b_2)\\ &= (b_1 \approx_B b_1) \otimes (b_1 \approx_B b_2)\\ &\le \bigvee_{x\in B} ((b_1 \approx_B x) \otimes (x \approx_B b_2))\\ &\overset{(*)}{\le} \bigvee_{x\in B} (\mu_h(b_1, x) \otimes \mu_h(x, b_2)) = \mu_h^2(b_1, b_2)\end{aligned}$$

where $(*)$ follows by $\approx_B$-reflexivity of μ_h.

2. For $x \in X$ and $b \in B$, let us see that $\mu_h(x, z) \otimes \mu_h(z, b) \le \mu_h(x, b)$ for any $z \in B$.

In case $z \in X$, since h is a contraction and ρ_X is $\otimes$-transitive, it follows that

$$\begin{aligned}\mu_h(x, z) \otimes \mu_h(z, b) &= \rho_X(x, h(z)) \otimes \rho_X(z, h(b))\\ &= \rho_X(x, z) \otimes \rho_X(z, h(b))\\ &\le \rho_X(x, h(b)) = \mu_h(x, b)\,.\end{aligned}$$

In case $z \notin X$, using the definition of μ_h, the fact that h is a contraction and the $\approx_B$-reflexivity of ρ_X, we have

$$\begin{aligned}\mu_h(x, z) \otimes \mu_h(z, b) &= \rho_X(x, h(z)) \otimes (z \approx_B b)\\ &\le \rho_X(x, h(z)) \otimes (h(z) \approx_B h(b))\\ &\le \rho_X(x, h(z)) \otimes \rho_X(h(z), h(b))\\ &\le \rho_X(x, h(b)) = \mu_h(x, b)\,.\end{aligned}$$

Therefore, $\mu_h^2(x, b) = \bigvee_{z\in B} (\mu_h(x, z) \otimes \mu_h(z, b)) \le \mu_h(x, b)$.

3. Due to property 1 and the definition of μ_h^3, it is clear that $\mu_h^2 \le \mu_h^3$. To prove the other inequality, that is $\mu_h^3 \le \mu_h^2$, we have to show that $\mu_h^2(b_1, z) \otimes \mu_h(z, b_2) \le \mu_h^2(b_1, b_2)$, for all $b_1, b_2, z \in B$ and according to the definition of μ_h^2, it suffices to prove that $\mu_h(b_1, x_1) \otimes \mu_h(x_1, x_2) \otimes \mu_h(x_2, b_2) \le \mu_h^2(b_1, b_2)$, for all $b_1, x_1, b_2, x_2 \in B$.

Property 2 allows to reduce this to the case in which $b_1, x_1 \notin X$, namely:

(i) $x_2 \in X$ and $b_1, x_1 \notin X$:

$$\begin{aligned}&\mu_h(b_1, x_1) \otimes \mu_h(x_1, x_2) \otimes \mu_h(x_2, b_2)\\ &\quad= (b_1 \approx_B x_1) \otimes (x_1 \approx_B x_2) \otimes \rho_X(x_2, h(b_2))\\ &\quad\le (b_1 \approx_B x_2) \otimes \rho_X(x_2, h(b_2))\\ &\quad= \mu_h(b_1, x_2) \otimes \mu_h(x_2, b_2) \le \mu_h^2(b_1, b_2)\,.\end{aligned}$$

(ii) $b_1, x_1, x_2 \notin X$:

$$\begin{aligned}&\mu_h(b_1, x_1) \otimes \mu_h(x_1, x_2) \otimes \mu_h(x_2, b_2)\\ &\quad= (b_1 \approx_B x_1) \otimes (x_1 \approx_B x_2) \otimes (x_2 \approx_B b_2)\\ &\quad\le (b_1 \approx_B x_2) \otimes (x_2 \approx_B b_2)\\ &\quad= \mu_h(b_1, x_2) \otimes \mu_h(x_2, b_2) \le \mu_h^2(b_1, b_2)\,.\end{aligned}$$

4. Straightforward from properties 1 and 3. ■

Proof of Theorem 7:

1) $\Rightarrow$ 2) Suppose that there exists a fuzzy relation ρ_B on B and a contraction h such that we have a Galois connection $(i,h)\colon \langle X,\approx_B,\rho_X\rangle \leftrightharpoons \langle B,\approx_B,\rho_B\rangle$. Consider the fuzzy

relation μ_h (see Definition 16) and let us prove that $\mu_h(b_1,b_2) \le \rho_B(b_1,b_2)$ for all $b_1,b_2 \in B$:

(i) If $b_1 \in X$, then $\mu_h(b_1,b_2) = \rho_X(b_1,h(b_2)) = \rho_B(i(b_1),b_2) = \rho_B(b_1,b_2)$.

(ii) If $b_1 \notin X$, then $\mu_h(b_1,b_2) = (b_1 \approx_B b_2) \le \rho_B(b_1,b_2)$.

As a consequence, $\mu_h^t \le \rho_B$ and therefore, since ρ_B is $\otimes$-$\approx_B$-antisymmetric, then μ_h^t so is.

2) ⇒ 1) Consider $\rho_B = \mu_h^2$, which is $\otimes$-$\approx_B$-antisymmetric, by the hypothesis. By Lemma 2, μ_h is $\approx_B$-reflexive and, by Lemma 3, property 1, μ_h^2 is $\approx_B$-reflexive as well. Finally, Lemma 3 ensures that it is also $\otimes$-transitive.

Finally, by property 2 of Lemma 3, $\mu_h^2(x,b) = \mu_h(x,b) = \rho_x(x,h(b))$ for all $x \in X$ and $b \in B$ and, therefore, $(i,h)\colon \langle X, \approx_B, \rho_X\rangle \leftrightharpoons \langle B, \approx_B, \rho_B\rangle$.

■

Proof of Proposition 3: Recall that μ_h^2 is $\otimes$-$\approx_B$-antisymmetric if for all $b_1,b_2 \in B$ we have that $\mu_h^2(b_1,b_2) \otimes \mu_h^2(b_2,b_1) \le (b_1 \approx_B b_2)$, and we will consider the three possible cases below for b_1 and b_2:

1) The case $b_1, b_2 \in X$.

Neither condition is needed, since by Lemma 3, if $b_1 \in X$, we have that $\mu_h^2(b_1,b_2) = \mu_h(b_1,b_2) = \rho_X(b_1,h(b_2))$. Therefore, in this case,

$$\begin{aligned}\mu_h^2(b_1,b_2)\otimes\mu_h^2(b_2,b_1) &= \rho_X(b_1,h(b_2)) \otimes \rho_X(b_2,h(b_1))\\ &= \rho_X(b_1,b_2) \otimes \rho_X(b_2,b_1) \le (b_1 \approx_B b_2)\,.\end{aligned}$$

2) The case $b_1 \in X, b_2 \notin X$.

We have the following chain of equalities:

$$\begin{aligned}\mu_h^2(b_2,b_1) &= \bigvee_{x\in B} \left(\mu_h(b_2,x) \otimes \mu_h(x,b_1)\right)\\ &= \bigvee_{x\in X} \big((b_2 \approx_B x) \otimes \rho_X(x,b_1)\big)\\ &\qquad \vee \bigvee_{x\in B\smallsetminus X} \big((b_2 \approx_B x) \otimes (x \approx_B b_1)\big)\\ &= \bigvee_{x\in X} \big((b_2 \approx_B x) \otimes \rho_X(x,b_1)\big)\end{aligned}$$

where the last equality holds because, for every $x \in B \smallsetminus X$, we have that $(b_2 \approx_B x) \otimes (x \approx_B b_1) \le (b_2 \approx_B b_1) = (b_2 \approx_B b_1) \otimes \rho_X(b_1,b_1)$, which is one of the terms of the first disjunction.

As a consequence, if $b_1 \in X$ and $b_2 \notin X$, the necessary and sufficient condition for μ_h^2 being antisymmetric is

$$\begin{aligned}\mu_h^2(b_1,b_2) \otimes \mu_h^2(b_2,b_1) = \rho_X(b_1,h(b_2)) \otimes\\ \bigvee_{x\in X} \big((b_2 \approx_B x) \otimes \rho_X(x,b_1)\big)\\ \le (b_1 \approx_B b_2)\end{aligned}$$

or, equivalently,

$$\rho_X(b_1,h(b_2)) \otimes (b_2 \approx_B x) \otimes \rho_X(x,b_1) \le (b_1 \approx_B b_2)$$

for all $x \in X$. By using the residuation property, this can be rewritten as

$$\rho_X(b_1,h(b_2)) \le (\rho_X(x,b_1) \otimes (b_2 \approx_B x)) \to (b_1 \approx_B b_2)$$

for all $b_1, x \in X$ and $b_2 \notin X$, which is condition 1.

3) The case $b_1, b_2 \notin X$.

We have that

$$\begin{aligned}&\mu_h^2(b_1,b_2) \otimes \mu_h^2(b_2,b_1) =\\ &\quad \bigvee_{x\in B} \left(\mu_h(b_1,x) \otimes \mu_h(x,b_2)\right) \otimes \bigvee_{y\in B} \left(\mu_h(b_2,y) \otimes \mu_h(y,b_1)\right)\\ &\quad = \bigvee_{x,y\in B} \left(\mu_h(b_1,x) \otimes \mu_h(x,b_2) \otimes \mu_h(b_2,y) \otimes \mu_h(y,b_1)\right).\end{aligned}$$

By definition of μ_h and standard properties, if either $x \notin X$ or $y \notin X$, the corresponding disjunction above is smaller than or equal to $b_1 \approx_B b_2$. Therefore, the necessary and sufficient condition for μ_h^2 to be $\otimes$-$\approx_B$-antisymmetric is

$$\begin{aligned}(b_1 \approx_B x) \otimes \rho_X(x,h(b_2)) \otimes (b_2 \approx_B y) \otimes \rho_X(y,h(b_1)) \le\\ (b_1 \approx_B b_2)\end{aligned}$$

for all $x, y \in X$, which is condition 2. ■

Inma P. Cabrera is an Associate Professor in the Department of Applied Mathematics of the University of Málaga. She received the MSc (1995) and the PhD (2001) in Mathematics both from the University of Málaga. Her research has been focused on algebraic triple systems and the applications of mathematics in computer science. Her current research area is algebraic foundations of (fuzzy) Formal Concept Analysis, Hyperstructures and Multilattice Theory, Galois Connections and Applications. She is coauthor of over twenty scientific contributions in a variety of journals and scientific conferences. She is a member of the Computer Science Committee of the Royal Spanish Mathematical Society.

Pablo Cordero Pablo Cordero is Full Professor in the Department of Applied Mathematics of the University of Málaga (Spain). He received the MSc in Mathematics in 1992 from the Universidad Complutense de Madrid, and the PhD in Computer Science in 1999 from the University of Málaga. His research is focused on Mathematics applied to Computer Science. Specifically, his research area is the logic-based treatment of information and knowledge and its algebraic foundations: Logic and Automated Reasoning, Fuzzy Logic, Formal Methods in Databases, Formal Concept Analysis, Hyperstructures and Multilattice Theory, Galois Connections and Applications, etc. He is coauthor of more than forty research articles in a variety of scientific journals and over a hundred contributions to conferences and workshops.

Francisca García-Pardo holds a Degree in Mathematics (2000), a MSc in Physics and Mathematics from Granada University (2011) and a PhD in Mathematics, summa cum laude, from University of Málaga (2016), where has been a Teaching Assistant since 2009. She belongs to the group of Applications of Mathematics in Computer Science since 2013, and has completed research stays with the LFI Group (Pierre et Marie Curie Univ., France), the research unit KERMIT (Ghent Univ., Belgium), and the ILR institute (Bonn Univ., Germany).

Manuel Ojeda-Aciego received his MSc in Mathematics in 1990, and PhD in Computer Science in 1996, both from the University of Málaga, Spain, where he is currently a Full Professor of Applied Mathematics. He has (co-)authored more than 130 papers in scientific journals and proceedings of international conferences. His current research interests include fuzzy answer set semantics, residuated and multi-adjoint logic programming, fuzzy formal concept analysis and algebraic structures for computer science. He is the president of the Computer Science Committee of the Royal Spanish Mathematical Society, he serves the Editorial Board of the Intl J on Uncertainty and Fuzziness in Knowledge-based Systems and the IEEE Tr on Fuzzy Systems, member of the Steering Committee of the international conferences Concept Lattices and their Applications (CLA) and Information Processing and Management of Uncertainty (IPMU).

Bernard De Baets holds an M.Sc. in Maths (1988), a Postgraduate degree in Knowledge Technology (1991) and a Ph.D. in Maths (1995), all summa cum laude from Ghent University (Belgium). He is a Senior Full Professor in Applied Maths (1999) at Ghent University, where he is leading KERMIT, the research unit Knowledge-Based Systems. He is a Government of Canada Award holder (1988), an Honorary Professor of Budapest Tech (2006), an IFSA Fellow (2011), and an Honorary Doctor of the University of Turku (2017). His publications comprise more than 400 papers in international journals and about 60 book chapters. He serves on the Editorial Boards of various international journals, in particular as co-editor-in-chief of Fuzzy Sets and Systems. B. De Baets is a member of the Board of Directors of EUSFLAT and of the Administrative Board of the Belgian OR Society.