On congruences, ideals and homomorphisms over multilattices

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Abstract

In this paper, we focus on the notions of congruence, ideal and homomorphism on the generalized structure of multilattice. We provide suitable definitions of these notions in order to guarantee the classical relationship between these concepts.

Keywords: Multilattice, congruence, L-fuzzy sets, non-deterministic algebra, homomorphism.

1 Introduction

The study of congruences is important both from a theoretical standpoint and for its applications in the field of logic-based approaches to uncertainty. Regarding applications, the notion of congruence is intimately related to the foundations of fuzzy reasoning and its relationships with other logics of uncertainty [9]. More focused on the theoretical aspects of Computer Science, some authors [1, 17] have pointed out the relation between congruences, fuzzy automata and determinism.

In spatial reasoning, the interest has been focused on spatial relationships and the imprecision attached to information and knowledge to be handled; two main components being knowledge representation and reasoning. In [3] we can see that the fuzzy set framework associated to the formalism provided by mathematical morphology and formal logics allows for deriving appropriate representations and reasoning tools.

There have also been studies on qualitative reasoning about the morphological relation of congruence. A spatial congruence relation is introduced in [6] which, moreover, provides an algebraic structure to host relations based on it.

The previous paragraphs have shown the usefulness of the theory of (crisp) congruences regarding practical applications. At this point, it is important to recall that the problem of providing suitable fuzzifications of crisp concepts is an important topic which has attracted the attention of a number of researchers. Since the inception of fuzzy sets and fuzzy logic, there have been approaches to consider underlying sets of truth-values more general than the unit interval; for instance, consider the L-fuzzy sets introduced in [10], where L is a complete lattice.

This paper originated as part of a research line aimed at investigating L-fuzzy sets where L has the structure of a *multilattice*. The concepts of ordered and algebraic multilattice were introduced by Benado in [2]. A multilattice is an algebraic structure in which the restrictions imposed on a (complete) lattice, namely, the "existence of least upper bounds and greatest lower bounds" are relaxed to the "existence of minimal upper bounds and maximal lower bounds".

Much more recently, Cordero *et al.* [14] proposed an alternative algebraic definition of multilattice which is more closely related to that of lattice, allowing for natural definitions of related structures such as multisemilattices and, in addition, is better suited for applications. For instance, Medina *et al.* [15] developed a general approach to fuzzy logic programming based on a multilattice as underlying set of truth-values for the logic.

A number of papers have been published on the lattice of fuzzy congruences on different algebraic structures [7, 8, 16, 19], and in this paper we continue the research in this direction initiated in [4], as a necessary step prior to considering the multilattice-based generalization of the concept of L-fuzzy congruence.

In this paper, we concentrate not only on congruences but, as well, on other notions which traditionally are related to them, namely, homomorphisms and ideals. Specifically, in Section 2, after introducing the preliminary definitions, we prove that the set of congruences on a multilattice forms a complete lattice. Then, in Section 3 we propose alternative definitions to the notions of ideal of a multilattice and homomorphism between multilattices, and we show that these new definitions allow to recover the classical relationships between congruences, ideals and homomorphisms which are not preserved by other definitions provided in the literature.

2 On the lattice of congruences on multilattices

Let us recall the concept of multilattice:

Given $(M, \leq)$ a partially ordered set (henceforth poset) and $B \subseteq M$, a *multi-supremum of* B is a minimal element of the set of upper bounds of B and $\mathrm{multisup}(B)$ denotes the set of multi-suprema of B. Dually, we define the *multi-infima* which will be denoted $\mathrm{multinf}(B)$.

Definition 1 *A poset, $(M, \leq)$, is an* ***ordered multilattice*** *if and only if it satisfies that, for all $a, b, x \in M$ with $a \leq x$ and $b \leq x$, there exists*[1] $z \in \mathrm{multisup}\{a, b\}$ *such that $z \leq x$ and its dual version for* $\mathrm{multinf}\{a, b\}$.

A multilattice is said to be ***full*** *if* $\mathrm{multisup}\{a, b\} \neq \varnothing$ *and* $\mathrm{multinf}\{a, b\} \neq \varnothing$ *for all $a, b \in M$.*

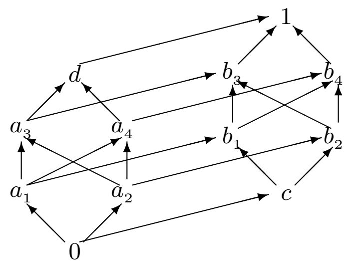
Similarly to lattice theory, if we define $a \vee b = \mathrm{multisup}\{a, b\}$ and $a \wedge b = \mathrm{multinf}\{a, b\}$, it is possible to define multilattices algebraically and, conversely, if we define $a \leq b$ if and only if $a \vee b = \{b\}$ it is possible to obtain the ordered version of multilattice. Both definitions of multilattice are proved to be equivalent (see [13, Theorem 2.11]).

Remark 2 *In the rest of the paper we will frequently write singletons without braces.*

Now, we will introduce a notation which will be useful hereafter. Let $\mathcal{R}$ be a binary relation in M and $X, Y \subseteq M$, then $X \widehat{\mathcal{R}} Y$ denotes that, for all $x \in X$, there exists $y \in Y$ such that $x\mathcal{R}y$ and for all $y \in Y$ there exists $x \in X$ such that $x\mathcal{R}y$.

Definition 3 *Let $(M, \vee, \wedge)$ be a multilattice, a* ***congruence*** *on M is any equivalence relation $\equiv$ such that if $a \equiv b$, then $a \vee c \mathrel{\widehat{\equiv}} b \vee c$ and $a \wedge c \mathrel{\widehat{\equiv}} b \wedge c$, for all $a, b, c \in M$.*

Example 4 *Let $(M, \vee, \wedge)$ be the multilattice which is described in the figure below.*



The partition

$$\{\{0, a_1, a_2, a_3, a_4, d\}, \{c, b_1, b_2, b_3, b_4, 1\}\}$$

defines a non-trivial congruence. However,

$$\mathcal{R}=\{\{0, a_1, b_1, c\}, \{a_2, b_2, a_3, b_3\}, \{a_4, b_4, d, 1\}\}$$

is not a congruence because $0\mathcal{R}a_1$ but $a_4 \in a_1 \vee a_2$ and there is not an element $x \in 0 \vee a_2 = a_2$ such that $x\mathcal{R}a_4$.

The following results are consequences from the definition, and will be useful later.

[1] Note that the definition is consistent with the existence of two incomparable elements *without* any multisupremum. In other words, $\mathrm{multisup}\{a, b\}$ can be empty.

Lemma 5 *Let $\equiv$ be a congruence relation in a multilattice M, let $[a]$ be the equivalence class of an element a, and consider $a, b \in M$:*

1. *If $b \in [a]$ then $\varnothing \neq a \vee b \subseteq [a]$ and $\varnothing \neq a \wedge b \subseteq [a]$*
2. *If there exist $z \in a \wedge b$ and $w \in a \vee b$ such that $z \equiv w$, then $a \equiv b$*
3. *If $z, w \in a \vee b$ with $z \neq w$ and $z \equiv w$ then $a \vee b \subseteq [a] = [b]$.*
4. *If $z, w \in a \wedge b$ with $z \neq w$ and $z \equiv w$ then $a \wedge b \subseteq [a] = [b]$.*

Proof:

(1) As $b \equiv a$ then $a \vee b \mathrel{\widehat{\equiv}} a \vee a = a$ which implies $\varnothing \neq a \vee b \subseteq [a]$. The other result is proved similarly.

(2) Let us assume that there exist $z \in a \wedge b$ and $w \in a \vee b$ such that $z \equiv w$. Then $a = a \wedge w \mathrel{\widehat{\equiv}} a \wedge z = z = b \wedge z \mathrel{\widehat{\equiv}} b \wedge w = b$. By transitivity, one obtains $a \equiv b$.

(3) If $z, w \in a \vee b$ with $z \neq w$ then $a, b \in z \wedge w$. Since $z \equiv w$, by Item *(1)*, $a, b \in z \wedge w \subseteq [z]$ and, therefore, $a \equiv b$. Finally, applying Item *(1)* again, $a \vee b \subseteq [a] = [b]$.

(4) Dual to *(3)*.

□

Lemma 6 *Let $\equiv$ be a congruence relation in a multilattice M, and consider $a, b, t \in M$. If $a \leq b$ with $a \equiv b$, then*

1. *For all $z \in a \wedge t$ we have that*

$$\varnothing \neq (b \wedge t) \cap z{\uparrow} \subseteq [z]$$

2. *For all $w \in b \vee t$ we have that*

$$\varnothing \neq (a \vee t) \cap w{\downarrow} \subseteq [w]$$

where $z{\uparrow} = \{x \mid x \geq z\}$ and $w{\downarrow} = \{x \mid x \leq w\}$.

Proof: If $z \in a \wedge t$, then $z \leq a \leq b$ and $z \leq t$ and, since $(M, \vee, \wedge)$ is a multilattice, there exists $w \in b \wedge t$ with $z \leq w$. Moreover, for any $w \in b \wedge t$ with $z \leq w$, it is easy to prove that $z \in a \wedge w$. As $a \equiv b$ we have that $z \in a \wedge w \mathrel{\widehat{\equiv}} b \wedge w = w$. The second item can be proved analogously. □

The following result can be viewed as a suitable generalisation to multilattices of a similar result about lattices given by Grätzer [11, page 26]. Its usefulness can be seen in that it reduces the set of requirements to be checked in order to prove that a given binary relation is a congruence relation.

Theorem 7 (See [4]) *Let $(M, \vee, \wedge)$ be a multilattice and $\mathcal{R}$ be a binary relation. Then $\mathcal{R}$ is a congruence relation if and only if the following conditions hold:*

1. *$\mathcal{R}$ is reflexive*
2. *$x\mathcal{R}y$ if and only if there exist $z \in x \wedge y$ and $w \in x \vee y$ with $z\mathcal{R}w$*
3. *If $x \leq y \leq z$ with $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$*
4. *If $x \leq y$ with $x\mathcal{R}y$, then $x \wedge t\, \widehat{\mathcal{R}}\, y \wedge t$ and $x \vee t\, \widehat{\mathcal{R}}\, y \vee t$.*

It is well-known that, for every set A, the set of equivalence relations on A, $Eq(A)$, with the inclusion ordering (in the powerset of $A \times A$) is a complete lattice in which the infimum is the meet and the supremum is the transitive closure of the join.

In [4] the authors proved that the set of congruences on a multilattice is a complete lattice under the assumption of m-distributivity. This requirement can be avoided by means of a more involved proof which is given below:

Theorem 8 *The set of the congruences in a multilattice M, $Con(M)$, is a sublattice of $Eq(M)$. Furthermore, $Con(M)$ is a complete lattice w.r.t. the inclusion ordering.*

Proof: Let $\{\equiv_i\}_{i \in \Lambda}$ be a set of congruences in M, consider $\equiv_\cap$ to be its intersection and $\equiv_{tc}$ be the transitive closure of their union.

Since $\equiv_\cap$ and $\equiv_{tc}$ are equivalence relations, they satisfy the conditions (1) and (3) of Theorem 7. On the other hand, condition (2) is a consequence of Lemma 5. Thus, we have just to check condition (4) in order to show that both relations $\equiv_\cap$ and $\equiv_{tc}$ are congruences.

Let us consider $x \le y$ with $x \equiv_\cap y$. Lemma 6 ensures that, if $z \in x \wedge t$, then there exists $w \in y \wedge t$ with $z \equiv_\cap w$.

Now, let us consider $w \in y \wedge t$, and let us prove that there exists $z \in x \wedge t$ such that $z \equiv_\cap w$. To this end, we will distinguish two cases:

a) If there exists $z \in x \wedge t$ such that $z \le w$ then, by Lemma 6, $z \equiv_\cap w$.

b) Otherwise, let us prove that $x \wedge t \subseteq [w]_\cap$. For all $i \in \Lambda$, there exists $z \in x \wedge t$ such that $z \equiv_i w$. By Lemma 6, there exists $w' \in y \wedge t$ such that $z \le w'$ and $z \equiv_i w' \equiv_i w$. So, $w, w' \in y \wedge t$ with $w \ne w'$ and $w \equiv_i w'$ and, by Lemma 5, $y \wedge t \subseteq [y]_i$. Now, from $x \equiv_i y$ we have $x \wedge t \mathrel{\widehat{\equiv}_i} y \wedge t$. Therefore, $w \in [y]_i$ and $x \wedge t \subseteq [y]_i$. Finally, as this argument is applicable to all $i \in \Lambda$, $x \wedge t \subseteq [w]_\cap$.

For $\vee$ we proceed similarly.

The transitive closure, $\equiv_{tc}$, is an equivalence relation. Now, we prove compatibility with the operations. Let $x \equiv_{tc} y$, that is, there exists a sequence $x_1, \dots, x_n$ such that $x_1 = x$, $x_n = y$ and $x_1 \equiv_{i_1} x_2 \equiv_{i_2} \cdots \equiv_{i_{n-1}} x_n$ with $i_1, i_2, \cdots, i_{n-1} \in \Lambda$. Then $x_1 \vee t \widehat{\equiv}_{i_1} x_2 \vee t \widehat{\equiv}_{i_2} \dots \widehat{\equiv}_{i_{n-1}} x_n \vee t$ and $x_1 \wedge t \widehat{\equiv}_{i_1} x_2 \wedge t \widehat{\equiv}_{i_2} \dots \widehat{\equiv}_{i_{n-1}} x_n \wedge t$. Therefore, $x \vee t \mathrel{\widehat{\equiv}_{tc}} y \vee t$ and $x \wedge t \mathrel{\widehat{\equiv}_{tc}} y \wedge t$. □

3 Ideals, homomorphisms, and congruences

There have been several proposed definitions for the notion of ideal of a multilattice. The definition of ideal in a multilattice is not canonical. For instance, one can find the notion of s-ideals introduced by Rachůnek, or the l-ideals of Burgess, or the m-ideals given by Johnston [12, 18]. In this section, we introduce an alternative definition which is more suitable for extending the classical results about congruences and homomorphisms.

Definition 9 *Let $(M, \vee, \wedge)$ be a multilattice. A non-empty set $I \subseteq M$ is said to be an **ideal** if the following conditions hold:*

1. *$i, j \in I$ implies $\varnothing \ne i \vee j \subseteq I$.*
2. *$i \in I$ implies $i \wedge a \subseteq I$ for all $a \in M$.*
3. *For all $a, b \in M$, if $a \wedge b \cap I \ne \varnothing$ then $a \wedge b \subseteq I$.*

The set of ideals in M is denoted by $\mathcal{I}(M)$.

The following lemma can be obtained directly from the definition above:

Lemma 10 *A non-empty intersection of ideals of a multilattice is an ideal.*

Theorem 11 *If $(M, \vee, \wedge)$ is a full multilattice then $(\mathcal{I}(M), \subseteq)$ is a complete lattice.*

Proof: The hypothesis of M being full guarantees that the arbitrary intersection of ideals is non-empty and, furthermore, M is an ideal. Therefore, $(\mathcal{I}(M), \subseteq)$ is a complete inf-semilattice with top element, hence, a complete lattice. □

Theorem 12 *Let $(M, \vee, \wedge)$ be a multilattice with bottom element 0, and let $\equiv$ be a congruence relation. Then $[0]$ is an ideal of M.*

Proof: Item 1 of the definition is a consequence of Lemma 5, whereas item 2 is a consequence of the definition of congruence.

For item 3, assume $x \in a \wedge b$ and $x \equiv 0$, consider $y \in a \wedge b$ with $x \ne y$ and let us prove that $y \equiv 0$. Since $x \le a$ and $y \le a$ there exists $a' \in x \vee y$ such that $a' \le a$. Analogously, there exists $b' \in x \vee y$ with $b' \le b$. Notice that $a' \ne b'$ because if $a' = b'$ then $x = y$. We have $a', b' \in x \vee y$ and, since $x \vee y \mathrel{\widehat{\equiv}} 0 \vee y = y$, thus $a' \equiv b'$; now, by Lemma 5(3), we have that $x \vee y \subseteq [y] = [x] = [0]$. □

In previous works, the notion of homomorphism is extended to the theory of multilattices as follows: $h\colon M \to M'$ is a homomorphism if $h(a \vee b) \subseteq h(a) \vee h(b)$ and

$h(a \wedge b) \subseteq h(a) \wedge h(b)$. As we will provide an alternative definition of a homomorphism, we will use the term *Benado-homomorphism* to refer to this definition, named after the person who introduced it. Now, as stated previously, we will introduce a new definition that fits better to the expected extension of the classical results.

Definition 13 *Let $h\colon M \to M'$ be a map between multilattices, h is said to be a (non-deterministic)* ***nd-homomorphism*** *if*

$$h(a \vee b) = \big(h(a) \vee h(b)\big) \cap h(M)$$
$$h(a \wedge b) = \big(h(a) \wedge h(b)\big) \cap h(M)$$

Theorem 14 *Let $h\colon M \to M'$ be a map where M is full. Then h is a Benado homomorphism if and only if it is an nd-homomorphism.*

Proof: It is sufficient to prove that, for all $a, b, c \in A$, if $h(c) \in h(a) \vee h(b)$ then there exists $c' \in a \vee b$ such that $h(c') = h(c)$. The same result for $\wedge$ follows by duality.

Firstly, $h(a \vee c) \subseteq h(a) \vee h(c) = h(c)$ and $h(b \vee c) \subseteq h(b) \vee h(c) = h(c)$. Since $a \vee c \neq \varnothing \neq b \vee c$, there exist $x \in a \vee c$ and $y \in b \vee c$ such that $h(x) = h(c) = h(y)$.

On the other hand, any element $z \in x \vee y$ satisfies $h(z) \in h(x \vee y) \subseteq h(x) \vee h(y) = h(c) \vee h(c) = h(c)$.

As $a \leq z$ and $b \leq z$, there exists $c' \in a \vee b$ such that $c' \leq z$. Therefore, $h(c) = h(z) = h(c' \vee z) \subseteq h(c') \vee h(z) = h(c') \vee h(c)$ which implies that $h(c') \leq h(c)$. Since also $h(c'), h(c) \in h(a) \vee h(b)$ we deduce $h(c') = h(c)$. $\square$

In the rest of the section we will focus on the relationship between congruences and homomorphisms.

Definition 15 *Let $h : A \to B$ be a mapping. The* ***kernel relation*** *of h is defined as follows*

$$a \,\mathcal{R}\, b \text{ if and only if } h(a) = h(b)$$

Obviously, for every mapping h, the kernel relation is an equivalence relation. It is remarkable that the particular definition given for nd-homomorphism is the key to the following interesting result.

Theorem 16

1. *The kernel relation of any nd-homomorphism between multilattices is a congruence.*

2. *Let $(M, \vee, \wedge)$ be a multilattice and $\equiv$ a congruence relation, then $M/_{\equiv}$ is a multilattice with*

   $$[a] \vee [b] = \{[x] \mid x \in a \vee b\}$$
   $$[a] \wedge [b] = \{[x] \mid x \in a \wedge b\}$$

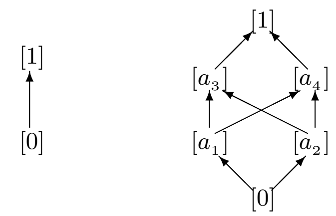
   Moreover, the mapping $p\colon M \to M/_{\equiv}$ such that $p(x) = [x]$ is a surjective nd-homomorphism.

3. *Every nd-homomorphism $h\colon M \to M'$ can be canonically decompose as $h = i \circ \overline{h} \circ p$ where $\overline{h}\colon M/_{\equiv} \to h(M)$ is the isomorphism defined as $\overline{h}([x]) = h(x)$ and $i\colon h(M) \to M'$ is the inclusion monomorphism.*

Example 17 *In the multilattice described in Example 4, we have the following congruence relations:*

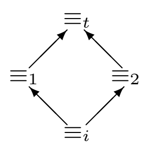
$\equiv$	$M/_{\equiv}$
$\equiv_t$	$\{\{0, a_1, a_2, a_3, a_4, d, c, b_1, b_2, b_3, b_4, 1\}\}$
$\equiv_1$	$\{\{0, a_1, a_2, a_3, a_4, d\},$ $\{c, b_1, b_2, b_3, b_4, 1\}\}$
$\equiv_2$	$\{\{0, c\}, \{a_1, b_1\}, \{a_2, b_2\},$ $\{a_3, b_3\}, \{a_4, b_4\}, \{d, 1\}\}$
$\equiv_i$	$\{\{0\}, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}, \{d\},$ $\{c\}, \{b_1\}, \{b_2\}, \{b_3\}, \{b_4\}, \{1\}\}$

The non-trivial quotient multilattices are the following:



$(M/\equiv_1, \cdot)$ $(M/\equiv_2, \cdot)$

Finally, the lattice $(Con(M), \subseteq)$ is:



Finally, the relation between ideals and homomorphisms is stated in the following result which, again, follows from the particular definition of ideal that we have introduced.

Theorem 18 *Let $h\colon M \to M'$ be a nd-homomorphism, and assume that M' has a bottom element* 0 *such that $0 \in h(M)$. Then $h^{-1}(0)$ is an ideal of M, called the* ***kernel ideal****.*

4 Conclusions and future work

We have introduced specific definitions of the notions of homomorphism and ideal for the theory of multilattices which, contrariwise to alternative definitions that can be found in the literature, allow to extend the classical relationship between the concepts of homomorphism, congruence and ideal.

As future work, we will study conditions on a multilattice which guarantee the possibility of defining congruences from an ideal, in the same way that distributivity allows to do so in the theory of lattices.

Another line for future research concerns the applications of these ideas to the more general framework of hyperstructures [5].

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