Fundamenta Informaticae XX (2004) 1–26
IOS Press

A multimodal logic approach to order of magnitude qualitative reasoning with comparability and negligibility relations

Alfredo Burrieza*
Dept. Filosofía. Universidad de Málaga. Spain
burrieza@uma.es

Manuel Ojeda-Aciego*
Dept. Matemática Aplicada. Universidad de Málaga. Spain
aciego@ctima.uma.es

Abstract. Non-classical logics have proven to be an adequate framework to formalize knowledge representation. In this paper we focus on a multimodal approach to formalize order-of-magnitude qualitative reasoning, extending the recently introduced system MQ, by means of a certain notion of negligibility relation which satisfies a number of intuitively plausible properties, as well as a minimal axiom system allowing for interaction among the different qualitative relations. The main aim is to show the completeness of the formal system introduced. Moreover, we consider some definability results and discuss possible directions for further research.

Keywords: multimodal logics; non-classical logics; order-of-magnitude qualitative reasoning

1. Introduction

Although the use of qualitative order of magnitude reasoning (OMR) has been an active research area in AI for some time, the analogous development of a logical approach has received little attention. Various multimodal approaches have been promulgated, for example, for qualitative spatial and temporal reasoning but, as far as we know, no such approach has been developed for OMR.

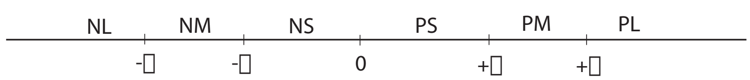
A typical OMR calculus is designed in such a way that it generalizes computations over precise values to computations over coarse values. The distinctive feature of OMR is that the coarse values are generally of different order of magnitude. Depending on the way the coarse values are defined, different

Address for correspondence: Dept. Matemática Aplicada. E.T.S.I. Informática. Univ. Málaga. 29071 Málaga, Spain
*Partially supported by project TIC2003-09001-C02-01.

OMR calculi can be generated: It is usual to distinguish between Absolute Order of Magnitude (AOM) and Relative Order of Magnitude (ROM) models. The former is represented by a partition of the real line, in which each element of $\mathbb{R}$ belongs to a qualitative class. The latter introduces a family of binary order of magnitude relations which establish different comparison relations between numbers. The underlying idea is that by reasoning in terms of qualitative ranges of variables, as opposed to precise numerical values, it is possible to compute information about the behavior of a system with very little information about the system and without doing expensive numerical simulation.

In [9] and extensions such as [5, 6, 7], coarse values are defined by means of ordering relations that express the distance between those values on a totally ordered domain in relation to the range they cover on that domain. Specifically, the seminal paper [9], distinguishes three types of qualitative relations, such as x *is close to* y, or x *is negligible w.r.t.* y or x *is comparable to* y; later on, some extensions were proposed in order to improve the original one with the inclusion of quantitative information, and allow for the control of the inference process [5, 6, 7].

There are attempts to integrate both approaches as well, so that an absolute partition is combined with a set of comparison relations between real numbers [12, 13]. For instance, it is usual to consider the $AOM(5)$ approach which, by considering five landmarks, divides the real line in seven equivalence classes and use the following labels to denote these equivalence classes of $\mathbb{R}$:



The labels correspond to "negative large", "negative medium", "negative small", "zero", "positive small", "positive medium" and "positive large", whereas the real numbers α and β are the landmarks used to delimit the equivalence classes (the particular criteria to choose these numbers would depend on the application in mind). In [12] three binary relations (*close to, comparable, negligible*) were defined in the spirit of [9], but using the labels corresponding to quantitative values, and preserving coherence between the relative model they define and the absolute model in which they are defined.

Our research line in this context is to develop a non-classical logic for handling qualitative reasoning with orders of magnitude. In [4], a minimal system for multimodal qualitative reasoning was introduced to handle, in some sense, the notion of comparability. Note that we use the term *multimodal* to refer to our approach to state that several independent modalities are included in the language.

To the best of our knowledge, no other formal logic has been developed to deal with order-of-magnitude reasoning. However, non-classical logics do have been used as a support of qualitative reasoning in several ways: among the formalisms for qualitative spatial reasoning, the Region Connection Calculus (RCC) [11, 1] has received particular attention; in [2, 14], multimodal logics were used to deal with qualitative spatio-temporal representations, and in [10] branching temporal logics have been used to describe the possible solutions of ordinary differential equations when we have limited information about a system.

Our aim in this paper is to include a certain notion of negligibility in the initial approach introduced in [4], in which an arbitrary linearly ordered set (usually thought of as a subset of the real numbers) was partitioned in classes consisting of positive observable, negative observable and non-observable (also called infinitesimal) numbers.

As a first approach to the logics of qualitative order-of-magnitude reasoning, we have based our minimal languages on the system $AOM(2)$, which is both simple enough to keep under control the complexity of the system and rich enough so as to permit the representation of a subset of the usual language of qualitative order-of-magnitude reasoning.

The intuitive representation of our underlying set of values (usually considered to be a subset of the real numbers, although this is not essential) is given below, in which two landmarks $-\alpha$ and $+\alpha$ are considered

In the picture, $-\alpha$ and $+\alpha$ represent respectively the greatest negative observable and the least positive observable. This choice makes sense, in particular, when considering physical metric spaces in which we always have a smallest unit which can be measured; however, it is not possible to identify a least or greatest non-observable number.

Once we have the equivalence classes in the real line, we can make comparisons between numbers by using binary relations such as

- x *is less than* y, in symbols $x < y$
- x *is less than and comparable to* y, in symbols $x \sqsubset y$
- x *is negligible w.r.t.* y, in symbols $x \prec y$

This paper is organized as follows: In Section 2 the syntax and the semantics of the proposed logic $\mathcal{L}(MQ)^N$ is introduced; in Section 3 an axiom system for $\mathcal{L}(MQ)^N$ is presented, which axiomatizes validity in frames based on an arbitrary linearly ordered set. In Section 4 the completeness proof is given, following a Henkin-style. In Section 5 we briefly explore the definability and relationships between some notions related to the property of density; general definability issues in $\mathcal{L}(MQ)^N$ are out of the scope of this paper and will be further studied elsewhere. Finally, in Section 6 some conclusions are drawn and prospects for future work are presented.

2. Syntax and Semantics of the Language $\mathcal{L}(MQ)^N$

In our syntax we will consider the connectives $\overrightarrow{\Box}$ and $\overleftarrow{\Box}$ to deal with the usual ordering $<$, the connectives $\overrightarrow{\blacksquare}$ and $\overleftarrow{\blacksquare}$ to deal with $\sqsubset$ and the connectives $\overrightarrow{\boxed{\mathrm{n}}}$ and $\overleftarrow{\boxed{\mathrm{n}}}$ to deal with $\prec$. The intuitive meanings of each modal connective is as follows:

$\overrightarrow{\Box}A$ means A *is true for all number greater than the current one*.

$\overrightarrow{\blacksquare}A$ is read A *is true for all number greater than and comparable with the current one.*

$\overleftarrow{\Box}A$ means A *is true for all number less than the current one*.

$\overleftarrow{\blacksquare}A$ means A *is true for all number less than and comparable with the current one*.

$\overrightarrow{\boxed{\mathrm{n}}}A$ means A *is true for all number from which the current one is negligible*.

$\overleftarrow{\boxed{\mathrm{n}}}A$ means A *is true for all number which is negligible from the current one*.

The intuitive description of the meaning of the negligibility-related modalities deserves some explaining comments. Depending on the particular context in which we are using the concept of negligibility, several possible definitions can arise. We have chosen to use an intrinsically directional notion of negligibility, in that negligible numbers are always to the left. There are other approaches in which the negligibility relation is not directional, so a point x can be negligible wrt points smaller than x and also wrt points greater than x, for instance, in [6, 13] it is the absolute value of an element what is considered before considering the negligibility relation.

As stated above, depending on the particular application a given approach might result either appropriate or inappropriate; specifically, one can wonder in which sense can one say that, for instance -1000 is negligible with respect to -1. It is not difficult to find real situations in which this interpretation makes sense, for example when interpreting the numbers above as exponents, since 10^{-1000} can be considered negligible with respect to 10^{-1}.

The syntax of our initial language for qualitative reasoning with comparability and negligibility is introduced below:

The alphabet of the language $\mathcal{L}(MQ)^N$ is defined by using:

- A stock of atoms or propositional variables, $\mathcal{V}$.
- The classical connectives $\neg, \wedge, \vee$ and $\rightarrow$ and the constants $\top$ and $\bot$.
- The unary modal connectives $\overrightarrow{\Box}$, $\overleftarrow{\Box}$, $\overrightarrow{\blacksquare}$, $\overleftarrow{\blacksquare}$, $\overrightarrow{\boxed{n}}$ and $\overleftarrow{\boxed{n}}$.
- The constants α^+ and α^-.
- The auxiliary symbols: (,).

Formulas are generated from $\mathcal{V} \cup \{\alpha^+, \alpha^-, \top, \bot\}$ by the construction rules of classical propositional logic adding the following rule: If A is a formula, then so are $\overrightarrow{\Box}A$, $\overleftarrow{\Box}A$, $\overrightarrow{\blacksquare}A$, $\overleftarrow{\blacksquare}A$, $\overrightarrow{\boxed{n}}A$ and $\overleftarrow{\boxed{n}}A$.

The *mirror image* of A is the result of replacing in A each occurrence of $\overrightarrow{\Box}$, $\overleftarrow{\Box}$, $\overrightarrow{\blacksquare}$, $\overleftarrow{\blacksquare}$, $\overrightarrow{\boxed{n}}$, $\overleftarrow{\boxed{n}}$, α^+, α^- by $\overleftarrow{\Box}$, $\overrightarrow{\Box}$, $\overleftarrow{\blacksquare}$, $\overrightarrow{\blacksquare}$, $\overleftarrow{\boxed{n}}$, $\overrightarrow{\boxed{n}}$, α^-, α^+, respectively. We shall use the symbols $\overrightarrow{\Diamond}$, $\overleftarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$, $\overleftarrow{\blacklozenge}$, $\overrightarrow{\Diamond_{n}}$ and $\overleftarrow{\Diamond_{n}}$ as abbreviations respectively of $\neg\overrightarrow{\Box}\neg$, $\neg\overleftarrow{\Box}\neg$, $\neg\overrightarrow{\blacksquare}\neg$, $\neg\overleftarrow{\blacksquare}\neg$, $\neg\overrightarrow{\boxed{n}}\neg$ and $\neg\overleftarrow{\boxed{n}}\neg$.

The intended meaning of our language is based on a multi-modal approach, therefore the semantics is given by using the concept of frame.

Definition 2.1. A *multimodal qualitative frame* for $\mathcal{L}(MQ)^N$ (or, simply, a *frame*) is a tuple $\Sigma = (\mathbb{S}, +\alpha, -\alpha, <, \prec)$, where

1. $(\mathbb{S}, <)$ is a linearly ordered set.
2. $+\alpha$ and $-\alpha$ are designated points in $\mathbb{S}$ (called *frame constants*) and allow to form the sets $\textsc{Obs}^+$, $\textsc{Inf}$, and $\textsc{Obs}^-$ that are defined as follows:

   $$\textsc{Obs}^- = \{x \in \mathbb{S} \mid x \leq -\alpha\}; \quad \textsc{Inf} = \{x \in \mathbb{S} \mid -\alpha < x < +\alpha\}; \quad \textsc{Obs}^+ = \{x \in \mathbb{S} \mid +\alpha \leq x\}$$

3. $\prec$ is a restriction of $<$, i.e. $\prec \subseteq <$, and satisfies:

   (i) If $x \prec y < z$, then $x \prec z$

   (ii) If $x < y \prec z$, then $x \prec z$

   (iii) If $x \prec y$, then either $x \notin \textsc{Inf}$ or $y \notin \textsc{Inf}$

We will use $x \sqsubset y$ as an abbreviation of "$x < y$ and $x, y \in EQ$, where $EQ \in \{\textsc{Inf}, \textsc{Obs}^+, \textsc{Obs}^-\}$".

It is worth noticing that as a consequence of items (i) and (ii) we have the transitivity of $\prec$; on the other hand, item (iii) states that two non-observable elements cannot be compared by the negligibility relation.

The conditions (i)–(iii) under the last item above aim at recovering a minimal set of standard intuitions about any qualitative notion of negligibility which are illustrated, in particular, in the following model based in the arithmetic of a pocket calculator:

Example 2.1. In a pocket calculator it is not possible to represent any number whose absolute value is less than 10^{-99}. Therefore, it makes sense to consider $-\alpha = -10^{-99}$ and $+\alpha = +10^{-99}$ since any number between -10^{-99} and 10^{-99} cannot be observed/represented.[1]

On the other hand, a number x can be said to be negligible with respect to y provided that the difference $y - x$ cannot be distinguished from y. Numerically, and assuming an 8+2 (digits and mantissa) display, this amounts to state that x is negligible wrt y iff $y - x > 10^8$.

Under this model, it is straightforward to check that properties (i)–(iii) hold.

Definition 2.2. Let Σ be a multimodal qualitative frame, a *multimodal qualitative model on* Σ (or Σ-*model*, for short) is an ordered pair $\mathcal{M} = (\Sigma, h)$, where h is a *meaning function* (or, *interpretation*) $h\colon \mathcal{V} \longrightarrow 2^{\mathbb{S}}$. Any interpretation can be uniquely extended to the set of all formulas in $\mathcal{L}(MQ)^N$ (also denoted by h) by means of the usual conditions for the classical boolean connectives and the constants $\top$ and $\bot$, and the following conditions for the modal operators and frame constants:

$$
\begin{array}{rcl}
h(\overrightarrow{\Box} A) & = & \{x \in \mathbb{S} \mid y \in h(A) \text{ for all } y \text{ such that } x < y\} \\
h(\overrightarrow{\blacksquare} A) & = & \{x \in \mathbb{S} \mid y \in h(A) \text{ for all } y \text{ such that } x \sqsubset y\} \\
h(\overrightarrow{\boxed{\scriptstyle n}} A) & = & \{x \in \mathbb{S} \mid y \in h(A) \text{ for all } y \text{ such that } x \prec y\} \\
h(\overleftarrow{\Box} A) & = & \{x \in \mathbb{S} \mid y \in h(A) \text{ for all } y \text{ such that } y < x\} \\
h(\overleftarrow{\blacksquare} A) & = & \{x \in \mathbb{S} \mid y \in h(A) \text{ for all } y \text{ such that } y \sqsubset x\} \\
h(\overleftarrow{\boxed{\scriptstyle n}} A) & = & \{x \in \mathbb{S} \mid y \in h(A) \text{ for all } y \text{ such that } y \prec x\} \\
h(\alpha^+) & = & \{+\alpha\} \\
h(\alpha^-) & = & \{-\alpha\}
\end{array}
$$

The concepts of truth and validity are defined in a straightforward manner.

3. Axiomatic system for $\mathcal{L}(MQ)^N$

In this section we define an axiomatic system for multimodal qualitative logic with negligibility. A list of axiom schemes and inference rules are presented in order to build the system. We consider all the tautologies of classical propositional logic together with the following axiom schemata:

Axiom schemata for white connectives:

K1 $\overrightarrow{\Box}(A \to B) \to (\overrightarrow{\Box} A \to \overrightarrow{\Box} B)$

K2 $A \to \overrightarrow{\Box}\overleftarrow{\Diamond} A$

K3 $\overrightarrow{\Box} A \to \overrightarrow{\Box}\overrightarrow{\Box} A$

K4 $\big(\overrightarrow{\Box}(A \vee B) \wedge \overrightarrow{\Box}(\overrightarrow{\Box} A \vee B) \wedge \overrightarrow{\Box}(A \vee \overrightarrow{\Box} B)\big) \to \big(\overrightarrow{\Box} A \vee \overrightarrow{\Box} B\big)$

[1] Of course, there are much more numbers which cannot be represented, but this is irrelevant for this example.

Axiom schema for $\overrightarrow{\blacksquare}$:

C1 $\overrightarrow{\blacksquare}(A \to B) \to (\overrightarrow{\blacksquare}A \to \overrightarrow{\blacksquare}B)$

Axiom schemata for constants, where ξ is either α^+ or α^-

c1 $\overleftarrow{\Diamond}\xi \lor \xi \lor \overrightarrow{\Diamond}\xi$

c2 $\xi \to (\overleftarrow{\Box}\neg\xi \land \overrightarrow{\Box}\neg\xi)$

c3 $\alpha^- \to \overrightarrow{\Diamond}\alpha^+$

c4 $\alpha^- \to \overrightarrow{\blacksquare}A$

c5 $(\overleftarrow{\Diamond}\alpha^- \land \overrightarrow{\Diamond}\alpha^+) \to \overrightarrow{\blacksquare}(\overleftarrow{\Diamond}\alpha^- \land \overrightarrow{\Diamond}\alpha^+)$

c6 $\overrightarrow{\Diamond}\alpha^- \to \overrightarrow{\blacksquare}(\alpha^- \lor \overrightarrow{\Diamond}\alpha^-)$

c7 $(\alpha^+ \land \overrightarrow{\blacksquare}A) \to \overrightarrow{\Box}A$

c8 $\overrightarrow{\blacksquare}A \to \overrightarrow{\Box}((\alpha^- \lor \overrightarrow{\Diamond}\alpha^-) \to A)$

c9 $(\overleftarrow{\Diamond}\alpha^+ \land \overrightarrow{\blacksquare}A) \to \overrightarrow{\Box}A$

c10 $(\overleftarrow{\Diamond}\alpha^- \land \overrightarrow{\Diamond}\alpha^+ \land \overrightarrow{\blacksquare}A) \to \overrightarrow{\Box}((\overleftarrow{\Diamond}\alpha^- \land \overrightarrow{\Diamond}\alpha^+) \to A)$

Mixed axiom:

M1 $\overrightarrow{\Box}A \to \overrightarrow{\blacksquare}A$

Axiom schemata for negligibility

N1 $\overrightarrow{\boxed{n}}(A \to B) \to (\overrightarrow{\boxed{n}}A \to \overrightarrow{\boxed{n}}B)$

N2 $A \to \overrightarrow{\boxed{n}}\overleftarrow{\Diamond_n}A$

N3 $\overrightarrow{\Box}A \to \overrightarrow{\boxed{n}}A$

N4 $\overrightarrow{\boxed{n}}A \to \overrightarrow{\Box}\overrightarrow{\boxed{n}}A$

N5 $\overrightarrow{\boxed{n}}A \to \overrightarrow{\boxed{n}}\overrightarrow{\Box}A$

N6 $(\overleftarrow{\Diamond}\alpha^- \land \overrightarrow{\Diamond}\alpha^+) \to \overrightarrow{\boxed{n}}(\alpha^+ \lor \overleftarrow{\Diamond}\alpha^+)$

We also consider as axioms the corresponding mirror images of all the axioms.[2]

Rules of inference:

(MP) Modus Ponens for $\to$. **(N$\overrightarrow{\Box}$)** If $\vdash A$ then $\vdash \overrightarrow{\Box}A$. **(N$\overleftarrow{\Box}$)** If $\vdash A$ then $\vdash \overleftarrow{\Box}A$.

Let us recall that the system MQ introduced in [4] consists of $K1$–$K4, M1, C1, c1$–$c10$, their mirror images and the rules of inference; the minimal system extending MQ with the negligibility relation is denoted MQ^N, and is MQ plus $N1$–$N6$ and their mirror images.

The concepts of *proof* and *theorem* are defined in a standard way.

4. Soundness and completeness of MQ^N

The proof of soundness is straightforward, since validity of the axioms and preservation of validity by inference rules is just a standard calculation. Thus, we need only to focus on completeness and, with this aim, a Henkin-style proof will be constructed.

The proof of completeness follows the step-by-step method as described in [3]; consequently, some results about *(maximal) consistent* sets of formulas are needed.

4.1. Preliminary lemmas

Some familiarity with the basic properties of maximal consistent sets (*mc-sets*) is assumed; in the proof of the properties of the relations between mc-sets defined below we shall use $\mathcal{MC}$ to denote the set of all mc-sets of formulas:

[2] Alternatively, we could have just considered only the mirror images corresponding to K1, K2, K4, C1, M1, N1–N6 and c4–10, since the rest of images can be derived.

Definition 4.1. Consider $\Gamma_1, \Gamma_2 \in \mathcal{MC}$. Then:

1. $\Gamma_1 \rhd \Gamma_2$ if and only if $\{A \mid \overrightarrow{\Box} A \in \Gamma_1\} \subseteq \Gamma_2$
2. $\Gamma_1 \blacktriangleright \Gamma_2$ if and only if $\{A \mid \overrightarrow{\blacksquare} A \in \Gamma_1\} \subseteq \Gamma_2$
3. $\Gamma_1 \unrhd \Gamma_2$ if and only if $\{A \mid \overrightarrow{\boxed{\scriptstyle n}} A \in \Gamma_1\} \subseteq \Gamma_2$.

Lemma 4.1. (Lindenbaum)
Any consistent set of formulas in MQ^N can be extended to an mc-set in MQ^N.

The three lemmas below state some general modal properties of the operators $\rhd$, $\blacktriangleright$ and $\unrhd$ whose proof is straightforward: the behaviour with respect to the relations just introduced, the transitivity and linearity of those orderings, and the existence of *mc*-sets with suitable properties.

Lemma 4.2. Consider $\Gamma_1, \Gamma_2 \in \mathcal{MC}$, then:

1. $\Gamma_1 \rhd \Gamma_2$ if and only if $\{A \mid \overleftarrow{\Box} A \in \Gamma_2\} \subseteq \Gamma_1$
2. $\Gamma_1 \rhd \Gamma_2$ if and only if $\{\overrightarrow{\Diamond} A \mid A \in \Gamma_2\} \subseteq \Gamma_1$
3. $\Gamma_1 \rhd \Gamma_2$ if and only if $\{\overleftarrow{\Diamond} A \mid A \in \Gamma_1\} \subseteq \Gamma_2$

Lemma 4.3. Consider $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{MC}$, then

1. If $\Gamma_1 \rhd \Gamma_2$ and $\Gamma_2 \rhd \Gamma_3$, then $\Gamma_1 \rhd \Gamma_3$.
2. If $\Gamma_1 \rhd \Gamma_2$ and $\Gamma_1 \rhd \Gamma_3$, then either $\Gamma_2 \rhd \Gamma_3$, or $\Gamma_3 \rhd \Gamma_2$, or $\Gamma_2 = \Gamma_3$.
3. If $\Gamma_2 \rhd \Gamma_1$ and $\Gamma_3 \rhd \Gamma_1$, then either $\Gamma_2 \rhd \Gamma_3$, or $\Gamma_3 \rhd \Gamma_2$, or $\Gamma_2 = \Gamma_3$.

Lemma 4.4. Assume $\Gamma_1 \in \mathcal{MC}$:

1. If $\overrightarrow{\Diamond} A \in \Gamma_1$, then there exists $\Gamma_2 \in \mathcal{MC}$ such that $\Gamma_1 \rhd \Gamma_2$ and $A \in \Gamma_2$.
2. If $\overleftarrow{\Diamond} A \in \Gamma_1$, then there exists $\Gamma_2 \in \mathcal{MC}$ such that $\Gamma_2 \rhd \Gamma_1$ and $A \in \Gamma_2$.

Remark 4.1. The statements of the three previous lemmas only contain the behaviour of the white modal connectives, however the black modalities and the modalities for negligibility have similar properties. For referring to these alternative formulations, we will write, e.g. Lemma 4.2($\blacktriangleright$) or Lemma 4.2($\unrhd$).

The following technical proposition introduces a number of theorems of $\mathcal{L}(MQ)^N$ which are used in some proofs to appear later:

Proposition 4.1. The following formulas are theorems of MQ^N, where A and B are wff:

1. $\overrightarrow{\Diamond}\alpha^- \vee \alpha^- \vee (\overleftarrow{\Diamond}\alpha^- \wedge \overrightarrow{\Diamond}\alpha^+) \vee \alpha^+ \vee \overleftarrow{\Diamond}\alpha^+$
2. $(\varphi \wedge \overrightarrow{\blacksquare} A) \rightarrow \overrightarrow{\Box}(\varphi \rightarrow A)$, where $\varphi \in \{\overrightarrow{\Diamond}\alpha^-, \overleftarrow{\Diamond}\alpha^+, \overleftarrow{\Diamond}\alpha^- \wedge \overrightarrow{\Diamond}\alpha^+\}$
3. $(\overrightarrow{\blacklozenge} A \wedge \overrightarrow{\Diamond} B) \rightarrow (\overrightarrow{\blacklozenge}(A \wedge B) \vee \overrightarrow{\blacklozenge}(\overrightarrow{\blacklozenge} A \wedge B) \vee \overrightarrow{\blacklozenge}(A \wedge \overrightarrow{\Diamond} B))$
4. $(\overleftarrow{\Diamond}\varphi \wedge \overrightarrow{\Diamond}\varphi) \rightarrow \varphi$, where $\varphi \in \{\overrightarrow{\Diamond}\alpha^-, \overleftarrow{\Diamond}\alpha^+, \overleftarrow{\Diamond}\alpha^- \wedge \overrightarrow{\Diamond}\alpha^+\}$

5. $(\overrightarrow{\Diamond\!\!\!\cdot\;} A \wedge \overrightarrow{\Diamond} B) \rightarrow (\overrightarrow{\Diamond\!\!\!\cdot\;}(A \wedge B) \vee \overrightarrow{\Diamond}(\overrightarrow{\Diamond} A \wedge B) \vee \overrightarrow{\Diamond\!\!\!\cdot\;}(A \wedge \overrightarrow{\Diamond} B))$
6. $(\alpha^+ \vee \overleftarrow{\Diamond} \alpha^+) \rightarrow (\overrightarrow{\Diamond} A \rightarrow \overrightarrow{\blacklozenge} A)$

The following two lemmas are specific of the system MQ, for they are concerned with $\rhd$ and $\blacktriangleright$. Although they were already stated in [4] their proof is firstly included here.

Lemma 4.5. Consider $\Gamma_1, \Gamma_2 \in \mathcal{MC}$ such that $\Gamma_1 \rhd \Gamma_2$, then $\Gamma_1 \blacktriangleright \Gamma_2$ holds if and only if one of the following conditions below is fulfilled:

1. $\{\overleftarrow{\Diamond} \alpha^- \wedge \overrightarrow{\Diamond} \alpha^+, \overleftarrow{\Diamond} \alpha^+, \overrightarrow{\Diamond} \alpha^-\} \cap \Gamma_1 \cap \Gamma_2 \neq \varnothing$
2. $\alpha^+ \in \Gamma_1$
3. $\alpha^- \in \Gamma_2$

Proof:
Consider $\Gamma_1 \rhd \Gamma_2$, and assume $\Gamma_1 \blacktriangleright \Gamma_2$. Let us see that some of the conditions 1–3 holds.

By maximality of Γ_1 and Proposition 4.1, we have that $\overrightarrow{\Diamond} \alpha^- \vee \alpha^- \vee (\overleftarrow{\Diamond} \alpha^- \wedge \overrightarrow{\Diamond} \alpha^+) \vee \alpha^+ \vee \overleftarrow{\Diamond} \alpha^+ \in \Gamma_1$. We proceed by cases:

- If $\overrightarrow{\Diamond} \alpha^- \in \Gamma_1$, by Axiom c6 we have $\overrightarrow{\blacksquare}(\alpha^- \vee \overrightarrow{\Diamond} \alpha^-)$, now using the fact that $\Gamma_1 \blacktriangleright \Gamma_2$ we have that $\alpha^- \vee \overrightarrow{\Diamond} \alpha^- \in \Gamma_2$ and, as a result, either $\alpha^- \in \Gamma_2$ (which is Condition 3) or $\overrightarrow{\Diamond} \alpha^- \in \Gamma_2$ (which leads to Condition 1).
- The case $\alpha^- \in \Gamma_1$ cannot hold, otherwise by Axiom c4 and $\Gamma_1 \blacktriangleright \Gamma_2$ we would obtain that any formula A, in particular $A = \perp$, is in Γ_2, which is a contradiction.
- If $\overleftarrow{\Diamond} \alpha^- \wedge \overrightarrow{\Diamond} \alpha^+ \in \Gamma_1$, then Axiom c5 leads to $\overleftarrow{\Diamond} \alpha^- \wedge \overrightarrow{\Diamond} \alpha^+ \in \Gamma_2$ by using $\Gamma_1 \blacktriangleright \Gamma_2$, then Condition 1 holds.
- For the case $\alpha^+ \in \Gamma_1$, there is nothing to prove, since Condition 2 trivially holds.
- Finally, if $\overleftarrow{\Diamond} \alpha^+ \in \Gamma_1$, then, by K2 and K3, we have $\overrightarrow{\Box}\overleftarrow{\Diamond} \alpha^+ \in \Gamma_1$ and, by $\Gamma_1 \rhd \Gamma_2$, we would also have $\overleftarrow{\Diamond} \alpha^+ \in \Gamma_2$ and Condition 1 holds.

Therefore, any of the alternatives leads to some of the Conditions 1–3.

Reciprocally, in order to show that $\Gamma_1 \blacktriangleright \Gamma_2$, consider $\overrightarrow{\blacksquare} A \in \Gamma_1$ and let us prove that $A \in \Gamma_2$:

- Assume Condition 1, and denote by φ an element in the intersection; in particular, we have both $\varphi \in \Gamma_1$, and $\varphi \in \Gamma_2$. Now, taking into account Proposition 4.1 the formula $(\varphi \wedge \overrightarrow{\blacksquare} A) \rightarrow \overrightarrow{\Box}(\varphi \rightarrow A)$ is a theorem of MQ, we obtain $\overrightarrow{\Box}(\varphi \rightarrow A) \in \Gamma_1$, then using the general hypothesis $\Gamma_1 \rhd \Gamma_2$, we have that $\varphi \rightarrow A \in \Gamma_2$ and thus $A \in \Gamma_2$.
- Assume Condition 2, this is $\alpha^+ \in \Gamma_1$, we would have $\alpha^+ \wedge \overrightarrow{\blacksquare} A \in \Gamma_1$ and, by Axiom c7, we obtain $\overrightarrow{\Box} A \in \Gamma_1$. Now using $\Gamma_1 \rhd \Gamma_2$ we obtain $A \in \Gamma_2$.
- Assume Condition 3, this is $\alpha^- \in \Gamma_2$. Recalling $\overrightarrow{\blacksquare} A \in \Gamma_1$, by Axiom c8, we get $\overrightarrow{\Box}((\overrightarrow{\Diamond} \alpha^- \vee \alpha^-) \rightarrow A) \in \Gamma_1$. On the other hand, by $\Gamma_1 \rhd \Gamma_2$ and the obvious fact that $\overrightarrow{\Diamond} \alpha^- \vee \alpha^- \in \Gamma_2$, we obtain $A \in \Gamma_2$.

QED

Lemma 4.6. Given $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{MC}$ we have:

1. If $\Gamma_1 \blacktriangleright \Gamma_2$, then $\Gamma_1 \rhd \Gamma_2$.
2. If $\Gamma_1 \blacktriangleright \Gamma_2$, $\Gamma_1 \rhd \Gamma_3$ and it is not the case that $\Gamma_1 \blacktriangleright \Gamma_3$, then $\Gamma_2 \rhd \Gamma_3$.
3. If $\Gamma_2 \blacktriangleright \Gamma_1$, $\Gamma_3 \rhd \Gamma_1$ and it is not the case that $\Gamma_3 \blacktriangleright \Gamma_1$, then $\Gamma_3 \rhd \Gamma_2$.
4. If $\Gamma_1 \rhd \Gamma_2 \rhd \Gamma_3$ and $\Gamma_1 \blacktriangleright \Gamma_3$, then $\Gamma_1 \blacktriangleright \Gamma_2 \blacktriangleright \Gamma_3$.

Proof:

1. The proof of the first item is straightforward by Axiom M1.

2. By contradiction, assume that $\Gamma_2 \not\rhd \Gamma_3$, that is, there exists $\overrightarrow{\Box} A \in \Gamma_2$ such that $A \notin \Gamma_3$, in addition to the hypotheses $\Gamma_1 \blacktriangleright \Gamma_2$, $\Gamma_1 \rhd \Gamma_3$ and it is not the case that $\Gamma_1 \blacktriangleright \Gamma_3$.

By using $\Gamma_1 \blacktriangleright \Gamma_2$, we get $\overrightarrow{\blacklozenge}\overrightarrow{\Box} A \in \Gamma_1$, by Lemma 4.2(2▶). Moreover, as it is not the case that $\Gamma_1 \blacktriangleright \Gamma_3$, there exists $\overrightarrow{\blacksquare} B \in \Gamma_1$ such that $B \notin \Gamma_3$. As a result, by standard properties of modalities, we have that $\overrightarrow{\blacklozenge}(\overrightarrow{\Box} A \wedge B) \in \Gamma_1$. Furthermore, since $\Gamma_1 \rhd \Gamma_3$, then $\overrightarrow{\Diamond}(\neg A \wedge \neg B) \in \Gamma_1$ by Lemma 4.2(2).

On the other hand, by Proposition 4.1 the formula

$$\left(\overrightarrow{\blacklozenge} C \wedge \overrightarrow{\Diamond} D\right) \to \left(\overrightarrow{\blacklozenge}(C \wedge D) \vee \overrightarrow{\blacklozenge}(\overrightarrow{\blacklozenge} C \wedge D) \vee \overrightarrow{\blacklozenge}(C \wedge \overrightarrow{\Diamond} D)\right)$$

is a theorem, then by instantiating $C = \overrightarrow{\Box} A \wedge B$ and $D = \neg A \wedge \neg B$, we obtain three different possibilities, all of which lead to a contradiction:

(a) $\overrightarrow{\blacklozenge}(\overrightarrow{\Box} A \wedge B \wedge \neg A \wedge \neg B) \in \Gamma_1$. Obviously contradictory.

(b) $\overrightarrow{\blacklozenge}(\overrightarrow{\blacklozenge}(\overrightarrow{\Box} A \wedge B) \wedge \neg A \wedge \neg B) \in \Gamma_1$. In particular, we obtain $\overrightarrow{\blacklozenge} \neg B \in \Gamma_1$; which contradicts $\overrightarrow{\blacksquare} B \in \Gamma_1$, the defining property of B, see above.

(c) $\overrightarrow{\blacklozenge}(\overrightarrow{\Box} A \wedge B \wedge \overrightarrow{\Diamond}(\neg A \wedge \neg B)) \in \Gamma_1$. Contradiction between $\overrightarrow{\Box} A$ and $\overrightarrow{\Diamond} \neg B$.

3. The proof is similar.

4. Consider $\Gamma_1 \rhd \Gamma_2 \rhd \Gamma_3$ and $\Gamma_1 \blacktriangleright \Gamma_3$. By transitivity we have $\Gamma_1 \rhd \Gamma_2$ and by Lemma 4.5 there are three possibilities:

(i) The intersection $\{\overleftarrow{\Diamond} \alpha^- \wedge \overrightarrow{\Diamond} \alpha^+, \overrightarrow{\Diamond} \alpha^+, \overleftarrow{\Diamond} \alpha^-\} \cap \Gamma_1 \cap \Gamma_3$ is non-empty.

Let φ be an element in the intersection above. Consider $\overrightarrow{\blacksquare} A \in \Gamma_1$, then relying on the fact that $(\varphi \wedge \overrightarrow{\blacksquare} A) \to \overrightarrow{\Box}(\varphi \to A)$ is a theorem (Proposition 4.1), we obtain $\overrightarrow{\Box}(\varphi \to A) \in \Gamma_1$ and, by $\Gamma_1 \rhd \Gamma_2$ we get $\varphi \to A \in \Gamma_2$. Let us prove now that $\varphi \in \Gamma_2$ in order to show that $A \in \Gamma_2$: By using $\Gamma_1 \rhd \Gamma_2 \rhd \Gamma_3$ and Lemma 4.2(items 2,3) we get $(\overleftarrow{\Diamond} \varphi \wedge \overrightarrow{\Diamond} \varphi) \in \Gamma_2$. Now, Proposition 4.1 also states that $(\overleftarrow{\Diamond} \varphi \wedge \overrightarrow{\Diamond} \varphi) \to \varphi$ is a theorem, then $\varphi \in \Gamma_2$, by modus ponens we obtain $A \in \Gamma_2$ and, thus, $\Gamma_1 \blacktriangleright \Gamma_2$.

Now, consider $\overrightarrow{\blacksquare} A \in \Gamma_2$, if we use again the theorem $(\varphi \wedge \overrightarrow{\blacksquare} A) \to \overrightarrow{\Box}(\varphi \to A)$, we have that $\overrightarrow{\Box}(\varphi \to A) \in \Gamma_2$. Now, since $\varphi \in \Gamma_3$ and $\Gamma_2 \rhd \Gamma_3$ we obtain $A \in \Gamma_3$, that is , $\Gamma_2 \blacktriangleright \Gamma_3$.

(ii) $\alpha^+ \in \Gamma_1$.

Consider $\overrightarrow{\blacksquare} A \in \Gamma_1$, by using Axiom c7 we get $\overrightarrow{\Box} A \in \Gamma_1$ and, from $\Gamma_1 \rhd \Gamma_2$, we obtain $A \in \Gamma_2$ and, thus, $\Gamma_1 \blacktriangleright \Gamma_2$.

Consider $\overrightarrow{\blacksquare} A \in \Gamma_2$. Using $\alpha^+ \in \Gamma_1$ and $\Gamma_1 \rhd \Gamma_2$ in Lemma 4.2(3) then $\overleftarrow{\Diamond} \alpha^+ \in \Gamma_2$. Now, by Axiom c9 we get $\overrightarrow{\Box} A \in \Gamma_2$ and, as $\Gamma_2 \rhd \Gamma_3$, we have $A \in \Gamma_3$ and, thus, $\Gamma_2 \blacktriangleright \Gamma_3$.

(iii) $\alpha^- \in \Gamma_3$.

By $\Gamma_2 \rhd \Gamma_3$, and by Lemma 4.2(2) we obtain $\overrightarrow{\Diamond} \alpha^- \in \Gamma_2$. Consider $\overrightarrow{\blacksquare} A \in \Gamma_1$, then by Axiom c8 we obtain that $\overrightarrow{\Box}((\alpha^- \vee \overrightarrow{\Diamond} \alpha^-) \to A) \in \Gamma_1$ and, by $\Gamma_1 \rhd \Gamma_2$ and $\alpha^- \vee \overrightarrow{\Diamond} \alpha^- \in \Gamma_2$, we obtain $A \in \Gamma_2$, thus $\Gamma_1 \blacktriangleright \Gamma_2$.

Consider $\overrightarrow{\blacksquare} A \in \Gamma_2$; again by Axiom c8 we obtain that $\overrightarrow{\Box}((\alpha^- \vee \overrightarrow{\Diamond} \alpha^-) \to A) \in \Gamma_2$, now by using $\Gamma_2 \rhd \Gamma_3$ and the fact that $\alpha^- \vee \overrightarrow{\Diamond} \alpha^- \in \Gamma_3$, we get $A \in \Gamma_3$. Therefore $\Gamma_2 \blacktriangleright \Gamma_3$.

QED

The following lemma is concerned specifically with the concept of negligibility.

Lemma 4.7. Consider $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{MC}$, then

1. If $\Gamma_1 \mathrel{|\!\rhd} \Gamma_2$, then $\Gamma_1 \rhd \Gamma_2$.
2. If $\Gamma_1 \rhd \Gamma_2$, $\Gamma_1 \mathrel{|\!\rhd} \Gamma_3$ and it is not the case that $\Gamma_1 \mathrel{|\!\rhd} \Gamma_2$, then $\Gamma_2 \rhd \Gamma_3$.
3. If $\Gamma_2 \rhd \Gamma_1$, $\Gamma_3 \mathrel{|\!\rhd} \Gamma_1$ and it is not the case that $\Gamma_2 \mathrel{|\!\rhd} \Gamma_1$, then $\Gamma_3 \rhd \Gamma_2$.
4. If $\Gamma_1 \rhd \Gamma_2 \rhd \Gamma_3$ and either $\Gamma_1 \mathrel{|\!\rhd} \Gamma_2$ or $\Gamma_2 \mathrel{|\!\rhd} \Gamma_3$, then $\Gamma_1 \mathrel{|\!\rhd} \Gamma_3$.

Proof:

1. The proof of this item is trivial, just consider axiom N3.

2. Given $\overrightarrow{\Box} A \in \Gamma_2$ we have to prove that $A \in \Gamma_3$. Assume, $A \notin \Gamma_3$, by $\Gamma_1 \mathrel{|\!\rhd} \Gamma_3$ and using Lemma 4.2(2$\mathrel{|\!\rhd}$), we would have $\overrightarrow{\Diamond_n} \neg A \in \Gamma_1$. On the other hand, using $\Gamma_1 \not\mathrel{|\!\rhd} \Gamma_2$, we know there exists a formula B such that $\overrightarrow{\boxed{n}} B \in \Gamma_1$ and $B \notin \Gamma_2$. Thus, $\overrightarrow{\Box} A \wedge \neg B \in \Gamma_2$ and, using $\Gamma_1 \rhd \Gamma_2$, we obtain $\overrightarrow{\Diamond}(\overrightarrow{\Box} A \wedge \neg B) \in \Gamma_1$. As a result, we obtain that $\overrightarrow{\Diamond_n} \neg A \wedge \overrightarrow{\Diamond}(\overrightarrow{\Box} A \wedge \neg B) \in \Gamma_1$.

We will prove that a contradiction arises from the assumption that $A \notin \Gamma_3$ by applying a case-based reasoning. The key issue is to take into account that the formula

$$\left(\overrightarrow{\Diamond_n} C \wedge \overrightarrow{\Diamond} D\right) \to \left(\overrightarrow{\Diamond_n}(C \wedge D) \vee \overrightarrow{\Diamond}(\overrightarrow{\Diamond} C \wedge D) \vee \overrightarrow{\Diamond_n}(C \wedge \overrightarrow{\Diamond} D)\right)$$

is a theorem of MQ^N (Proposition 4.1), therefore some of the following three conditions should hold (where C and D have been substituted, respectively, by $\neg A$ and $\overrightarrow{\Box} A \wedge \neg B$):

(a) $\overrightarrow{\Diamond_n}(\neg A \wedge \overrightarrow{\Box} A \wedge \neg B) \in \Gamma_1$. This contradicts the fact that $\overrightarrow{\boxed{n}} B \in \Gamma_1$.

(b) $\overrightarrow{\Diamond}(\overrightarrow{\Diamond} \neg A \wedge \overrightarrow{\Box} A \wedge \neg B)) \in \Gamma_1$. This possibility is clearly contradictory.

(c) $\overrightarrow{\Diamond_n}(\neg A \wedge \overrightarrow{\Diamond}(\overrightarrow{\Box} A \wedge \neg B)) \in \Gamma_1$. This leads to $\overrightarrow{\Diamond_n} \overrightarrow{\Diamond} \neg B \in \Gamma_1$ and, by axiom N5, $\overrightarrow{\Diamond_n} \neg B \in \Gamma_1$, contradicting $\overrightarrow{\boxed{n}} B \in \Gamma_1$.

As a result, it cannot be the case that $A \notin \Gamma_3$ and, therefore, $\Gamma_2 \triangleright \Gamma_3$.

3. Similar to the previous case.

4. Firstly, assume $\Gamma_1 \overset{n}{\triangleright} \Gamma_2$ and $\overrightarrow{\boxed{n}}A \in \Gamma_1$. By axiom N5, we also have $\overrightarrow{\boxed{n}}\overrightarrow{\Box}A \in \Gamma_1$ and $\overrightarrow{\Box}A \in \Gamma_2$; now, using $\Gamma_2 \triangleright \Gamma_3$, we obtain $A \in \Gamma_3$. Assuming $\Gamma_2 \overset{n}{\triangleright} \Gamma_3$, the same idea (but now using axiom N4) leads to $\Gamma_1 \overset{n}{\triangleright} \Gamma_3$. QED

4.2. Completeness of MQ^N

As stated above, we will provide a proof of completeness by using the step-by-step method. We have to start the construction upon the concept of *pre-frame*, which is a generalization of a frame in that the landmarks are not required.

Definition 4.2. A *pre-frame* is a tuple obtained by eliminating either one or both frame constants from a frame, that is, a pre-frame is either $(\mathbb{S}, <, \prec)$ or $(\mathbb{S}, +\alpha, <, \prec)$ or $(\mathbb{S}, -\alpha, <, \prec)$.

The following definitions are needed in order to formally describe the construction method of each step in the completeness proof.

Definition 4.3.

1. Given a (pre-)frame Σ, a *trace* of Σ is a function $f_\Sigma : \mathbb{S} \longrightarrow 2^{\mathcal{L}(MQ)^N}$ such that the set $f_\Sigma(x)$ is a maximal consistent set for all $x \in \mathbb{S}$.
2. Let f_Σ be a trace of $\Sigma = (\mathbb{S}, +\alpha, -\alpha, <)$. Then f_Σ is called:
   - *Coherent* if it satisfies for all x, y:

     (a) $\alpha^+ \in f_\Sigma(+\alpha)$ and $\alpha^- \in f_\Sigma(-\alpha)$

     (b) If $x < y$, then $f_\Sigma(x) \triangleright f_\Sigma(y)$

     (c) If $x \sqsubset y$, then $f_\Sigma(x) \blacktriangleright f_\Sigma(y)$

     (d) If $x \prec y$, then $f_\Sigma(x) \overset{n}{\triangleright} f_\Sigma(y)$

   - $\overrightarrow{\Diamond}$*-prophetic* if it is coherent and for all formula A and all $x \in \mathbb{S}$:

     $$\text{if } \overrightarrow{\Diamond}A \in f_\Sigma(x), \text{ there exists } y \text{ such that } x < y \text{ and } A \in f_\Sigma(y) \tag{1}$$

     The definition of $\overrightarrow{\blacklozenge}$*-prophetic* and $\overrightarrow{\Diamond^{n}}$*-prophetic* is similar, but considering the order relations $\sqsubset$ and $\prec$, respectively.
   - $\overleftarrow{\Diamond}$*-historic* if it is coherent and for all formula A and all $x \in \mathbb{S}$:

     $$\text{if } \overleftarrow{\Diamond}A \in f_\Sigma(x), \text{ there exists } y \text{ such that } y < x \text{ and } A \in f_\Sigma(y) \tag{2}$$

     The definition of $\overleftarrow{\blacklozenge}$*-historic* and $\overleftarrow{\Diamond^{n}}$*-historic* is similar.
   - The expressions (1) and (2) are called $\overrightarrow{\Diamond}$-prophetic (resp. $\overleftarrow{\Diamond}$-historic) *conditionals for* f_Σ *wrt* $\overrightarrow{\Diamond}A$ (resp. $\overleftarrow{\Diamond}A$) *and* x. The same terminology is applied to connectives $\overrightarrow{\blacklozenge}$, $\overrightarrow{\Diamond^{n}}$, $\overleftarrow{\blacklozenge}$, $\overleftarrow{\Diamond^{n}}$.
   - f_Σ is said to be *full* if it is prophetic and historic.

The key concepts of *extension* of a frame and *active* or *exhausted* conditional are given in the definition below:

Definition 4.4.

1. Given two frames $\Sigma_1 = (\mathbb{S}_1, +\alpha_1, -\alpha_1, <_1, \prec_1), \Sigma_2 = (\mathbb{S}_2, +\alpha_2, -\alpha_2, <_2, \prec_2)$, we say that Σ_2 is *an extension of* Σ_1 if the following conditions are satisfied: $\mathbb{S}_1 \subseteq \mathbb{S}_2$, $<_1 \subseteq <_2$, $\prec_1 \subseteq \prec_2$, $+\alpha_1 = +\alpha_2, -\alpha_1 = -\alpha_2$.

   Similarly, we say that a pre-frame Υ_1 is an extension of the pre-frame Υ_2.

2. Let f_Σ be a trace of a frame $\Sigma = (\mathbb{S}, +\alpha, -\alpha, <, \prec)$.

   - A $\overrightarrow{\Diamond}$-prophetic conditional for f_Σ (with respect to $\overrightarrow{\Diamond} A$ and x) is said to be *active* if $\overrightarrow{\Diamond} A \in f_\Sigma(x)$ but there is no y such that $x < y$ and $A \in f_\Sigma(y)$; otherwise, if there exists y such that $x < y$ and $A \in f_\Sigma(y)$ the conditional is said to be *exhausted*.[3]
   - The definition of *active* and *exhausted* $\overrightarrow{\blacklozenge}$-prophetic ($\overrightarrow{\Diamond\!\!\!\cdot}$-prophetic) conditional are given in a similar manner.
   - For conditionals of type *historic* the definitions are similar.

The idea of the proof of completeness is to show that for any consistent formula A, a frame $\Sigma = (\mathbb{S}, \alpha^+, \alpha^-, <, \prec)$ and a full trace f_Σ can be defined, such that $A \in f_\Sigma(x)$ for some $x \in \mathbb{S}$. This frame Σ is constructed step-by-step and, in order to obtain an initial frame to work with, a procedure is needed in which, beginning with a pre-frame, an initial frame Σ_0 is obtained.

Obtaining an initial frame

We consider $\Upsilon_0 = (\mathbb{S}', <', \prec')$ where $\mathbb{S}' = \{x_0\}, <' = \prec' = \varnothing$, for which the trace f_{Υ_0} is defined as $f_{\Upsilon_0}(x_0) = \Gamma_0$ where Γ_0 is a maximal consistent set containing A, which exists by Lindenbaum's lemma. The next step depends on whether Γ_0 contains either α^+, or α^- or none of them (these alternatives are pairwise incompatible because of the consistency of Γ_0 and Axioms c2 and c3).

- Assume that $\alpha^- \in \Gamma_0$, then we force $x_0 = -\alpha$ and, as a result, we have $\overrightarrow{\Diamond} \alpha^+ \in \Gamma_0 = f_{\Upsilon_0}(x_0)$ because of Axiom c3. By Lemma 4.4(1) there exists Γ_1 such that $\alpha^+ \in \Gamma_1$. Now, we consider the frame $\Sigma_0 = (\mathbb{S}_0, <_0, \prec_0)$ as follows:

  – $\mathbb{S}_0 = \{x_0, +\alpha\}$
  – $<_0 = \{(x_0, +\alpha)\}$
  – $\prec_0 = \{(x_0, +\alpha)\}$, if $f_{\Upsilon_0}(x) \trianglerighteq \Gamma_1$; otherwise define $\prec_0 = \varnothing$.

  and the corresponding trace is defined as $f_{\Sigma_0} = f_{\Upsilon_0} \cup \{(\alpha^+, \Gamma_1)\}$, which is clearly coherent.

- The case $\alpha^+ \in \Gamma_0$ is similar by using the mirror image of Axiom c3.

[3]In other words, a conditional is said to be active if the conditional expression is not satisfied, whereas is said to be exhausted if the consequent is satisfied.

- Finally, assume that neither α^+ nor α^- is in Γ_0. We need to apply two steps as the previously described, one for introducing each frame constant. This can be done in a coherence-preserving way by using the theorem $\overrightarrow{\Diamond}\alpha^- \vee \alpha^- \vee (\overleftarrow{\Diamond}\alpha^- \wedge \overrightarrow{\Diamond}\alpha^+) \vee \alpha^+ \vee \overleftarrow{\Diamond}\alpha^+$ (see Proposition 4.1) then, taking into account that neither α^+ nor α^- is in Γ_0, we have $\overrightarrow{\Diamond}\alpha^- \vee (\overleftarrow{\Diamond}\alpha^- \wedge \overrightarrow{\Diamond}\alpha^+) \vee \overleftarrow{\Diamond}\alpha^+ \in \Gamma_0$. Lemma 4.4 allows to introduce a new frame constant, and we are in one of the previous cases.

Now, we can consider that we have an initial frame to work with. This frame Σ_0 is the basis of the construction of the frame Σ stated above as the countable union of a countable sequence of finite frames, $\Sigma_0, \Sigma_1, \ldots, \Sigma_n, \ldots$; with this aim:

- We will consider an indexed denumerable infinite set[4] $\mathcal{S} = \{x_i \mid i \in \mathbb{N}\}$ whose elements will be used to build the frames in the sequence Σ_i; we will consider the class $\Xi_{\mathcal{S}}$ of frames $(\mathbb{S}, +\alpha, -\alpha, <, \prec)$, where $\mathbb{S}$ is a finite subset of $\mathcal{S}$.

- We will also consider an enumeration of formulas $A_0, A_1, \ldots, A_n, \ldots$ of the language $\mathcal{L}(MQ)^N$, so that we can also assign a code number to each prophetic (historic) conditional in the usual way.

From the initial frame onwards

Assume that $\Sigma_n = (\mathbb{S}_n, <_n, \prec_n)$ and f_{Σ_n} are defined. If no conditional is active, then $\Sigma_{n+1} = \Sigma_n$, $f_{\Sigma_{n+1}} = f_{\Sigma_n}$ and the construction is finished. Otherwise, i.e., if there are prophetic (or historic) conditionals for f_{Σ_n} which are active, then we choose the conditional (C) with the lowest code number and then, by the *exhausting lemma* below, construct an extension $\Sigma_{n+1} = (\mathbb{S}_{n+1}, <_{n+1}, \prec_{n+1}) \in \Xi_{\mathcal{S}}$ of Σ_n and an extension $f_{\Sigma_{n+1}}$ of f_{Σ_n} such that the conditional (C) for $f_{\Sigma_{n+1}}$ is exhausted.

Although the trace of each of these finite frames is coherent, in general, it fails to be either prophetic or historic. However, the trace f_Σ of Σ, defined as the countable union of the countable sequence of finite frames Σ_i, can be proven to be full. Thus, the consistent formula A is verified by applying the *trace lemma*.

To finish the proof of completeness we have just to state and prove the two lemmas referenced above. The first one follows easily by induction on the complexity of A.

Lemma 4.8. (Trace lemma)
Let f_Σ be a full trace of a multimodal qualitative frame Σ. Let h be an interpretation assigning to each propositional variable p the set $h(p) = \{x \in \mathbb{S} \mid p \in f_\Sigma(x)\}$. Then, for any formula A we have $h(A) = \{x \in \mathbb{S} \mid A \in f_\Sigma(x)\}$.

Regarding the proof of the exhausting lemma, it is worth to notice that we will use a stronger version of coherent traces Σ wrt $\trianglerighteq$, in the sense that $x \prec y$ <u>if and only if</u> $f_\Sigma(x) \trianglerighteq f_\Sigma(y)$. Such a trace will be said to be *strongly $\prec$-coherent*.

In the construction we will guarantee that any finite frame Σ_k of the sequence $\Sigma_0, \Sigma_1, \ldots$ is strongly $\prec$-coherent (note that this holds trivially for Σ_0).

[4] Note that x_0 (and possibly x_1) have been used in the construction of the initial frame Σ_0.

Lemma 4.9. (Exhausting lemma)
Let f_{Σ_k} be a strongly $\prec$-coherent trace of a frame $\Sigma_k \in \Xi_{\mathcal{S}}$, and suppose that there is a prophetic (historic) conditional, (C), for f_{Σ_k} which is active. Then there is a frame $\Sigma_{k+1} \in \Xi_{\mathcal{S}}$ and a strongly $\prec$-coherent trace $f_{\Sigma_{k+1}}$ extending f_{Σ_k}, such that (C) is a conditional[5] for $f_{\Sigma_{k+1}}$ which is exhausted.

Proof:

Exhausting active $\overrightarrow{\blacklozenge}$-prophetic conditionals

Consider the case of an active $\overrightarrow{\blacklozenge}$-prophetic conditional:

If $\overrightarrow{\blacklozenge} A \in f_{\Sigma_k}(x)$, then there exists y such that $x \sqsubset y$ and $A \in f_{\Sigma_k}(y)$

That is, we have $\overrightarrow{\blacklozenge} A \in f_{\Sigma_k}(x)$ but there does not exist y satisfying the consequent of the conditional.

Our goal is to select a new frame Σ_{k+1} in the class $\Xi_{\mathcal{S}}$ which is an extension of Σ_k and, moreover, define a new trace $f_{\Sigma_{k+1}}$, extending f_{Σ_k}, for which the previous conditional is exhausted. This is proved by induction on the number l of successors of x in $\mathbb{S}_k$.

To begin with, let us consider a maximal consistent set Γ such that $f_{\Sigma_k}(x) \blacktriangleright \Gamma$ and $A \in \Gamma$, which exists by Lemma 4.4(1►).

1. If $l = 0$, then Σ_{k+1} is defined as follows:

$$\begin{aligned}
\mathbb{S}_{k+1} &= \mathbb{S}_k \cup \{y\}, \text{ where } y \in \mathcal{S} \setminus \mathbb{S}_k \\
<_{k+1} &= <_k \cup \{(x, y)\} \cup \{(z, y) \mid z <_k x\} \\
\prec_{k+1} &= \prec_k \cup \{(z, y) \mid z \leq_k x \text{ and } f_{\Sigma_{k+1}}(z) \vartriangleright \Gamma\} \\
f_{\Sigma_{k+1}} &= f_{\Sigma_k} \cup \{(y, \Gamma)\}
\end{aligned}$$

   Lemma 4.3(1,1►) guarantees the coherence of the definitions for both white and black triangles. The strong $\prec$-coherence of $f_{\Sigma_{n+1}}$ is ensured by construction and Lemma 4.7(4).

   Note also that Σ_{k+1} as defined is a finite frame, that is, belongs to the class $\Xi_{\mathcal{S}}$. Specifically, Lemma 4.7(4) guarantees conditions (i) and (ii) of Definition 2.1 for $\prec_{k+1}$; moreover, condition (iii) is immediate because the new point y clearly satisfies $+\alpha \leq_{k+1} x <_{k+1} y$.

2. If $l > 0$, let x' be the successor of x in Σ_k. The definitions of $\mathbb{S}_{k+1}$ and $f_{\Sigma_{k+1}}$ are the same as above. Now, we have two subcases to consider:

   (a) $x \sqsubset_k x'$.
   If $\overrightarrow{\blacklozenge} A \in f_{\Sigma_k}(x')$, then we apply the inductive case.
   If $\overrightarrow{\blacklozenge} A \notin f_{\Sigma_k}(x')$, as it is obvious that $A \notin f_{\Sigma_k}(x')$; therefore, we have that $\neg A \wedge \neg \overrightarrow{\blacklozenge} A \in f_{\Sigma_k}(x')$. Now, by coherence of $f_{\Sigma_k}(x)$, we have $f_{\Sigma_k}(x) \blacktriangleright f_{\Sigma_k}(x')$, as we also have $f_{\Sigma_k}(x) \blacktriangleright \Gamma$, by Lemma 4.3(2►) we have three possibilities out of which only $\Gamma \blacktriangleright f_{\Sigma_k}(x')$ is not contradictory.

[5] Given a conditional for f_{Σ}, if we simply replace the label Σ with Σ' where $\Sigma \subseteq \Sigma'$, we have a conditional for $f_{\Sigma'}$ but with the same code number as the conditional for f_{Σ}. Then we can say that in both cases we refer to the same conditional.

Therefore, we have $f_{\Sigma_k}(x) \blacktriangleright \Gamma \blacktriangleright f_{\Sigma_k}(x')$. This means that it is possible to select a point $y \in \mathcal{S} \smallsetminus \mathbb{S}_k$ to be located between x and x' where we will consider the set Γ, by preserving coherence, that is, the relations $<_{k+1}$ and $\prec_{k+1}$ are defined as follows (recall that $\mathbb{S}_{k+1}$ and $f_{\Sigma_{k+1}}$ have the same definition as in the previous case):

$$\begin{array}{rcl} <_{k+1} & = & <_k \cup \{(x,y),(y,x')\} \cup \{(z,y) \mid z <_k x\} \cup \{(y,z) \mid x' <_k z\} \\ \prec_{k+1} & = & \prec_k \cup \{(z,y) \mid z \leq_k x \text{ and } f_{\Sigma_k}(z) \mathrel{\rhd_{\mathrm{n}}} \Gamma\} \cup \{(y,z) \mid x' \leq_k z \text{ and } \Gamma \mathrel{\rhd_{\mathrm{n}}} f_{\Sigma_k}(z)\} \end{array}$$

Lemma 4.6(1) and Lemma 4.3(1) ensure the coherence of $f_{\Sigma_{k+1}}$ w.r.t. $\rhd$. Lemma 4.3(1$\blacktriangleright$) is needed to guarantee the coherence w.r.t $\blacktriangleright$. Strong $\prec$-coherence of $f_{\Sigma_{k+1}}$ is obtained by construction and Lemma 4.7(4).

Finally, let us prove that Σ_{k+1} is in $\Xi_{\mathcal{S}}$. Conditions (i) and (ii) are given by Lemma 4.7(4); to check condition (iii) we only need to show that for the new point y and any z the following property holds: if either $z \prec_{k+1} y$ or $y \prec_{k+1} z$ it cannot be the case that $z, y \in$ INF.

Take any z such that $z \prec_{k+1} y$ and assume $z \in$ INF. By coherence of $f_{\Sigma_{k+1}}$ w.r.t $\rhd$, we would have $f_{\Sigma_{k+1}}(\alpha^-) \rhd f_{\Sigma_{k+1}}(z) \rhd f_{\Sigma}(\alpha^+)$; thus, clearly, $\overleftarrow{\Diamond}\alpha^- \wedge \overrightarrow{\Diamond}\alpha^+ \in f_{\Sigma_{k+1}}(z)$. Now, by using Axiom N6, as $z \prec_{k+1} y$, by coherence, we have $\alpha^+ \vee \overleftarrow{\Diamond}\alpha^+ \in f_{\Sigma_{k+1}}(y)$ and, again by coherence, this means that $y \notin$ INF. The case $y \prec_{k+1} z$ is similar

(b) $x \not\sqsubset_k x'$.

Under these conditions let us prove that $-\alpha <_k x <_k +\alpha$ and, furthermore, $x' = +\alpha$.

As $\overrightarrow{\blacklozenge} A \in f_{\Sigma_k}(x)$, by using Axiom c4 and coherence of f_{Σ_k} it follows that $x \neq -\alpha$; as a result, by $x \not\sqsubset_k x'$, we obtain $x' = +\alpha$ and $-\alpha <_k x <_k +\alpha$ (recall that x' is the successor of x).

Now, in order to apply Lemma 4.6(2), let us prove that the relation $f_{\Sigma_k}(x) \blacktriangleright f_{\Sigma_k}(x')$ does not hold. By contradiction, using the coherence of f_{Σ_k} we would have $\alpha^+ \in f_{\Sigma_k}(x')$ and, by the mirror image of Axiom c4 (taking $A = \perp$) and Lemma 4.2(1) we would obtain $\perp \in f_{\Sigma_k}(x)$ in contradiction with the consistency of $f_{\Sigma_k}(x)$.

Moreover, recall that we have $f_{\Sigma_k}(x) \blacktriangleright \Gamma$ and $f_{\Sigma_k}(x) \rhd f_{\Sigma_k}(x')$, thus, by Lemma 4.6(2), we have $\Gamma \rhd f_{\Sigma_k}(x')$. This fact allows to define $<_{k+1}$ as above with a point y (associated to Γ) between x and x', note that in this case $x \sqsubset_{k+1} y \not\sqsubset_{k+1} x'$.

Regarding the negligibility ordering, the relation $\prec_{k+1}$ is defined as in the previous case.

Finally, Lemma 4.3(1,1$\blacktriangleright$) guarantees the coherence of the trace $f_{\Sigma_{k+1}}$ w.r.t. $\rhd$ and $\blacktriangleright$, and strong $\prec$-coherence is given by construction and Lemma 4.7(4). The same justification as above serves to show that Σ_{k+1}, as defined, is in $\Xi_{\mathcal{S}}$.

Exhausting active $\overrightarrow{\Diamond_{\mathrm{n}}}$-prophetic conditionals

Consider an active $\overrightarrow{\Diamond_{\mathrm{n}}}$-prophetic conditional wrt $\overrightarrow{\Diamond_{\mathrm{n}}} A$ and x, that is, we have $\overrightarrow{\Diamond_{\mathrm{n}}} A \in f_{\Sigma_k}(x)$, but there does not exist y such that $x \prec y$ and $A \in f_{\Sigma_k}(y)$.

By Lemma 4.4(1$\rhd_{\mathrm{n}}$), there exists Γ such that $f_{\Sigma_k}(x) \mathrel{\rhd_{\mathrm{n}}} \Gamma$ and $A \in \Gamma$. Now, we will select an extension of Σ_k containing a new point $y \in \mathcal{S} \smallsetminus \mathbb{S}_k$, to which Γ is assigned by preserving coherence. We will proceed inductively on the number of successors of x in $\mathbb{S}_k$.

For the case $l = 0$, the extension is defined straightforwardly.

If $l > 0$, then make $\mathbb{S}_{k+1} = \mathbb{S}_k \cup \{y\}$, where $y \in \mathcal{S} \smallsetminus \mathbb{S}_k$ and define $f_{\Sigma_{k+1}} = f_{\Sigma_k} \cup \{(y, \Gamma)\}$; let us consider the set $\{x^* \mid x \prec_k x^*\}$, and reason by cases depending on whether the set is empty or not:

1. $\{x^* \mid x \prec_k x^*\} = \varnothing$.

   Let x' be the last element in the frame $\mathbb{S}_k$, now define $<_{k+1}$ and $\prec_{k+1}$ as follows:

   $$\begin{aligned} <_{k+1} &= <_k \cup \{(x', y)\} \cup \{(z, y) \mid z <_k x'\} \\ \prec_{k+1} &= \prec_k \cup \{(x, y)\} \cup \{(z, y) \mid z \leq_k x' \text{ and } f_{\Sigma_k}(z) \not\trianglerighteq \Gamma\} \end{aligned}$$

   We only need to prove the coherence of $f_{\Sigma_{k+1}}$: the coherence wrt $\triangleright$ is justified by Lemma 4.7(1) and Lemma 4.3(1), and strong $\prec$-coherence by construction and Lemma 4.7(4). The case of $\blacktriangleright$ is proved by using the obvious relation $+\alpha \leq_k x'$; therefore by Lemma 4.5(1 or 2) and Lemma 4.3(1$\blacktriangleright$) we obtain the coherence.

2. $\{x^* \mid x \prec_k x^*\} \neq \varnothing$.

   Let us consider $\tilde{x} = \min\{x^* \mid x \prec_k x^*\}$ (which can be shown to exist); moreover, let x' be the immediate successor of x.

   If it was the case that $\overrightarrow{\diamondsuit} A \in f_{\Sigma_k}(\tilde{x})$, then we would apply induction; otherwise, we would have $\neg \overrightarrow{\diamondsuit} A \in f_{\Sigma_k}(z)$ for all z such that $x <_k z$ (by Axiom N4). As a result, and taking into account that $\neg A \in f_{\Sigma_k}(\tilde{x})$, by Lemma 4.7(1) and Lemma 4.3(2) we have that either $\Gamma \triangleright f_{\Sigma_k}(\tilde{x})$ or $f_{\Sigma_k}(\tilde{x}) \triangleright \Gamma$.

   - Let us assume $\Gamma \triangleright f_{\Sigma_k}(\tilde{x})$:
     Recall that for all z satisfying $x <_k z <_k \tilde{x}$ we have that $f_{\Sigma_k}(x) \not\trianglerighteq f_{\Sigma_k}(z)$ by strong $\prec$-coherence[6], and so $f_{\Sigma_k}(z) \triangleright \Gamma$ (by Lemma 4.7(2)). Thus $<_{k+1}$ and $\prec_{k+1}$ can be defined as follows:

     $$\begin{aligned} &<_{k+1} = <_k \cup \{(x, y), (y, \tilde{x})\} \cup \{(z, y) \mid z <_k x\} \cup \{(y, z) \mid \tilde{x} <_k z\} \\ &\prec_{k+1} = \prec_k \cup \{(x, y)\} \cup \{(z, y) \mid z <_k \tilde{x} \text{ and } f_{\Sigma_k}(z) \not\trianglerighteq \Gamma\} \cup \{(y, z) \mid \Gamma \not\trianglerighteq f_{\Sigma_k}(z)\} \end{aligned} \quad (3)$$

     The coherence of $f_{\Sigma_{k+1}}$ w.r.t $\triangleright$ and strong coherence w.r.t. $\trianglerighteq$ are justified as above. The case $\blacktriangleright$ is handled by Lemmas 4.5(1), 4.6(4), and 4.3(1$\blacktriangleright$).

   - Now, assume $f_{\Sigma_k}(\tilde{x}) \triangleright \Gamma$, and consider the set $\{x^* \mid \tilde{x} <_k x^* \text{ such that } \Gamma \triangleright f_{\Sigma_k}(x^*)\}$. Two possible cases arise:
     - The set is empty.
       In this case, we consider the last element of $\mathbb{S}_k$ and proceed as in case 1 above.
     - The set is non-empty.
       Let x' be the first element of the set, then the extended orderings $<_{k+1}$ and $\prec_{k+1}$ are defined as in (3) but changing $\tilde{x}$ by x'.

   Finally, it is not difficult to check that the definition provided for Σ_{k+1} belongs to $\Xi_{\mathcal{S}}$.

[6] This is where strong $\prec$-coherence is used.

Exhausting active $\overrightarrow{\Diamond}$-prophetic conditionals

Consider now an active $\overrightarrow{\Diamond}$-prophetic conditional:

$$\text{if } \overrightarrow{\Diamond} A \in f_{\Sigma_k}(x), \text{there exists a } y \text{ such that } x < y \text{ and } A \in f_{\Sigma_k}(y)$$

that is, we have $\overrightarrow{\Diamond} A \in f_{\Sigma_k}(x)$, but there does not exist y such that both $x < y$ and $A \in f_{\Sigma_k}(y)$.

The idea again is to select a convenient extension of Σ_k as in the previous cases; we will give an inductive proof on the number of successors l of x in Σ_k.

1. If $l = 0$, then $+\alpha \leq_k x$ and, by coherence, $\alpha^+ \vee \overleftarrow{\Diamond} \alpha^+ \in f_{\Sigma_k}(x)$. On the other hand, the formula $(\alpha^+ \vee \overleftarrow{\Diamond} \alpha^+) \to (\overrightarrow{\Diamond} A \to \overrightarrow{\blacklozenge} A)$ is a theorem (Proposition 4.1); therefore maximality leads to $\overrightarrow{\blacklozenge} A \in f_{\Sigma_k}(x)$, which reduces to a previous case.

2. For the case $l > 0$ we apply a case-based argumentation depending on whether $\overrightarrow{\blacklozenge} A \in f_{\Sigma_k}(x)$, or $\overrightarrow{\diamonddot} A \in f_{\Sigma_k}(x)$, or both $\overrightarrow{\blacklozenge} A \notin f_{\Sigma_k}(x)$ and $\overrightarrow{\diamonddot} A \notin f_{\Sigma_k}(x)$.

   The cases $\overrightarrow{\blacklozenge} A \in f_{\Sigma_k}(x)$ and $\overrightarrow{\diamonddot} A \in f_{\Sigma_k}(x)$ have been already solved, so let us assume $\overrightarrow{\blacklozenge} A, \overrightarrow{\diamonddot} A \notin f_{\Sigma_k}(x)$, and let x' be the successor of x in the frame.

   Now, if $\overrightarrow{\Diamond} A \in f_{\Sigma_k}(x')$, then apply the inductive case; otherwise it is clearly the case that $\neg A \wedge \neg \overrightarrow{\Diamond} A \in f_{\Sigma_k}(x')$.

   On the other hand, by Lemma 4.4(1), there exists a maximal consistent set Γ such that $f_{\Sigma_k}(x) \rhd \Gamma$ and $A \in \Gamma$. We select a point $y \in \mathcal{S} \smallsetminus \mathbb{S}_k$ to be located between x and x' to which we will associate Γ. Specifically, we have

$$\begin{array}{rcl}
\mathbb{S}_{k+1} & = & \mathbb{S}_k \cup \{y\}, \text{ where } y \in \mathcal{S} \smallsetminus \mathbb{S}_k \\
<_{k+1} & = & <_k \cup \{(x,y),(y,x')\} \cup \{(z,y) \mid z <_k x\} \cup \{(y,z) \mid x' <_k z\} \\
\prec_{k+1} & = & \prec_k \cup \{(z,y) \mid z <_k x \text{ and } f_{\Sigma_{k+1}}(z) \mathrel{\rhd\!\!\!\rhd} \Gamma\} \cup \{(y,z) \mid x' \leq_k z \text{ and } \Gamma \mathrel{\rhd\!\!\!\rhd} f_{\Sigma_{k+1}}(z)\} \\
f_{\Sigma_{k+1}} & = & f_{\Sigma_k} \cup \{(y,\Gamma)\}
\end{array}$$

   In order to prove the coherence of the trace of $f_{\Sigma_{k+1}}$ we take into account the following:

   (i) Regarding $\rhd$, we apply Lemma 4.3(1 and 2) to obtain condition (a) of coherence.

   (ii) Regarding $\blacktriangleright$, to prove condition (b), we only have to consider the induced relation $\sqsubset$ between y and the rest of points in the frame Σ_{k+1}. Recall that we are under the assumption that $\overrightarrow{\blacklozenge} A \notin f_{\Sigma_k}(x)$, therefore it is not the case that $f_{\Sigma_k}(x) \blacktriangleright \Gamma$.

   To begin with, we have information enough to prove that $x = -\alpha$. With this purpose, let us proceed by considering the relation of x with x':

   - It cannot be the case that $x \sqsubset_k x'$, otherwise by coherence of f_{Σ_k} we would have $f_{\Sigma_k}(x) \blacktriangleright f_{\Sigma_k}(x')$. Recall that $f_{\Sigma_k}(x) \rhd \Gamma \rhd f_{\Sigma_k}(x')$, now by Lemma 4.6(4$\blacktriangleright$) we obtain $f_{\Sigma_k}(x) \blacktriangleright \Gamma$, which is a contradiction.

- Now, if $x \neq -\alpha$, by $x \not\sqsubset_k x'$, we would have that $-\alpha <_k x <_k +\alpha = x'$. By coherence of f_{Σ_k} we obtain $\overleftarrow{\Diamond}\alpha^- \wedge \overrightarrow{\Diamond}\alpha^+ \in f_{\Sigma_k}(x)$ and, and by $f_{\Sigma_k}(x) \rhd \Gamma \rhd f_{\Sigma_k}(x')$ it is easily obtained that $\overleftarrow{\Diamond}\alpha^- \wedge \overrightarrow{\Diamond}\alpha^+ \in \Gamma$. Finally, by Lemma 4.5(1) we obtain $f_{\Sigma_k}(x) \blacktriangleright \Gamma$, which is a contradiction and, thus, $x = -\alpha$.

Since $x = -\alpha$, it is obvious that $x \not\sqsubset_{k+1} y$ holds, and we will only have to consider the successors of y in the frame Σ_{k+1}. We have two possibilities:

(2.1) $y \not\sqsubset_{k+1} x'$.

In this case coherence is immediately guaranteed.

(2.2) $y \sqsubset_{k+1} x'$.

It cannot be the case that $x' \neq +\alpha$, by using $x = -\alpha$ we also have $-\alpha <_k x' <_k +\alpha$; therefore, $-\alpha <_{k+1} y <_{k+1} x' <_{k+1} +\alpha$.

Recall that we have already proved part (a) of coherence for $f_{\Sigma_{k+1}}$, thus $f_{\Sigma_{k+1}}(-\alpha) \rhd \Gamma \rhd f_{\Sigma_{k+1}}(x') \rhd f_{\Sigma_{k+1}}(+\alpha)$, and from this we easily obtain $\overleftarrow{\Diamond}\alpha^- \wedge \overrightarrow{\Diamond}\alpha^+ \in \Gamma$ and $\overleftarrow{\Diamond}\alpha^- \wedge \overrightarrow{\Diamond}\alpha^+ \in f_{\Sigma_{k+1}}(x')$, thus, by Lemma 4.5(1), we obtain $\Gamma \blacktriangleright f_{\Sigma_{k+1}}(x')$. Finally, Lemma 4.3(1$\blacktriangleright$) allows to finish the proof of coherence of the trace.

(iii) Regarding $\rhd\!\!\!\rhd$, the strong coherence is proved as previously.

It is not difficult to check that the definition provided for Σ_{k+1} belongs to $\Xi_{\mathcal{S}}$.

The cases of historic conditionals are handled similarly.

QED

5. Quasi-density properties in $\mathcal{L}(MQ)^N$

The existence of several modal connectives in our language allows for considering variations of properties like density, continuity or others. The study of this kind of properties from a purely logical standpoint is worth to be done; specifically, we address here some particular cases of the problem of definability of properties in $\mathcal{L}(MQ)^N$ obtained as variations of the standard concept of density; other definability-related issues will be studied elsewhere.

Definition 5.1. Let $\mathbb{K}$ be a set of multimodal qualitative frames. The set $\mathbb{K}$ is said to be *definable* by a schema of formulas A if for every frame Σ we have that $\Sigma \in \mathbb{K}$ if and only if A is valid in Σ.

Let P be a property of multimodal qualitative frames, and $\mathbb{K}$ the class of all multimodal qualitative frames satisfying P, then P is said to be *definable* if $\mathbb{K}$ is definable.

A first approach to quasi-density can be expressed as follows: a frame $\Sigma = (\mathbb{S}, +\alpha, -\alpha, <, \prec)$ is said to be *quasi-dense with respect to* $\sqsubset$ (or $\sqsubset$-quasi-dense) if the following property holds:

$$\text{for all } x, y \in \mathbb{S},\ \text{if } x \sqsubset y,\ \text{then there exists } z \text{ such that } x \sqsubset z \sqsubset y \qquad (\sqsubset\text{qd})$$

It is clear that the usual density

$$\text{for all } x, y \in \mathbb{S},\ \text{if } x < y,\ \text{then there exists } z \text{ such that } x < z < y \qquad (\text{dens})$$

implies $\sqsubset$-quasi-density; however, the other implication needs not hold: a counterexample can be obtained by considering a frame whose only points are the landmarks.

Regarding definability, the formula

$$\overrightarrow{\blacksquare}\,\overrightarrow{\blacksquare}\,A \rightarrow \overrightarrow{\blacksquare}\,A \tag{$\sqsubset$QD}$$

can be proved to define $\sqsubset$-quasi-density. More formally, we can state the following proposition

Proposition 5.1. The class of frames $(\mathbb{S}, +\alpha, -\alpha, <)$ satisfying ($\sqsubset$qd) is definable in $\mathcal{L}(MQ)$.

If we are interested in recovering the usual density property from $\sqsubset$-quasi-density, it is sufficient to impose the requirement that INF is a non-empty set and does not have either least or greatest points. From an axiomatic standpoint, we can state the following:

Proposition 5.2. The formula $\overrightarrow{\Box}\,\overrightarrow{\Box}\,A \rightarrow \overrightarrow{\Box}\,A$ (which defines density) can be deduced from MQ+($\sqsubset$QD) plus the following axiom schemata:

$$(\overleftarrow{\Diamond}\,\alpha^- \land \overrightarrow{\Diamond}\,\alpha^+) \rightarrow \overrightarrow{\blacklozenge}\,(\overleftarrow{\Diamond}\,\alpha^- \land \overrightarrow{\Diamond}\,\alpha^+)$$
$$(\overleftarrow{\Diamond}\,\alpha^- \land \overrightarrow{\Diamond}\,\alpha^+) \rightarrow \overleftarrow{\blacklozenge}\,(\overleftarrow{\Diamond}\,\alpha^- \land \overrightarrow{\Diamond}\,\alpha^+)$$
$$\overleftarrow{\Diamond}\,(\overleftarrow{\Diamond}\,\alpha^- \land \overrightarrow{\Diamond}\,\alpha^+) \lor (\overleftarrow{\Diamond}\,\alpha^- \land \overrightarrow{\Diamond}\,\alpha^+) \lor \overrightarrow{\Diamond}\,(\overleftarrow{\Diamond}\,\alpha^- \land \overrightarrow{\Diamond}\,\alpha^+)$$

Proof:
The formula $\overrightarrow{\Box}\,\overrightarrow{\Box}\,A \rightarrow \overrightarrow{\Box}\,A$ can be proven by applying a reasoning by cases strategy based on the theorem $\overrightarrow{\Diamond}\,\alpha^- \lor \alpha^- \lor (\overleftarrow{\Diamond}\,\alpha^- \land \overrightarrow{\Diamond}\,\alpha^+) \lor \alpha^+ \lor \overleftarrow{\Diamond}\,\alpha^+$ (see Proposition 4.1); specifically, it suffices to prove $\left(\overrightarrow{\Box}\,\overrightarrow{\Box}\,A \land \varphi\right) \rightarrow \overrightarrow{\Box}\,A$ where φ is any of the disjuncts (the details of the derivations are omitted).

QED

Other combinations, e.g. by merging the use of $<$ and $\sqsubset$ in the property defining density, generate the properties

for all $x, y \in \mathbb{S},$ if $x \sqsubset y,$ then there exists z such that $x \sqsubset z < y$

for all $x, y \in \mathbb{S},$ if $x \sqsubset y,$ then there exists z such that $x < z \sqsubset y$

which can be shown to be defined, respectively, by the formulas $\overrightarrow{\blacksquare}\,\overrightarrow{\Box}\,A \rightarrow \overrightarrow{\blacksquare}\,A$ and $\overrightarrow{\Box}\,\overrightarrow{\blacksquare}\,A \rightarrow \overrightarrow{\blacksquare}\,A$.

Interestingly enough, these properties are nothing but alternative formulations of $\sqsubset$-quasi-density, that is, each of the properties above is equivalent to property ($\sqsubset$qd).

These alternative cases of quasi-density can be enriched by preventing the landmarks to be affected by the quasi-density relation:

$$\text{for all } x, y \in \mathbb{S}, \text{ if } x < y \text{ and } x \neq -\alpha, \text{ then there exists } z \text{ such that } x \sqsubset z < y \tag{mqd-i}$$
$$\text{for all } x, y \in \mathbb{S}, \text{ if } x < y \text{ and } y \neq +\alpha, \text{ then there exists } z \text{ such that } x < z \sqsubset y \tag{mqd-ii}$$

Both cases of weak mixed quasi-density are definable in $\mathcal{L}(MQ)$, respectively, by the formulas

$$(\overrightarrow{\blacksquare}\,\overrightarrow{\Box}\,A \land \neg\alpha^-) \rightarrow \overrightarrow{\Box}\,A \tag{MQD-i}$$
$$\overrightarrow{\Box}\,\overrightarrow{\blacksquare}\,A \rightarrow \overrightarrow{\Box}(\neg\alpha^+ \rightarrow A) \tag{MQD-ii}$$

as the following proposition shows:

Proposition 5.3.

1. Property (mqd-i) is definable by the formula (MQD-i).
2. Property (mqd-ii) is definable by the formula (MQD-ii).

Proof:
We will only prove the first item above, the proof for the second item is similar.

Let us consider (Σ, h), where Σ satisfies (mqd-i) and an element $x \neq -\alpha$ in Σ such that $x \notin h(\overrightarrow{\Box} A)$. Then there exists y satisfying $x < y$ and $y \notin h(A)$. By property (mqd-i), we would have an element z such that $x \sqsubset z < y$. Therefore $z \notin h(\overrightarrow{\Box} A)$ and, hence, $x \notin h(\overrightarrow{\blacksquare}\overrightarrow{\Box} A)$. Thus, the formula $(\overrightarrow{\blacksquare}\overrightarrow{\Box} A \wedge \neg\alpha^{-}) \to \overrightarrow{\Box} A$ is true in (Σ, h).

Reciprocally, for a frame Σ not satisfying property (mqd-i), we have at least two elements x, y in Σ such that $x < y$ and $x \neq -\alpha$. Now, let us define a model on Σ in which $h(p) = \mathbb{S} \smallsetminus \{y\}$; by analyzing the different possibilities for x it is trivial to check that in any case the model refutes the instance $(\overrightarrow{\blacksquare}\overrightarrow{\Box} p \wedge \neg\alpha^{-}) \to \overrightarrow{\Box} p$ in x. QED

When considering the mirror images of the formulas (MQD-i) and (MQD-ii) one obtains the curious fact that the mirror image of (MQD-i) defines the same property than (MQD-ii) and, similarly, the mirror image of (MQD-ii) defines the same property than (MQD-i).

Regarding the negligibility connectives we can, for instance, consider the two properties below:

$$\text{If } x \prec y\text{, then there exists } z \text{ such that } x \prec z < y \tag{nqd-i}$$

$$\text{If } x \prec y\text{, then there exists } z \text{ such that } x < z \prec y \tag{nqd-ii}$$

On the one hand, for a given x, property (nqd-i) states that the set of the elements from which x is negligible does not have a first element. On the other hand, property (nqd-ii) expresses that there is at least an intermediate element between x and the set of elements from which x is negligible.

The formulas that define the properties stated above are, respectively,

$$\overrightarrow{\boxed{n}}\overrightarrow{\Box} A \to \overrightarrow{\boxed{n}} A \tag{NQD-i}$$

$$\overrightarrow{\Box}\overrightarrow{\boxed{n}} A \to \overrightarrow{\boxed{n}} A \tag{NQD-ii}$$

that is, formally we have the proposition below:

Proposition 5.4. The following classes of frames are definable:

1. $\mathbb{K}_1 = \{(\mathbb{S}, +\alpha, -\alpha, <, \prec) \mid \prec \text{ satisfies (nqd-i)}\}$
2. $\mathbb{K}_2 = \{(\mathbb{S}, +\alpha, -\alpha, <, \prec) \mid \prec \text{ satisfies (nqd-ii)}\}$

Proof:

1. Let us prove that $\overrightarrow{\boxed{n}}\overrightarrow{\Box} A \to \overrightarrow{\boxed{n}} A$ defines $\mathbb{K}_1$.

Consider $\Sigma \in \mathbb{K}_1$ and let (Σ, h) be any model on Σ and x any element in Σ. Now we proceed by contraposition: if $x \notin h(\overrightarrow{\boxed{n}} A)$ then there exists y such that $x \prec y$ and $y \notin h(A)$. Now, by (nqd-i), we have some z such that $x \prec z < y$. Thus $z \notin h(\overrightarrow{\Box} A)$ and so $x \notin h(\overrightarrow{\boxed{n}}\overrightarrow{\Box} A)$.

Conversely, consider $\Sigma = (\mathbb{S}, +\alpha, -\alpha, <, \prec) \notin \mathbb{K}_1$. Then there are some $x, y \in \mathbb{S}$ such that $x \prec y$ and it does not exist $z \in \mathbb{S}$ such that $x \prec z < y$. Now, the model (Σ, h) where $h(p) = \mathbb{S} \smallsetminus \{y\}$ refutes the formula $\overrightarrow{\boxed{\scriptstyle n}}\,\overrightarrow{\Box} p \to \overrightarrow{\boxed{\scriptstyle n}} p$ at x and so $\overrightarrow{\boxed{\scriptstyle n}}\,\overrightarrow{\Box} A \to \overrightarrow{\boxed{\scriptstyle n}} A$ is not valid in Σ and therefore it is not valid in $\mathbb{K}_1$.
2. The proof of this case is similar. QED

Once again it is remarkable that the mirror image of (NQD-i) defines the same property as (NQD-ii), and the mirror of (NQD-ii) defines the same property as (NQD-i). Thus, from an axiomatic point of view, the extension of MQ^N with (NQD-i) and its mirror image is equivalent to the extension with (NQD-i) and (NQD-ii).

There are other properties which are worth to be studied, for instance, the strong versions of (nqd-i) and (nqd-ii) in which the middle element is required to be *comparable* with one of the extreme points:

$$\text{If } x \prec y \text{, then there exists } z \text{ such that } x \prec z \sqsubset y \qquad \text{(ncqd-i)}$$

$$\text{If } x \prec y \text{, then there exists } z \text{ such that } x \sqsubset z \prec y \qquad \text{(ncqd-ii)}$$

these properties can be proven to be definable by the following schemata:

$$\overrightarrow{\boxed{\scriptstyle n}}\,\overrightarrow{\blacksquare} A \to \overrightarrow{\boxed{\scriptstyle n}} A \qquad \text{(NCQD-i)}$$

$$\overrightarrow{\blacksquare}\,\overrightarrow{\boxed{\scriptstyle n}} A \to \overrightarrow{\boxed{\scriptstyle n}} A \qquad \text{(NCQD-ii)}$$

6. Conclusions and future work

A sound and complete extension of the minimal system MQ of multimodal qualitative reasoning with notions of comparability and negligibility has been introduced.

Although the system presented in this work (considering just two landmarks) is considerably simpler than those stated at the beginning of this section, still it is useful as a stepping stone for considering more complex systems, for which the logic has to be enriched by adding new modal operators capable to treat a bigger number of milestones, equivalence classes and/or qualitative relations.

As future work, it is planned to investigate other notions or additional plausible properties of negligibility, together with the analysis of which properties are definable and which are not, and the consideration of the corresponding extensions of MQ^N.

A deeper analysis of the notion of *quasi-density* and a study of an analogous notion of *quasi-continuity* and, moreover, further extensions of the system in order to obtain sound and complete versions with denseness and continuity axioms are envisaged.

We are also investigating the possibility of providing an algebraic presentation of the semantics of MQ^N, in order to get a better understanding of the properties of the qualitative relations which facilitate their proofs.

Last but not least, from the computational standpoint, we are planning to develop theorem provers for MQ and MQ^N based either on tableaux or on resolution.

References

[1] B. Bennett. Modal logics for qualitative spatial reasoning. *Bull. of the IGPL*, 3:1–22, 1995.

[2] B. Bennett, A.G. Cohn, F. Wolter, and M. Zakharyaschev. Multi-Dimensional Modal Logic as a Framework for Spatio-Temporal Reasoning. *Applied Intelligence*, 17(3): 239–251, 2002

[3] J. P. Burgess. Basic tense logic. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic: Volume II: Extensions of Classical Logic*, pages 89–133. 1984.

[4] A. Burrieza and M. Ojeda-Aciego. A multimodal logic approach to order of magnitude qualitative reasoning. *Lect. Notes in Artificial Intelligence* 3040, pages 431-440, 2004.

[5] P. Dague. Numeric reasoning with relative orders of magnitude. In *Proc. 11th National Conference on Artificial Intelligence*, pages 541–547. The AAAI Press/The MIT Press, 1993.

[6] P. Dague. Symbolic reasoning with relative orders of magnitude. In *Proc. 13th Intl. Joint Conference on Artificial Intelligence*, pages 1509–1515. Morgan Kaufmann, 1993.

[7] M.L. Mavrovouniotis and G. Stephanopoulos. Reasoning with orders of magnitude and approximate relations. In *Proc. 6th National Conference on Artificial Intelligence*. The AAAI Press/The MIT Press, 1987.

[8] N. Piera, M. Sánchez, and L. Travé-Massuyès. Qualitative operators for order of magnitude calculus: robustness and precision. In *Proc. of 13th IMACS World Congress on Computation and Applied Mathematics*, pages 1–6. IMACS, 1991.

[9] O. Raiman. Order of magnitude reasoning. *Artificial Intelligence*, 51:11–38, 1991.

[10] B. Shults and B.J. Kuipers. Proving properties of continuous systems: qualitative simulation and temporal logic. *Artificial Intelligence*, 92:91–129, 1997.

[11] D. Randell, Z. Cui, and A. Cohn. A spatial logic based on regions and connections. *Proc. of the 3rd Intl Conf on Principles of Knowledge Representation and Reasoning* (KR'92), pg 165–176, 1992.

[12] M. Sánchez, F. Prats, and N. Piera. Una formalización de relaciones de comparabilidad en modelos cualitativos. *Boletín de la AEPIA (Bulletin of the Spanish Association for AI)*, 6:15–22, 1996.

[13] L. Travé-Massuyès, F. Prats, M. Sánchez, and N. Agell. Consistent relative and absolute order-of-magnitude models. In *Proc. Qualitative Reasoning 2002 Conference*, 2002.

[14] F. Wolter and M. Zakharyaschev. Qualitative spatio-temporal representation and reasoning: a computational perspective. In G. Lakemeyer and B. Nebel, editors, *Exploring Artificial Intelligence in the New Millenium*. Morgan Kaufmann, 2002.

A. Proof of Proposition 4.1

Proposition 4.1. The following formulas are theorems of MQ^N, where A and B are wff:

T1. $\overrightarrow{\Diamond}\alpha^- \vee \alpha^- \vee (\overleftarrow{\Diamond}\alpha^- \wedge \overrightarrow{\Diamond}\alpha^+) \vee \alpha^+ \vee \overleftarrow{\Diamond}\alpha^+$

T2. $(\varphi \wedge \overrightarrow{\blacksquare}A) \rightarrow \overrightarrow{\Box}(\varphi \rightarrow A)$, where $\varphi \in \{\overrightarrow{\Diamond}\alpha^-, \overleftarrow{\Diamond}\alpha^+, \overleftarrow{\Diamond}\alpha^- \wedge \overrightarrow{\Diamond}\alpha^+\}$

T3. $(\overrightarrow{\blacklozenge}A \wedge \overrightarrow{\Diamond}B) \rightarrow (\overrightarrow{\blacklozenge}(A \wedge B) \vee \overrightarrow{\blacklozenge}(\overrightarrow{\blacklozenge}A \wedge B) \vee \overrightarrow{\blacklozenge}(A \wedge \overrightarrow{\Diamond}B))$

T4. $(\overleftarrow{\Diamond}\varphi \wedge \overrightarrow{\Diamond}\varphi) \rightarrow \varphi$, where $\varphi \in \{\overrightarrow{\Diamond}\alpha^-, \overleftarrow{\Diamond}\alpha^+, \overleftarrow{\Diamond}\alpha^- \wedge \overrightarrow{\Diamond}\alpha^+\}$

T5. $(\overrightarrow{\Diamond_{\mathrm{n}}}A \wedge \overrightarrow{\Diamond}B) \rightarrow (\overrightarrow{\Diamond_{\mathrm{n}}}(A \wedge B) \vee \overrightarrow{\Diamond}(\overrightarrow{\Diamond}A \wedge B) \vee \overrightarrow{\Diamond_{\mathrm{n}}}(A \wedge \overrightarrow{\Diamond}B))$

T6. $(\alpha^+ \vee \overleftarrow{\Diamond}\alpha^+) \rightarrow (\overrightarrow{\Diamond}A \rightarrow \overrightarrow{\blacklozenge}A)$

Proof:
We will only provide a complete proof for **T3**, which is based on **T1**, **T4** and the following auxiliary theorems and derived rules which are standard in modal logic:

LM 1 $(\overrightarrow{\blacksquare}A \wedge \overrightarrow{\blacklozenge}B) \rightarrow \overrightarrow{\blacklozenge}(A \wedge B)$

LM 2 $\overrightarrow{\Diamond}(A \wedge B) \rightarrow (\overrightarrow{\Diamond}A \wedge \overrightarrow{\Diamond}B)$

LM3 $(A \wedge \overrightarrow{\Diamond}B) \rightarrow \overrightarrow{\Diamond}(\overleftarrow{\Diamond}A \wedge B)$

P$\overrightarrow{\Diamond}_1$ If $\vdash A \rightarrow B$ then $\vdash \overrightarrow{\Diamond}A \rightarrow \overrightarrow{\Diamond}B$

***Proof of* T1.** A formal proof in MQ^N, modulo propositional calculus (**PC**) is given below:

1.	$\overleftarrow{\Diamond}\alpha^- \vee \alpha^- \vee \overrightarrow{\Diamond}\alpha^-$	from **c1**$[\xi/\alpha^-]$
2.	$\overleftarrow{\Diamond}\alpha^+ \vee \alpha^+ \vee \overrightarrow{\Diamond}\alpha^+$	from **c1**$[\xi/\alpha^+]$
3.	$\overrightarrow{\Diamond}\alpha^- \vee \alpha^- \vee (\overleftarrow{\Diamond}\alpha^- \wedge \overrightarrow{\Diamond}\alpha^+) \vee \alpha^+ \vee \overleftarrow{\Diamond}\alpha^+$	from 1, 2 by **PC**

***Proof of* T4.**

1.	$(\overleftarrow{\Diamond}\overrightarrow{\Diamond}\alpha^- \wedge \overrightarrow{\Diamond}\overrightarrow{\Diamond}\alpha^-) \rightarrow \overrightarrow{\Diamond}\overrightarrow{\Diamond}\alpha^-$	tautology
2.	$(\overleftarrow{\Diamond}\overrightarrow{\Diamond}\alpha^- \wedge \overrightarrow{\Diamond}\overrightarrow{\Diamond}\alpha^-) \rightarrow \overrightarrow{\Diamond}\alpha^-$	from 1 by **K3** and **PC**
3.	$(\overleftarrow{\Diamond}\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\Diamond}\overleftarrow{\Diamond}\alpha^+) \rightarrow \overleftarrow{\Diamond}\alpha^+$	[similar to 1–2]
4.	$(\overleftarrow{\Diamond}(\overleftarrow{\Diamond}\alpha^- \wedge \overrightarrow{\Diamond}\alpha^+) \wedge \overrightarrow{\Diamond}(\overleftarrow{\Diamond}\alpha^- \wedge \overrightarrow{\Diamond}\alpha^+)) \rightarrow (\overleftarrow{\Diamond}\overleftarrow{\Diamond}\alpha^- \wedge \overrightarrow{\Diamond}\overrightarrow{\Diamond}\alpha^+)$	from **LM2** and **PC**
5.	$\overleftarrow{\Diamond}\overleftarrow{\Diamond}\alpha^- \rightarrow \overleftarrow{\Diamond}\alpha^-$	from mirror of **K3** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and **PC**
6.	$\overrightarrow{\Diamond}\overrightarrow{\Diamond}\alpha^+ \rightarrow \overrightarrow{\Diamond}\alpha^+$	**K3** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and **PC**
7.	$(\overleftarrow{\Diamond}\varphi \wedge \overrightarrow{\Diamond}\varphi) \rightarrow \varphi$	from 2, 3, and 4–6 by **PC**

***Proof of* T3.** By **T1** we can split the proof into five parts, according to each possible disjunct.

In order to improve readability some abbreviations will be used for formulas which occur several times: for instance $\alpha^- \vee \overrightarrow{\Diamond}\alpha^-$ means that we are in an observable negative which we denote *obs*$^-$, similarly *inf* denotes $\overleftarrow{\Diamond}\alpha^- \wedge \overrightarrow{\Diamond}\alpha^+$.

Case I: $\overrightarrow{\Diamond}\alpha^- \to$ **T3**

1.	$\overrightarrow{\Diamond}\alpha^- \to \overrightarrow{\blacksquare}\mathit{obs}^-$	from **c6**
2.	$(\overrightarrow{\Diamond}\alpha^- \wedge \overrightarrow{\blacklozenge}A \wedge \overrightarrow{\Diamond}B) \to (\overrightarrow{\blacksquare}\mathit{obs}^- \wedge \overrightarrow{\blacklozenge}A \wedge \overrightarrow{\Diamond}B)$	from 1 by **PC**
3.	$\overrightarrow{\blacksquare}(\mathit{obs}^- \wedge \overrightarrow{\blacklozenge}A) \to \overrightarrow{\blacklozenge}(\mathit{obs}^- \wedge A)$	from **LM1**
4.	$(\overrightarrow{\blacksquare}\mathit{obs}^- \wedge \overrightarrow{\blacklozenge}A \wedge \overrightarrow{\Diamond}B) \to (\overrightarrow{\blacklozenge}(\mathit{obs}^- \wedge A) \wedge \overrightarrow{\Diamond}B)$	from 3 by **PC**
5.	$\overrightarrow{\blacklozenge}(\mathit{obs}^- \wedge A) \to \overrightarrow{\Diamond}(\mathit{obs}^- \wedge A)$	from **M1** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and **PC**
6.	$(\overrightarrow{\Diamond}\alpha^- \wedge \overrightarrow{\blacklozenge}A \wedge \overrightarrow{\Diamond}B) \to (\overrightarrow{\Diamond}(\mathit{obs}^- \wedge A) \wedge \overrightarrow{\Diamond}B)$	from 2, 4, 5 by **PC**
7.	$\big(\overrightarrow{\Diamond}(\mathit{obs}^- \wedge A) \wedge \overrightarrow{\Diamond}B\big) \to \big(\overrightarrow{\Diamond}(\mathit{obs}^- \wedge A \wedge B) \vee \overrightarrow{\Diamond}(\overrightarrow{\Diamond}(\mathit{obs}^- \wedge A) \wedge B) \vee \overrightarrow{\Diamond}(\mathit{obs}^- \wedge A \wedge \overrightarrow{\Diamond}B)\big)$	from **K4** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and **PC**
8.	$(\overrightarrow{\Diamond}\alpha^- \wedge \overrightarrow{\blacklozenge}A \wedge \overrightarrow{\Diamond}B) \to \big(\overrightarrow{\Diamond}(\mathit{obs}^- \wedge A \wedge B) \vee \overrightarrow{\Diamond}(\overrightarrow{\Diamond}(\mathit{obs}^- \wedge A) \wedge B) \vee \overrightarrow{\Diamond}(\mathit{obs}^- \wedge A \wedge \overrightarrow{\Diamond}B)\big)$	from 6 and 7 by **PC**
9.	$\overrightarrow{\Diamond}(\mathit{obs}^- \wedge A \wedge B) \to \overrightarrow{\blacklozenge}(A \wedge B)$	from **c8** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and **PC**
10.	$\overrightarrow{\Diamond}(\mathit{obs}^- \wedge A \wedge \overrightarrow{\Diamond}B) \to \overrightarrow{\blacklozenge}(A \wedge \overrightarrow{\Diamond}B)$	from **c8** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and **PC**
11.	$\overrightarrow{\Diamond}(\mathit{obs}^- \wedge A) \to \overrightarrow{\blacklozenge}A$	from **c8** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and **PC**
12.	$\overrightarrow{\Diamond}(\mathit{obs}^- \wedge A) \to \overrightarrow{\Diamond}\mathit{obs}^-$	from **LM2** and **PC**
13.	$(\overrightarrow{\Diamond}(\mathit{obs}^- \wedge A) \wedge B) \to (\mathit{obs}^- \wedge \overrightarrow{\blacklozenge}A \wedge B)$	from 11 and 12 by **PC**
14.	$\overrightarrow{\Diamond}(\overrightarrow{\Diamond}(\mathit{obs}^- \wedge A) \wedge B) \to \overrightarrow{\Diamond}(\mathit{obs}^- \wedge \overrightarrow{\blacklozenge}A \wedge B)$	from 13 by $\mathbf{P}\overrightarrow{\Diamond}_1$
15.	$\overrightarrow{\Diamond}(\mathit{obs}^- \wedge \overrightarrow{\blacklozenge}A \wedge B) \to \overrightarrow{\blacklozenge}(\overrightarrow{\blacklozenge}A \wedge B)$	from **c8** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and **PC**
16.	$\overrightarrow{\Diamond}(\overrightarrow{\Diamond}(\mathit{obs}^- \wedge A) \wedge B) \to \overrightarrow{\blacklozenge}(\overrightarrow{\blacklozenge}A \wedge B)$	from 14 and 15 by **PC**
17.	$\overrightarrow{\Diamond}\alpha^- \to$ **T3**	from 8 and 9, 10, and 16 by **PC**

Case II: $\alpha^- \to$ **T3**

1.	$\alpha^- \to \neg\overrightarrow{\blacklozenge}A$	from **c4** by $\overrightarrow{\blacklozenge}$ and **PC**
2.	$\alpha^- \to$ **T3**	from 1 by **PC**

Case III: *inf* $\to$ **T3**

1.	$\mathit{inf} \to \overrightarrow{\blacksquare}\mathit{inf}$	from **c5**
2.	$(\overrightarrow{\blacksquare}\mathit{inf} \wedge \overrightarrow{\blacklozenge}A) \to \overrightarrow{\blacklozenge}(\mathit{inf} \wedge A)$	from **LM1**
3.	$(\mathit{inf} \wedge \overrightarrow{\blacklozenge}A \wedge \overrightarrow{\Diamond}B) \to (\mathit{inf} \wedge \overrightarrow{\blacklozenge}(\mathit{inf} \wedge A) \wedge \overrightarrow{\Diamond}B)$	from 1, 2 by **PC**
4.	$\overrightarrow{\blacklozenge}(\mathit{inf} \wedge A) \to \overrightarrow{\Diamond}(\mathit{inf} \wedge A)$	from **M1**, by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and **PC**
5.	$(\mathit{inf} \wedge \overrightarrow{\blacklozenge}A \wedge \overrightarrow{\Diamond}B) \to (\mathit{inf} \wedge \overrightarrow{\Diamond}(\mathit{inf} \wedge A) \wedge \overrightarrow{\Diamond}B)$	from 3, 4 by **PC**
6.	$(\overrightarrow{\Diamond}(\mathit{inf} \wedge A) \wedge \overrightarrow{\Diamond}B) \to \big(\overrightarrow{\Diamond}(\mathit{inf} \wedge A \wedge B) \vee \overrightarrow{\Diamond}(\mathit{inf} \wedge A \wedge \overrightarrow{\Diamond}B) \vee \overrightarrow{\Diamond}(\overrightarrow{\Diamond}(\mathit{inf} \wedge A) \wedge B)\big)$	from **K4** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and **PC**
7.	$(\mathit{inf} \wedge \overrightarrow{\blacklozenge}A \wedge \overrightarrow{\Diamond}B) \to \big(\overrightarrow{\Diamond}(\mathit{inf} \wedge A \wedge B) \vee \overrightarrow{\Diamond}(\mathit{inf} \wedge A \wedge \overrightarrow{\Diamond}B) \vee \overrightarrow{\Diamond}(\overrightarrow{\Diamond}(\mathit{inf} \wedge A) \wedge B)\big)$	from 5, 6 and **PC**
8.	$(\mathit{inf} \wedge \overrightarrow{\Diamond}(\mathit{inf} \wedge A \wedge B)) \to \overrightarrow{\blacklozenge}(A \wedge B)$	from **c10** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and **PC**

9. $(\mathit{inf} \wedge \overrightarrow{\Diamond}(\mathit{inf} \wedge A \wedge \overrightarrow{\Diamond} B)) \to \overrightarrow{\blacklozenge}(A \wedge \overrightarrow{\Diamond} B)$ from **c10** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and **PC**
10. $\overrightarrow{\Diamond}(\mathit{inf} \wedge A) \to \overrightarrow{\Diamond}\mathit{inf}$ from **LM2** and **PC**
11. $(\mathit{inf} \wedge \overrightarrow{\Diamond}(\overrightarrow{\Diamond}(\mathit{inf} \wedge A) \wedge B)) \to \overrightarrow{\Diamond}(\overleftarrow{\Diamond}\mathit{inf} \wedge \overrightarrow{\Diamond}(\mathit{inf} \wedge A) \wedge B)$ from **LM3**
12. $(\overleftarrow{\Diamond}\mathit{inf} \wedge \overrightarrow{\Diamond}(\mathit{inf} \wedge A) \wedge B)) \to (\overleftarrow{\Diamond}\mathit{inf} \wedge \overrightarrow{\Diamond}\mathit{inf} \wedge \overrightarrow{\Diamond}(\mathit{inf} \wedge A) \wedge B))$ from 10 by **PC**
13. $\overrightarrow{\Diamond}(\overleftarrow{\Diamond}\mathit{inf} \wedge \overrightarrow{\Diamond}(\mathit{inf} \wedge A) \wedge B)) \to \overrightarrow{\Diamond}(\overleftarrow{\Diamond}\mathit{inf} \wedge \overrightarrow{\Diamond}\mathit{inf} \wedge \overrightarrow{\Diamond}(\mathit{inf} \wedge A) \wedge B))$ from 12 by $\mathbf{P}\overrightarrow{\Diamond}_1$
14. $(\mathit{inf} \wedge \overrightarrow{\Diamond}(\overrightarrow{\Diamond}(\mathit{inf} \wedge A) \wedge B)) \to (\mathit{inf} \wedge \overrightarrow{\Diamond}(\overleftarrow{\Diamond}\mathit{inf} \wedge \overrightarrow{\Diamond}\mathit{inf} \wedge \overrightarrow{\Diamond}(\mathit{inf} \wedge A) \wedge B))$
from 11, 13 by **PC**
15. $(\overleftarrow{\Diamond}\mathit{inf} \wedge \overrightarrow{\Diamond}\mathit{inf}) \to \mathit{inf}$ **T4**
16. $(\mathit{inf} \wedge \overrightarrow{\Diamond}(\overrightarrow{\Diamond}(\mathit{inf} \wedge A) \wedge B)) \to (\mathit{inf} \wedge \overrightarrow{\Diamond}(\mathit{inf} \wedge \overrightarrow{\Diamond}(\mathit{inf} \wedge A) \wedge B))$
from 14, 15 by **PC** and $\mathbf{P}\overrightarrow{\Diamond}_1$
17. $(\mathit{inf} \wedge \overrightarrow{\Diamond}(\mathit{inf} \wedge A)) \to \overrightarrow{\blacklozenge} A$ from **c10** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and **PC**
18. $(\mathit{inf} \wedge \overrightarrow{\Diamond}(\mathit{inf} \wedge A) \wedge B) \to (\mathit{inf} \wedge \overrightarrow{\blacklozenge} A \wedge B)$ from 17 by **PC**
19. $\overrightarrow{\Diamond}(\mathit{inf} \wedge \overrightarrow{\Diamond}(\mathit{inf} \wedge A) \wedge B) \to \overrightarrow{\Diamond}(\mathit{inf} \wedge \overrightarrow{\blacklozenge} A \wedge B)$ from 18 by $\mathbf{P}\overrightarrow{\Diamond}_1$
20. $(\mathit{inf} \wedge \overrightarrow{\Diamond}(\overrightarrow{\Diamond}(\mathit{inf} \wedge A) \wedge B)) \to (\mathit{inf} \wedge \overrightarrow{\Diamond}(\mathit{inf} \wedge \overrightarrow{\blacklozenge} A \wedge B))$ from 19 by **PC**
21. $(\mathit{inf} \wedge \overrightarrow{\Diamond}(\mathit{inf} \wedge \overrightarrow{\blacklozenge} A \wedge B)) \to \overrightarrow{\blacklozenge}(\overrightarrow{\blacklozenge} A \wedge B)$ from **c10** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and **PC**
22. $(\mathit{inf} \wedge \overrightarrow{\Diamond}(\overrightarrow{\Diamond}(\mathit{inf} \wedge A) \wedge B)) \to \overrightarrow{\blacklozenge}(\overrightarrow{\blacklozenge} A \wedge B)$ from 20, 21 by **PC**
23. $\mathit{inf} \to$ **T3** from 7 and 8, 9 and 22 by **PC**

Case IV: $\alpha^+ \to$ **T3**:

1. $\overrightarrow{\blacklozenge} A \to \overrightarrow{\Diamond} A$ from **M1** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$, and **PC**
2. $(\alpha^+ \wedge \overrightarrow{\blacklozenge} A \wedge \overrightarrow{\Diamond} B) \to (\overrightarrow{\Diamond} A \wedge \overrightarrow{\Diamond} B)$ from 1 by **PC**
3. $\big(\overrightarrow{\Diamond} A \wedge \overrightarrow{\Diamond} B\big) \to \big(\overrightarrow{\Diamond}(A \wedge B) \vee \overrightarrow{\Diamond}(\overrightarrow{\Diamond} A \wedge B) \vee \overrightarrow{\Diamond}(A \wedge \overrightarrow{\Diamond} B)\big)$ from **K4** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$, and **PC**
4. $(\alpha^+ \wedge \overrightarrow{\blacklozenge} A \wedge \overrightarrow{\Diamond} B) \to \big(\overrightarrow{\Diamond}(A \wedge B) \vee \overrightarrow{\Diamond}(\overrightarrow{\Diamond} A \wedge B) \vee \overrightarrow{\Diamond}(A \wedge \overrightarrow{\Diamond} B)\big)$ from 2, 3 by **PC**
5. $(\alpha^+ \wedge \overrightarrow{\Diamond}(A \wedge B)) \to \overrightarrow{\blacklozenge}(A \wedge B)$ from **c7** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and **PC**
6. $(\alpha^+ \wedge \overrightarrow{\Diamond}(A \wedge \overrightarrow{\Diamond} B)) \to \overrightarrow{\blacklozenge}(A \wedge \overrightarrow{\Diamond} B)$ from **c7** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and **PC**
7. $(\alpha^+ \wedge \overrightarrow{\Diamond}(\overrightarrow{\Diamond} A \wedge B)) \to \overrightarrow{\Diamond}(\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\Diamond} A \wedge B)$ from **LM3**
8. $(\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\Diamond} A) \to \overrightarrow{\blacklozenge} A$ from **c9** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and **PC**
9. $(\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\Diamond} A \wedge B) \to (\overrightarrow{\blacklozenge} A \wedge B)$ from 8 by **PC**
10. $\overrightarrow{\Diamond}(\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\Diamond} A \wedge B) \to \overrightarrow{\Diamond}(\overrightarrow{\blacklozenge} A \wedge B)$ from 9 by $\mathbf{P}\overrightarrow{\Diamond}_1$
11. $(\alpha^+ \wedge \overrightarrow{\Diamond}(\overrightarrow{\Diamond} A \wedge B)) \to (\alpha^+ \wedge \overrightarrow{\Diamond}(\overrightarrow{\blacklozenge} A \wedge B))$ from 7, 10 by **PC**
12. $(\alpha^+ \wedge \overrightarrow{\Diamond}(\overrightarrow{\blacklozenge} A \wedge B)) \to \overrightarrow{\blacklozenge}(\overrightarrow{\blacklozenge} A \wedge B))$ from **c7** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and **PC**
13. $(\alpha^+ \wedge \overrightarrow{\Diamond}(\overrightarrow{\Diamond} A \wedge B)) \to \overrightarrow{\blacklozenge}(\overrightarrow{\blacklozenge} A \wedge B))$ from 11, 12 by **PC**
14. $\alpha^+ \to$ **T3** from 4 and 5, 6 and 13 by **PC**

Case V: $\overleftarrow{\Diamond}\alpha^+ \to \mathbf{T3}$:

1. $\overrightarrow{\blacklozenge} A \to \overrightarrow{\Diamond} A$ — from **M1** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and **PC**
2. $(\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\blacklozenge} A \wedge \overrightarrow{\Diamond} B) \to (\overrightarrow{\Diamond} A \wedge \overrightarrow{\Diamond} B)$ — from 1 by **PC**
3. $\big(\overrightarrow{\Diamond} A \wedge \overrightarrow{\Diamond} B\big) \to \big(\overrightarrow{\Diamond}(A \wedge B) \vee \overrightarrow{\Diamond}(\overrightarrow{\Diamond} A \wedge B) \vee \overrightarrow{\Diamond}(A \wedge \overrightarrow{\Diamond} B)\big)$ — from **K4** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and **PC**
4. $(\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\blacklozenge} A \wedge \overrightarrow{\Diamond} B) \to \big(\overrightarrow{\Diamond}(A \wedge B) \vee \overrightarrow{\Diamond}(\overrightarrow{\Diamond} A \wedge B) \vee \overrightarrow{\Diamond}(A \wedge \overrightarrow{\Diamond} B)\big)$ — from 2, 3 by **PC**
5. $(\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\Diamond}(A \wedge B)) \to \overrightarrow{\blacklozenge}(A \wedge B)$ — from **c9** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and **PC**
6. $(\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\Diamond}(A \wedge \overrightarrow{\Diamond} B)) \to \overrightarrow{\blacklozenge}(A \wedge \overrightarrow{\Diamond} B)$ — from **c9** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and **PC**
7. $(\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\Diamond}(\overrightarrow{\Diamond} A \wedge B)) \to \overrightarrow{\Diamond}(\overleftarrow{\Diamond}\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\Diamond} A \wedge B)$ — from **LM3**
8. $\overleftarrow{\Diamond}\overleftarrow{\Diamond}\alpha^+ \to \overleftarrow{\Diamond}\alpha^+$ — from mirror of **K3** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and **PC**
9. $\overrightarrow{\Diamond}(\overleftarrow{\Diamond}\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\Diamond} A \wedge B) \to \overrightarrow{\Diamond}(\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\Diamond} A \wedge B)$ — from 8 by **PC** and $\mathbf{P}\overrightarrow{\Diamond}_1$
10. $(\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\Diamond} A) \to \overrightarrow{\blacklozenge} A$ — from **c9** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and **PC**
11. $(\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\Diamond} A \wedge B) \to (\overrightarrow{\blacklozenge} A \wedge B)$ — from 10 by **PC**
12. $\overrightarrow{\Diamond}(\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\Diamond} A \wedge B) \to \overrightarrow{\Diamond}(\overrightarrow{\blacklozenge} A \wedge B)$ — from 11 by $\mathrm{P}\overrightarrow{\Diamond}_1$
13. $(\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\Diamond}(\overrightarrow{\Diamond} A \wedge B)) \to (\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\Diamond}(\overrightarrow{\blacklozenge} A \wedge B))$ — from 7, 9, 12 by **PC**
14. $\overrightarrow{\Diamond}(\overrightarrow{\blacklozenge} A \wedge B)) \to \overrightarrow{\blacklozenge}(\overrightarrow{\blacklozenge} A \wedge B)$ — from **c9** by $\overrightarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and **PC**
15. $(\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\Diamond}(\overrightarrow{\Diamond} A \wedge B)) \to\to \overrightarrow{\blacklozenge}(\overrightarrow{\blacklozenge} A \wedge B)$ — from 13, 14 by **PC**
16. $\overleftarrow{\Diamond}\alpha^+ \to \mathbf{T3}$ — from 4, and 5, 6 and 15 by **PC**

QED