On fuzzy homomorphisms between hyperrings

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Abstract

We focus on the study of the structure of hyperrings; in this paper, we recall the basics of crisp homomorphisms between hyperstructures, particularly, between hyperrings and, then, the notion of fuzzy homomorphism between hyperrings is established and its main properties are analysed.

Keywords: Hyperstructures, fuzzy homomorphisms, fuzzy ideals, fuzzy congruences.

1 Introduction

The study of hyperstructures started seventy-five years ago with Marty's paper [13] which firstly used a multiple-valued operator. Nowadays, the theory of hyperstructures is being thoroughly studied, focusing in particular classes such as hypergroups, hyper-near rings, semi-hyperrings, join spaces, etc.

Otherwise, fuzzy sets were introduced by Zadeh [15] and since then there has been a number of authors who started the development of fuzzy algebra.

The two types of extensions presented in the previous paragraphs have started to be studied jointly, giving rise to the so-called fuzzy hyperalgebra. Several areas have benefitted from the developments in the area of hyperstructures and fuzzy set theory, in particular, artificial intelligence and soft computing:

- hyperstructures can be used as a useful tool to modelling computing with uncertainty, in that the result of an operation is not a single value and can be considered a set of possible values;
- on the other hand, some ideas underlying fuzzy set theory form the crux on which the development of the different approaches to multiple-valued and fuzzy logics has been based.

The use of hyperstructures in conjunction with fuzzy logic has been shown to be fruitful in some areas certainly related to artificial intelligence and soft computing, such as fuzzy logic programming with multilattices [14].

In this work, we focus on a particular topic related to fuzzy hyperalgebra, which develops fuzzy versions of hyperstructures. Specifically, we study the theory of hyperrings and fuzzy homomorphisms between them.

The structure of the paper is the following: after stating the preliminary definitions, we recall the basics of the theory of crisp homomorphisms between hyperstructures, recalling specially the results and isomorphism theorems which relate homomorphisms, congruences and ideals. Then, the main contribution of the paper is presented: the extension of the previous relations to the fuzzy case.

2 Preliminary definitions

Firstly, let us introduce some preliminary concepts:

Definition 1 *A* ***hypergroupoid*** *is a pair* $(A,+)$ *consisting of a non-empty set* A *together with a hyperoperation* $+ : A \times A \to 2^A \smallsetminus \varnothing$.

Definition 2 *A hypergroupoid* $(A,+)$ *is a* ***canonical hypergroup*** *if the following properties hold:*

(i) *for every* $x,y,z \in A \quad x+(y+z) = (x+y)+z$

(ii) *for every* $x,y \in A \quad x+y = y+x$

(iii) *there exists* $0 \in A$ *such that* $0+x = x$ *for all* $x \in A$

(iv) *for every* $x \in A$ *there exists a unique element* $x' \in A$ *such that* $0 \in x+x'$; *(we shall write* $-x$ *for* x' *and we call it the opposite of* x.*)*

(v) *for every* $x,y,z \in A$ *if* $z \in x+y$ *then* $y \in x-z$.

In the above definition, for $X, Y \subseteq A$, we denote

$$X + Y = \bigcup_{\substack{x\in X\\ y\in Y}} x + y$$

The following equalities follow easily from the axioms: $-(-x) = x$ and $-(x + y) = -x - y$

Note that in the rest of the paper we will frequently write singletons without braces.

Let us now introduce the definition of hyperring we will work with. We will use that given by Krasner, in which the sum is a hyperoperation with the structure of canonical hypergroup and the product is an associative operation, together with distributivity and adequate conditions for the neutral element of the sum.

Definition 3 (Krasner [12]) *A **hyperring** is an algebraic structure $(A, +, \cdot, 0)$ which satisfies:*

1. *$(A, +)$ is a canonical hypergroup with the neutral element 0.*
2. *Relating to the multiplication, $(A, \cdot)$ is a semigroup having 0 as a bilaterally absorbing element.*
3. *The multiplication is distributive respect to the hyperoperation $+$*

3 On crisp homomorphisms

We begin by discussing the different versions of the concept of homomorphism on hypegroupoids (also called multigroupoids) appearing in the literature. Some authors that deal with these and other hyperstructures use the following definitions of homomorphism [4].

Definition 4 *Let $(A, \cdot)$ and $(B, \cdot)$ be hypergroupoids. A map $h: A \to B$ is said to be:*

- ***Benado-homomorphism** if $h(ab) \subseteq h(a)h(b)$, for all $a, b \in A$.*
- ***Algebraic-homomorphism** if $h(ab) = h(a)h(b)$, for all $a, b \in A$.*

Regarding the terminology, we depart here a bit from the usual one. The first one was the original definition by Benado [1], which has been used in several recent papers [4, 5, 10]. However, it is noticeable that, finally, the authors concentrate mostly on the equality-based definition.

The terminology used in those papers is to call homomorphism to Benado's ones and call *good* (or *strong*) homomorphism to algebraic ones. We have adopted the term *algebraic* instead of good or strong because this type of homomorphism immediate allows the lifting of classical homomorphisms to the so-called powerset extension. Obviously, the advantage of using algebraic homomorphisms is that one can transfer properties from the powerset to the hypergroupoid very easily, so that the presentation of multivalued concepts is greatly simplified.

The term homomorphism should induce the properties of the initial hyperalgebra on the image set. It can be easily checked that this is the case for algebraic-homomorphisms but, in general, it is not true for Benado-homomorphisms.

The notion of homomorphism can be easily extended to the structure of hyperrings, since the product operation is classical. The formal definition is given below:

Definition 5 *Let $(A, +, \cdot, 0)$ and $(B, +, \cdot, 0)$ be hyperrings. A map $h: A \to B$ is said to be a hyperring **homomorphism** if $h(a + b) = h(a) + h(b)$ and $h(ab) = h(a)h(b)$, for all $a, b \in A$ and $h(0) = 0$.*

Ideals are crucial in ring theory, and they have also been studied in the context of hyperrings; its definition in this case is given below:

Definition 6 *Let $(A, +, \cdot, 0)$ be a hyperring. A subset $I \subseteq A$ is said to be an **ideal** of A is the following conditions hold:*

- *$i - j \subseteq I$ for all $i, j \in I$.*
- *$aI \subseteq I$ for all $a \in A$.*
- *$Ia \subseteq I$ for all $a \in A$.*

Finally, we recall below the notion of congruence relation on a hyperring that we will extend.

Definition 7 *Let $(A, +, \cdot, 0)$ be a hyperring. A **congruence** on A is an equivalence relation $\equiv$ which for all $a, b, c, d \in A$ satisfies that if $a \equiv b$ and $c \equiv d$ then*

- *for all $x \in a + c$ there exists $y \in b + d$ such that $x \equiv y$,*
- *for all $y \in b + d$ there exists $x \in a + c$ such that $x \equiv y$ and*
- *$ac \equiv bd$.*

In classical settings, it is usual to consider the kernel relation associated to a homomorphism; this idea has been used in the framework of hyperoperations as follows:

Definition 8 *Any hyperring homomorphism $h: A \to B$ defines a congruence relation, namely* kernel relation *$\equiv_h$, defined as*

$$a \equiv_h b \text{ if and only if } h(a) = h(b)$$

The relationship between the concepts of homomorphism, congruence and ideals in the framework of hyperstructures is similar to that in the classical case.

4 Fuzzy homomorphisms on hyperrings

A fuzzy relation is a mapping φ from $A \times B$ into $[0,1]$, that is to say, any fuzzy subset of $A \times B$. The powerset extension of a fuzzy relation is defined as, $\widehat{\varphi}\colon 2^A \times 2^B \to [0,1]$ with

$$\widehat{\varphi}(X,Y) = \Big(\bigwedge_{x\in X}\bigvee_{y\in Y}\varphi(x,y)\Big) \wedge \Big(\bigwedge_{y\in Y}\bigvee_{x\in X}\varphi(x,y)\Big)$$

The composition of fuzzy relations φ and ψ is defined as follows:

$$(\psi\circ\varphi)(a,c) = \bigvee_{b\in B}\varphi(a,b)\wedge\psi(b,c)$$

A fuzzy relation ρ on $A \times A$ is said to be

1. **reflexive** if $\rho(x,x) = 1$, for every $x \in A$
2. **symmetric** if $\rho(x,y) = \rho(y,x)$, for all $x, y \in A$
3. **transitive** if for all $x, a, y \in A$ we have

$$\rho(x,a)\wedge\rho(a,y) \le \rho(x,y)$$

A reflexive, symmetric and transitive fuzzy relation on A is called a **fuzzy equivalence**. A fuzzy equivalence ρ on A is called a **fuzzy equality** if for any $x, y \in A$, $\rho(x,y) = 1$ implies $x = y$.

Definition 9 ([2]) *A fuzzy equivalence relation ρ on a hyperring $(A, +, \cdot, 0)$ is said to be a **fuzzy congruence relation** if, for all $a_1, a_2, b_1, b_2 \in A$:*

- $\rho(a_1,b_1)\wedge\rho(a_2,b_2) \le \widehat{\rho}(a_1+a_2, b_1+b_2)$ *and*
- $\rho(a_1,b_1)\wedge\rho(a_2,b_2) \le \rho(a_1a_2, b_1b_2)$.

The fuzzification of the concept of function that we adopt has been introduced in [11], also studied in [7, 8, 9], and more recently in [3]. We will introduce the extension of the notion of perfect fuzzy function.

Definition 10 ([7]) *Let ρ and σ be fuzzy equalities defined on the sets A and B, respectively. A **partial fuzzy function** φ from A to B is a mapping $\varphi\colon A \times B \to [0,1]$ satisfying the following conditions for all $a, a' \in A$ and $b, b' \in B$:*

ext1 $\varphi(a,b)\wedge\rho(a,a') \le \varphi(a',b)$

ext2 $\varphi(a,b)\wedge\sigma(b,b') \le \varphi(a,b')$

part $\varphi(a,b)\wedge\varphi(a,b') \le \sigma(b,b')$

If, in addition, the following condition holds:

f1 *For all $a \in A$ there is $b \in B$ such that $\varphi(a,b) = 1$*

*then we say that φ is a **perfect fuzzy function**.*

It is not difficult to show that the element b in condition (f1) above is unique. As a result, every perfect fuzzy function defines a crisp mapping from A to B called **crisp description of** φ.

Definition 11 *Let $(A, +, \cdot, 0)$ and $(B, +, \cdot, 0)$ be hyperrings endowed with fuzzy equalities ρ and σ, respectively, such that $\sigma(a,b) = \sigma(-a,-b)$*

*A perfect fuzzy function $\varphi \in [0,1]^{A\times B}$ is said to be a **fuzzy homomorphism** if for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$, the following conditions hold:*

Sum-compat $\varphi(a_1,b_1)\wedge\varphi(a_2,b_2) \le \widehat{\varphi}(a_1+a_2, b_1+b_2)$

Prod-compat $\varphi(a_1,b_1)\wedge\varphi(a_2,b_2) \le \varphi(a_1a_2, b_1b_2)$

Neutral $\varphi(0,0) = 1$

*Moreover, φ is said to be **complete** if the two following conditions hold:*

1. *if $\bigvee_{y\in Y}\varphi(a,y) = 1$, then there exists $y \in Y$ such that $\varphi(a,y) = 1$.*
2. *if $\bigvee_{x\in X}\varphi(x,b) = 1$, then there exists $x \in X$ such that $\varphi(x,b) = 1$.*

Remark: Hereafter, unless stated otherwise, we will always consider that we are working with a complete fuzzy homomorphism φ between hyperrings A and B and fuzzy equalities ρ and σ, respectively.

Proposition 12 *Given φ between A and B, the crisp description h of φ is a hyperring homomorphism.*

The notion of fuzzy homomorphism between hyperrings behaves properly with respect to the composition of fuzzy relations, in that the composition of fuzzy homomorphisms is a fuzzy homomorphism. Furthermore, the composition is associative and there exists an identity for this composition. As a result, the class of hyperrings together with the fuzzy homomorphisms between them forms a category.

Let us concentrate now on the relationship between fuzzy homomorphism and congruences.

Definition 13 *The **fuzzy kernel relation** induced by φ in A is defined as $\rho_\varphi(a,a') = \varphi(a, h(a'))$.*

We adopt here the term *kernel* as an extension of the crisp case because of the inequality

$$\varphi(a,b) \wedge \varphi(a',b) \leq \rho_\varphi(a,a')$$

Moreover, in [3], the authors prove the following result.

Proposition 14 *Let φ a perfect fuzzy function from A to B. For all $a, a' \in A$,*

$$\rho_\varphi(a,a') = \bigvee_{b\in B} \varphi(a,b) \wedge \varphi(a',b)$$

In the case of fuzzy homomorphisms between hyperrings, we prove that this fuzzy equivalence relation is also a fuzzy congruence on the initial hyperring.

Theorem 15 *Consider φ between A and B. The fuzzy kernel relation ρ_φ is a fuzzy congruence relation which includes the fuzzy equality ρ in A.*

PROOF: Let us see that ρ_φ is compatible with the sum operation.

$$\begin{aligned}
&\widehat{\rho}_\varphi(a_1+a_3, a_2+a_4) = \\
&= \bigwedge_{a\in a_1+a_3} \bigvee_{a'\in a_2+a_4} \rho_\varphi(a,a') \wedge \bigwedge_{a'\in a_2+a_4} \bigvee_{a\in a_1+a_3} \rho_\varphi(a,a') \\
&= \widehat{\varphi}(a_1+a_3, h(a_2+a_4)) \qquad \text{as } \rho_\varphi(a,a') = \varphi(a,h(a')) \\
&= \widehat{\varphi}(a_1+a_3, h(a_2)+h(a_4)) \qquad \text{by Prop. 12} \\
&\geq \varphi(a_1,h(a_2)) \wedge \varphi(a_3,h(a_4)) \\
&= \rho_\varphi(a_1,a_2) \wedge \rho_\varphi(a_3,a_4)
\end{aligned}$$

The compatibility with the multiplication follows from

$$\begin{aligned}
\rho_\varphi(a_1a_3, a_2a_4) = \varphi(a_1a_3, h(a_2a_4)) \\
&= \varphi(a_1a_3, h(a_2)h(a_4)) \qquad \text{by Prop. 12} \\
&\geq \varphi(a_1,h(a_2)) \wedge \varphi(a_3,h(a_4)) \\
&= \rho_\varphi(a_1,a_2) \wedge \rho_\varphi(a_3,a_4)
\end{aligned}$$

Now, let us show that $\rho \leq \rho_\varphi$:

$$\begin{aligned}
\rho(a,a') &= \rho(a,a') \wedge \varphi(a',h(a')) \\
&\leq \varphi(a,h(a')) = \rho_\varphi(a,a') \qquad \text{by (ext1)}
\end{aligned}$$

□

In the rest of this section we will show the canonical decomposition theorem for a complete fuzzy homomorphism and a fuzzy congruence relation. For suitable extensions on the notions of injectivity and surjectivity we will rely on the definitions given in [7].

Definition 16 *A perfect fuzzy function $\varphi \in [0,1]^{A\times B}$ is said to be:*

- ***surjective*** *if for all $b \in B$ there exists $a \in A$ such that $\varphi(a,b) = 1$.*
- ***injective*** *if $\varphi(a,b) \wedge \varphi(a',b) \leq \rho(a,a')$, for all $a, a' \in A$ and $b \in B$.*
- ***bijective*** *if it is injective and surjective.*

The image set is $\text{Im } \varphi = \{b \in B \mid$ *there exists* $a \in A$ *with* $\varphi(a,b) = 1\}$.

In order to define the different homomorphisms involved in the decomposition theorem, we have to introduce the quotient set associated to a fuzzy equivalence relation.

Definition 17 *Let $(A, +, \cdot, 0)$ be a hyperring and ρ be a fuzzy equivalence relation in A. An* ***equivalence class*** *of an element $a \in A$ is defined as*

$$\rho(a) \in [0,1]^A \quad \textit{with} \quad \rho(a)(a') = \rho(a,a')$$

The ***quotient set*** *is defined as $A/_\rho = \{\rho(a) \mid a \in A\}$ and a fuzzy equality $\overline{\rho}$ can be defined in $A/_\rho$ as $\overline{\rho}(\rho(a), \rho(a')) = \rho(a,a')$.*

The ***fuzzy projection*** *π from A to $A/_\rho$ is defined as $\pi(a, \rho(a')) = \rho(a,a')$.*

Proposition 18 *Let $(A, +, \cdot, 0)$ be a hyperring, ρ a fuzzy equality in A and ρ_A be a fuzzy congruence relation in A that includes ρ. The* ***fuzzy projection*** *π from A to $A/_{\rho_A}$ is a surjective fuzzy homomorphism where the hyperoperations in $A/_{\rho_A}$ are given by*

$$\begin{aligned}
\rho_A(a_1) + \rho_A(a_2) = \{\rho_A(d) \mid d \in a_1 + a_2\} \\
\rho_A(a_1) \cdot \rho_A(a_2) = \{\rho_A(d) \mid d \in a_1a_2\}
\end{aligned}$$

the zero element is $\rho_A(0)$ and the fuzzy equality is $\overline{\rho_A}$.

Remark: In order to prove that the canonical inclusion from the image of a homomorphism is an injective fuzzy homomorphism, we recall the following result from [7]: given φ between A and B, there exists a unique crisp function f such that $\varphi(a,b) = \sigma(f(a),b)$. This f actually coincides with the crisp description h of φ, which satisfies $\varphi(a,h(a)) = 1$.

Lemma 19 *Given φ between A and B, then the inclusion ι from $\text{Im } \varphi$ to B defined as $\iota(b,b') = \sigma(b,b')$ is an injective fuzzy homomorphism.*

Theorem 20 *Any complete fuzzy homomorphism φ from A to B can be canonically decomposed as $\varphi = \iota \circ \overline{\varphi} \circ \pi$ where π is the fuzzy projection from A to $A/_{\rho_\varphi}$, ι is the inclusion from $\text{Im } \varphi$ to B, and $\overline{\varphi}$ is the isomorphism from $A/_{\rho_\varphi}$ to $\text{Im } \varphi$ defined as $\overline{\varphi}(\rho_\varphi(a), b) = \varphi(a,b)$, and the operations and the fuzzy equality in $\text{Im } \varphi$ being the corresponding restrictions of those in B.*

PROOF: We will prove (ext1), (inj) and (surj) for $\overline{\varphi}$ since the rest of properties are straightforward:

ext1 $\overline{\varphi}(\rho_\varphi(a),b)\wedge\overline{\rho_\varphi}(\rho_\varphi(a),\rho_\varphi(a'))=$
$\varphi(a,b)\wedge\rho_\varphi(a,a')=\varphi(a,b)\wedge\varphi(a,h(a'))=$
$\varphi(a,b)\wedge\varphi(a,h(a'))\wedge\varphi(a',h(a'))\leq$
$\sigma(b,h(a'))\wedge\varphi(a',h(a'))\leq\varphi(a',b)=\overline{\varphi}(\rho_\varphi(a'),b)$

inj $\overline{\varphi}(\rho_\varphi(a),b)\wedge\overline{\varphi}(\rho_\varphi(a'),b)=\varphi(a,b)\wedge\varphi(a',b)\leq\rho_\varphi(a,a')=\overline{\rho_\varphi}(\rho_\varphi(a),\rho_\varphi(a'))$.

surj For all $b\in\text{Im }\varphi$ there exists $a\in A$ such that $\varphi(a,b)=1$ and then $\overline{\varphi}(\rho_\varphi(a),b)=1$

Now, let us check that $\varphi=\iota\circ\overline{\varphi}\circ\pi$:

$$\begin{aligned}(\iota\circ\overline{\varphi}\circ\pi)(a,b)&=\\ &=\bigvee_{\substack{\rho_\varphi(a')\in A/\rho_\varphi\\ b'\in\text{Im }\varphi}}\Big(\pi(a,\rho_\varphi(a'))\wedge\overline{\varphi}(\rho_\varphi(a'),b')\wedge\iota(b',b)\Big)\\ &=\bigvee_{\substack{a'\in A\\ b'\in\text{Im}\varphi}}\Big(\rho_\varphi(a,a')\wedge\varphi(a',b')\wedge\sigma(b',b)\Big)\\ &\overset{(ext2)}{\leq}\bigvee_{a'\in A}\Big(\rho_\varphi(a,a')\wedge\varphi(a',b)\Big)\\ &\overset{(def\rho_\varphi)}{=}\bigvee_{a'\in A}\Big(\varphi(a,h(a'))\wedge\varphi(a',b)\Big)\\ &\overset{(f1)}{=}\bigvee_{a'\in A}\Big(\varphi(a,h(a'))\wedge\varphi(a',h(a'))\wedge\varphi(a',b)\Big)\\ &\overset{(part)}{\leq}\bigvee_{a'\in A}\Big(\varphi(a,h(a'))\wedge\sigma(h(a'),b)\Big)\overset{(ext2)}{\leq}\varphi(a,b)\end{aligned}$$

Conversely, $\varphi(a,b)=\sigma(h(a),b)=\pi(a,\rho_\varphi(a))\wedge\overline{\varphi}(\rho_\varphi(a),h(a))\wedge\sigma(h(a),b)\leq(\iota\circ\overline{\varphi}\circ\pi)(a,b)$. □

5 Fuzzy hyperideals and homomorphisms

This section is devoted to the fuzzy extension of the classical relation between crisp ideals and homomorphisms.

First of all, we adopt the definition of fuzzy hyperideal defined in [16]. Nevertheless, the superfluous conditions have been removed.

Definition 21 *Let $(A,+,\cdot,0)$ be a hyperring. A fuzzy subset μ of A is a **fuzzy hyperideal** if it satisfies: for all $a,b\in A$,*

1. *$\mu(a)\wedge\mu(b)\leq\mu(x)$ for all $x\in a-b$*
2. *$\mu(a)\vee\mu(b)\leq\mu(ab)$*

The kernel relation in a hyperring can be expressed in terms of the kernel of the corresponding homomorphism. In this section we prove that the same occurs in the fuzzy case, due to the convenient properties including in our definition of fuzzy homomorphism.

Let $(A,+,\cdot,0)$ and $(B,+,\cdot,0)$ be hyperrings endowed with fuzzy equalities ρ and σ, respectively, such that $\sigma(a,b)=\sigma(-a,-b)$

Let us consider now the fuzzy kernel relation induced by φ in A, $\rho_\varphi\in[0,1]^{A\times A}$, defined as $\rho_\varphi(a,a')=\varphi(a,h(a'))$, where h is the crisp description of φ.

Note that $\rho_\varphi(0)\in[0,1]^A$ is a a fuzzy subset of A and

$$\begin{aligned}\rho_\varphi(0)(a)=\rho_\varphi(0,a)=\varphi(0,h(a))=\sigma(0,h(a))\\ =\sigma(h(a),0)=\varphi(a,0)\end{aligned}$$

Theorem 22 *Consider φ between A and B and the fuzzy kernel relation ρ_φ. Then $\rho_\varphi(0)$ is a fuzzy hyperideal.*

Proof: Firstly, observe that $\varphi(a,b)=\sigma(h(a),b)=\sigma(-h(a),-b)=\sigma(h(-a),-b)=\varphi(-a,-b)$ for all $a\in A$ and $b\in B$. On the other hand,

$$\begin{aligned}\rho_\varphi(0)(a)\wedge\rho_\varphi(0)(b)=\varphi(0,h(a))\wedge\varphi(0,h(b))=\\ \varphi(0,h(a))\wedge\varphi(0,-h(b))\leq\widehat{\varphi}(0,h(a)-h(b))=\\ \widehat{\varphi}(0,h(a-b))=\bigwedge_{y\in h(a-b)}\varphi(0,y)=\\ \bigwedge_{x\in a-b}\varphi(0,h(x))=\bigwedge_{x\in a-b}\rho_\varphi(0)(x)\end{aligned}$$

So, $\rho_\varphi(0)(a)\wedge\rho_\varphi(0)(b)\leq\rho_\varphi(0)(x)$ for all $x\in a-b$.

Related to the multiplication,

$$\begin{aligned}\rho_\varphi(0)(a)=\rho_\varphi(0,a)=\rho_\varphi(0,a)\wedge\rho_\varphi(b,b)\\ \leq\rho_\varphi(0,a\cdot b)=\rho_\varphi(0)(a\cdot b)\end{aligned}$$

and analogously, $\rho_\varphi(0)(b)\leq\rho_\varphi(0)(a\cdot b)$. Thus,

$$\rho_\varphi(0)(a)\vee\rho_\varphi(0)(b)\leq\rho_\varphi(0)(a\cdot b)\qquad\square$$

Once proven that the kernel of a fuzzy homomorphism between hyperrings is a fuzzy hyperideal, we consider whether the existing relationship between the fuzzy hyperideal and the congruence defined by the fuzzy homomorphisms is what one could expect.

In the crisp case, given an ideal I, a congruence is defined [6] by

$$a\equiv b\pmod I\quad\text{if and only if}\quad(a-b)\cap I\neq\varnothing$$

The natural form of fuzzifying this construction would be the following: if μ is a fuzzy hyperideal of A, the fuzzy relation $\rho_\mu\in[0,1]^{A\times A}$ should be

$$\rho_\mu(a,a')=\bigvee_{x\in a-a'}\mu(x)$$

In a nutshell, the fuzzy subset $\rho_\varphi(0)$ plays the role of the kernel of the homomorphism in the crisp case.

Proposition 23 *Consider φ between A and B and the fuzzy kernel relation ρ_φ. Then,*

$$\rho_\varphi(a,a') = \bigvee_{x\in a-a'} \rho_\varphi(0)(x)$$

PROOF: Firstly we prove $\rho_\varphi(a,a') \leq \bigvee_{x\in a-a'} \rho_\varphi(0)(x)$

From Proposition 14,

$$\rho_\varphi(a,a') = \varphi(a,h(a')) = \bigvee_{b\in B} \varphi(a,b)\wedge\varphi(a',b) \quad (1)$$

On the other hand, for all $b \in B$,

$$\begin{aligned}\varphi(a,b)\wedge\varphi(a',b) &\leq \widehat{\varphi}(a-a', b-b)\\ &= \bigwedge_{x\in a-a'}\bigvee_{y\in b-b} \varphi(x,y) \wedge \bigwedge_{y\in b-b}\bigvee_{x\in a-a'} \varphi(x,y)\\ &\leq \bigwedge_{y\in b-b}\bigvee_{x\in a-a'} \varphi(x,y) \leq \bigvee_{x\in a-a'} \varphi(x,0)\\ &= \bigvee_{x\in a-a'} \rho_\varphi(0)(x)\end{aligned}$$

Thus, $\bigvee_{b\in B} \varphi(a,b)\wedge\varphi(a',b) \leq \bigvee_{x\in a-a'} \rho_\varphi(0)(x)$ and by (1), one obtains the inequality required.
Now, we check that $\bigvee_{x\in a-a'} \varphi(x,0) \leq \rho_\varphi(a,a')$. It suffices to prove that $\rho_\varphi(a,a')$ is an upper bound.

$$\rho_\varphi(0)(x) = \rho_\varphi(x,0) = \rho_\varphi(x,0)\wedge\rho_\varphi(a',a')$$
$$\leq \widehat{\rho}_\varphi(x+a',a') = \bigwedge_{z\in x+a'} \rho_\varphi(z,a') \leq \rho_\varphi(a,a') \qquad \square$$

6 Conclusions

In this paper, we have introduced a suitable notion of fuzzy homomorphism between hyperrings and have studied the results and isomorphism theorems which relate fuzzy homomorphisms between hyperrings, fuzzy congruences and fuzzy hyperideals.

References

[1] M. Benado. Les ensembles partiellement ordonnés et le théorème de raffinement de Schreier. I. *Čehoslovack. Mat. Ž.*, 4(79):105–129, 1954.

[2] I. P. Cabrera, P. Cordero, G. Gutiérrez, J. Martínez, and M. Ojeda-Aciego. Fuzzy congruence relations on nd-groupoids. *Intl J on Computer Mathematics* 86:1684-1695, 2009.

[3] M. Ćirić, J. Ignjatović, and S. Bogdanović. Uniform fuzzy relations and fuzzy functions. *Fuzzy Sets and Systems*, 160(8):1054–1081, 2009.

[4] P. Corsini. A new connection between hypergroups and fuzzy sets. *Southeast Asian Bull. Math.*, 27(2):221–229, 2003.

[5] B. Davvaz. Some results on congruences on semihypergroups. *Bulletin of the Malaysian Mathematical Sciences Society. Second Series*, 23(1):53–58, 2000.

[6] B. Davvaz. Isomorphism theorems of hyperrings. *Indian J. Pure Applied Mathematics*, 35(3):321–331, 2004.

[7] M. Demirci. Fuzzy functions and their applications. *J. Math. Anal. Appl*, 252:495–517, 2000.

[8] M. Demirci. Gradation of being fuzzy function. *Fuzzy Sets and Systems*, 119(3):383–392, 2001.

[9] M. Demirci. Constructions of fuzzy functions based on fuzzy equalities. *Soft Computing*, 7(3):199–207, 2003.

[10] G. Gentile. Multiendomorphisms of hypergroupoids. *International Mathematical Forum*, 1(9-12):563–576, 2006.

[11] F. Klawonn. Fuzzy points, fuzzy relations and fuzzy function. In V. Novák and I. Perfilieva, editors, *Discovering World with Fuzzy Logic*, pages 431–453. Physica-Verlag, 2000.

[12] M. Krasner. A class of hyperrings and hyperfields. *International Journal of Mathematics and Mathematical Sciences*, 6(2):307–311, 1983.

[13] F. Marty. Sur une généralisation de la notion de groupe. In *8th Congress Math. Scandinaves*, pages 45–49, 1934.

[14] J. Medina and M. Ojeda-Aciego and J. Ruiz-Calviño. Fuzzy logic programming via multilattices. *Fuzzy Sets and Systems.* N. 158, Vol. 6, Pág. 674-688, 2007.

[15] L. A. Zadeh. Fuzzy sets. *Information and Control*, 8:338–353, 1965.

[16] J. Zhan, B. Davvaz, and K. P. Shum. A new view of fuzzy hypernear-rings. *Inf. Sci.*, 178(2):425–438, 2008.