

CONGRUENCE RELATIONS ON HYPERGROUPOIDS AND ND-GROUPOIDS

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Abstract

In this work we prove that the set of congruences on an nd-groupoid under suitable conditions is a complete lattice which is a sublattice of the lattice of equivalence relations on the nd-groupoid. The study of these conditions allowed to construct a counterexample to the statement that the set of (fuzzy) congruences on a hypergroupoid is a complete lattice.

Keywords: Hypergroupoid, nd-groupoid, congruence, L -fuzzy sets.

1 Introduction

This paper follows the current trend of providing suitable fuzzifications of crisp concepts, as a theoretical tool to the development of new method of reasoning under uncertainty, imprecision and lack of information. Moreover, although originally fuzzy sets were presented as mappings with codomain $[0, 1]$, the unit interval was soon replaced by more general structures, for instance a complete lattice, as in the L -fuzzy sets introduced by Goguen [7].

This paper follows one of our research lines which is aimed at investigating L -fuzzy sets where L has the structure of a multilattice [3]. Roughly speaking, a multilattice is an algebraic structure in which the restrictions imposed on a lattice, namely, the “existence of least upper bounds and greatest lower bounds” are relaxed to the “existence of minimal upper bounds and maximal lower bounds”.

Recently, Cordero *et al.* [9, 10] proposed an alternative algebraic definition of multilattice which is more closely related to that of lattice, allowing for natural

definitions of related structures such that multiseimilattices and, in addition, is better suited for applications. For instance, Medina *et al.* [12] developed a general approach to fuzzy logic programming in which the authors use a multilattice as underlying set of truth-values for the logic.

Attending to the description given above, the main difference that one notices when working with multilattices is that the operators which compute suprema and infima are no longer single-valued, since there may be several multi-suprema or multi-infima, or may be none. This immediately leads to the theory of hyperstructures, that is, set-valued operations.

Hyperstructure theory was introduced in [11] when Marty defined hypergroups, began to analyse their properties and applied them to groups, rational fractions and algebraic functions. Nowadays, a number of different hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics.

In this paper, we will focus on the most general hyperstructures, namely hypergroupoid and nd-groupoid. In a nutshell, let us recall that a hypergroupoid is simply a binary operator $H \times H \rightarrow \mathcal{P}(H) \setminus \{\emptyset\}$, whereas a non-deterministic groupoid (nd-groupoid, for short) is a binary operator $H \times H \rightarrow \mathcal{P}(H)$, that is, a hypergroupoid in which the restriction of the codomain being the non-empty subsets is dropped.

Our interest in nd-groupoids and hypergroupoids arises from the fact that, in a multilattice, the operators which compute the multi-suprema and multi-infima are precisely nd-groupoids or, if we have for granted that at least a multi-supremum always exists, a hypergroupoid. Most of the results will be stated mainly in terms of multiseimilattices.

Several papers have been published on the lattice of fuzzy congruences on different algebraic structures [1, 5, 6, 14, 15], and in [4] we initiated research in this

direction. Specifically, we focused on the theory of congruences on a multilattice, as this is a necessary step prior to considering the multilattice-based generalization of the concept of L -fuzzy congruence. In this paper, we consider the particular case of the congruences on an nd-groupoid.

The fact that the structure of nd-groupoid is simpler than that of a multilattice does not necessarily mean that the theory is simpler as well. Specifically, the set of congruences on an nd-groupoid is not a lattice unless we assume some extra properties. This problem led us to review some related literature and, as a result, we found one counter-example in the context of congruences on a hypergroupoid.

2 Preliminaries

The concept of multilattice [3] was introduced by Benado in 1954, and is an extension of the concept of lattice by means the so-called multi-suprema and multi-infima, which are formally defined below:

Definition 2.1 Given (M, \leq) a partially ordered set (poset, hereafter) and $B \subseteq M$, a **multi-supremum of B** is a minimal element of the set of upper bounds of B and $\text{Multi-sup}(B)$ denote the set of all the multi-suprema of B . Dually, we define the **multi-infima**.

Now, we introduce the definition of multilattice.

Definition 2.2 A poset (M, \leq) is said to be a **multi-semilattice** if it satisfies that, for all $a, b, x \in M$ with $a \leq x$ and $b \leq x$, there exists $z \in \text{Multi-sup}(\{a, b\})$ such that $z \leq x$.

Similarly to what happens in the theory of lattices, a poset (M, \leq) is said to be a **multilattice** if it is a multilattice and also its dual (M, \geq) .

Note that the definition is consistent with the existence of two incomparable elements *without* any multisupremum. In other words, $\text{Multi-sup}(\{a, b\})$ can be empty.

Let us move now to the context of congruences on different algebraic structures.

It is well-known that given a lattice L , the set of all congruence relations on L , partially ordered by set inclusion, is a complete lattice [8]. In the recent literature, several authors have presented different approaches to fuzzy congruence relations on some algebraic structures [5, 6, 13, 15]. The most general frame which almost includes multilattices seems to be that of a hypergroupoid. In [2], the notion of fuzzy congruence relation on hypergroupoid is introduced.

Note that we will assume the following notational convention: If $X, Y \subseteq H$ then $X \equiv Y$ denotes that, for

all $x \in X$, there exists $y \in Y$ such that $x \equiv y$ and for all $y \in Y$ there exists $x \in X$ such that $x \equiv y$.

Definition 2.3 Let (H, \cdot) be a hypergroupoid, that is, a mapping $\cdot : H \times H \rightarrow \mathcal{P}(H) \setminus \{\emptyset\}$. A **congruence relation on H** is an equivalence relation \equiv such that for all $a, b, c \in H$, if $a \equiv b$, then $ac \equiv bc$.

Definition 2.4 (Zadeh, [16]) Let H be a nonempty set. A **fuzzy relation ρ on H** is a fuzzy subset of $H \times H$ (i.e. ρ is a function from $H \times H$ to $[0, 1]$). Now, ρ is said to be a **fuzzy equivalence relation** if

1. $\rho(x, x) = \sup_{y, z \in H} \rho(y, z)$ (fuzzy reflexivity),
2. $\rho(y, x) = \rho(x, y)$ (fuzzy symmetry),
3. $\rho(x, y) \geq \sup_{z \in H} \min(\rho(x, z), \rho(z, y))$ (fuzzy transitivity).

Definition 2.5 (Bakhshi and Borzooei [2]) Let (H, \cdot) be a hypergroupoid and ρ a fuzzy relation on H .

1. ρ is **fuzzy left (right) compatible** if for all $u \in ax$ ($u \in xa$) there exists $v \in ay$ ($v \in ya$) and for all $v \in ay$ ($v \in ya$) there exists $u \in ax$ ($u \in xa$) such that $\rho(u, v) \geq \rho(x, y)$, for all $x, y, a \in H$ and **fuzzy compatible** if it is both fuzzy left and right compatible.
2. ρ is a **fuzzy (left, right) congruence relation on H** if it is a fuzzy (left, right) compatible equivalence relation on H .

Under the additional assumption of commutativity with respect to the usual composition of binary relations, Bakhshi and Borzooei [2], apparently proved that the set of all fuzzy congruence relations on an hypergroupoid (H, \cdot) is a complete lattice where, in particular, the infimum of two congruences is its intersection. The following example proves that, without additional hypotheses, the statement is false even in a crisp setting.

Example 2.1 Let H be the set $\{a, b, c, u_0, u_1, v_0, v_1\}$ provided with a commutative hyperoperation $*$ which is defined as follows:

$$a * a = a * b = b * b = \{a, b\}; \quad a * c = \{u_0, u_1\};$$

$$b * c = \{v_0, v_1\} \quad \text{and } x * y = \{c\}, \quad \text{elsewhere}$$

Consider $R, S : H \times H \rightarrow \{0, 1\}$ two binary relations, where R is the least equivalence relation containing $\{(a, b), (u_0, v_0), (u_1, v_1)\}$ and S the least equivalence relation containing $\{(a, b), (u_0, v_1), (u_1, v_0)\}$. A simple (but tedious) check shows that $R \circ S = S \circ R$ and they are both compatible with the hyperoperation

*, therefore, both R and S are congruence relations. However, one can check that the intersection $R \cap S$ is not a congruence relation.

The previous example motivated the search for a sufficient condition which granted the structure of complete lattice for the set of congruences on a hypergroupoid and, by extension, on an nd-groupoid.

The idea underlying this missing property comes from a previous work by the authors [4], in which we studied congruence relations on a multilattice M as a natural inception for the fuzzy case. In that paper, we proved that the set of equivalence relations compatible with both operations of a multilattice forms a complete lattice, under a suitable form of distributivity.

In the following section we extend the concept of hypergroupoid by formally introducing and studying the properties of nd-groupoids, together with a suitable generalization of the property of distributivity, albeit in a framework with just one operation!!

3 Congruence relations in nd-groupoids

We are specially interested in a generalization of hypergroupoid that we will call non deterministic groupoid (nd-groupoid, for short).

We consider a set A endowed with an **nd-operation**, $\cdot : A \times A \rightarrow \mathcal{P}(A)$ and thus the pair (A, \cdot) is called **nd-groupoid**. Notice that the definition allows the assignment of the empty set to a pair of elements, that is $a \cdot b = \emptyset$, this mere fact, albeit simple, represents an important difference with hypergroupoids.

Notation: As usual, we will use the notational conventions:

- If $a \in A$ and $X \subseteq A$ $aX = \{ax \mid x \in X\}$ and $Xa = \{xa \mid x \in X\}$. In particular, $a\emptyset = \emptyset a = \emptyset$
- We will use multiplicative notation and, thus, the dot is omitted.
- When the result of the nd-operation is a singleton, we will often omit the braces.

As stated in the introduction, our interest in extending the concept of hypergroupoid is justified by the algebraic characterization of multilattices and multi-semilattices, since the operators for multi-suprema and multi-infima are both examples of nd-groupoids.

With this idea in mind, we introduce below the extension to the framework of nd-groupoids of some well-known properties. Assume that (A, \cdot) is an nd-groupoid:

- **Idempotency:** $aa = a$ for all $a \in A$.
- **Commutativity:** $ab = ba$ for all $a, b \in A$.
- **Left m-associativity:** $(ab)c \subseteq a(bc)$ when $ab = b$, for all $a, b, c \in A$.
- **Right m-associativity:** $a(bc) \subseteq (ab)c$ when $bc = c$, for all $a, b, c \in A$.
- **m-associativity:** if it is left and right m-associative.

Note that the prefix ‘m’ has its origin in the concept of multilattice.

We will focus our interest in the binary relation usually named *natural ordering*, which is defined by

$$a \leq b \text{ if and only if } ab = b$$

In general, this relation is not an ordering, but we have sufficient conditions which guarantee the properties of an ordering. Specifically, it is reflexive if the nd-groupoid is idempotent, the relation is antisymmetric if the nd-groupoid is commutative and, finally, it is transitive if the nd-groupoid is m-associative.

The two following properties of nd-groupoids have an important role in multilattice theory:

- C_1 : $c \in ab$ implies that $a \leq c$ and $b \leq c$.
- C_2 : $c, d \in ab$ and $c \leq d$ imply that $c = d$.

These two properties are named **comparability**. Similarly to lattice theory, we can define algebraically the concept of multi-semilattice as an nd-groupoid that satisfies idempotency, commutativity, m-associativity and comparability laws. Both definitions of multi-semilattice can be proved to be equivalent considering $a \cdot b = \text{Multi-sup}\{a, b\}$ and \leq the natural ordering (see [10, Theorem 2.11]).

In the following, we will consider congruence relations on an nd-groupoid. This concept is defined in the same manner as in Definition 2.3.

The following result is an immediate consequence from the definition.

Lemma 3.1 *Let (A, \cdot) be an idempotent nd-groupoid and \equiv be a congruence relation. If $a \equiv b$ then $\emptyset \neq ab \equiv a$.*

Theorem 3.2 *Let (A, \cdot) be an nd-groupoid satisfying idempotency and property C_1 . An equivalence relation \equiv is a congruence if and only if the following holds:*

$$\forall a, b, c \in A, \text{ if } a \leq b \text{ and } a \equiv b, \text{ then } ac \equiv bc.$$

Proof The necessity is obvious, thus we will just prove the sufficiency.

If $a \equiv b$ then, by Lemma 3.1, there exists $z \in ab$ such that $a \equiv z \equiv b$. Property C_1 ensures that $a \leq z$ and $b \leq z$ and then, by the condition, $ac \equiv zc \equiv bc$. \square

In the rest of the paper, we focus on the search of properties that ensure the condition of the previous theorem.

Proposition 3.3 *Let (A, \cdot) be an m -associative nd -groupoid that satisfies C_1 and, for $a, b, c \in A$, consider $a \leq b$ and $z \in bc$:*

1. *There exists $w \in ac$ such that $w \leq z$.*
2. *Furthermore, if (A, \cdot) is commutative and C_2 holds, then $a \equiv b$ implies that every element w as in the previous item satisfies that $w \equiv z$.*

Proof

1. By hypothesis $a \leq b$ and, by C_1 , since $z \in bc$, we obtain $b \leq z$. Therefore $a \leq z$ because, by m -associativity of the nd -operation \cdot , the relation \leq is transitive.

Now, again by C_1 , since $z \in bc$, $c \leq z$ and, by m -associativity, $z = az = a(cz) \subseteq (ac)z$. In particular, we have that $z \in (ac)z$, this implies the existence of $w \in ac$ such that $z = wz$, that is, $w \leq z$.

2. Consider $w \in ac$ such that $wz = z$ (i.e., $w \leq z$). By C_1 we have that $a \leq w$ and $w = aw \equiv bw$. Now, it is sufficient to prove that $z \in bw$.

Since $b \leq z$ and $w \leq z$, by m -associativity, $z = bz = b(wz) \subseteq (bw)z$, that is, $z \in (bw)z$. Consider $z' \in bw$ such that $z' \leq z$. By C_1 , $b \leq z'$ and $c \leq w \leq z'$. Once again, m -associativity ensures that there exists $z'' \in bc$ such that $z'' \leq z'$ and therefore $z'' \leq z$. Now, by C_2 , $z'' = z$. From $z \leq z'$ and $z' \leq z$, by commutativity, the relation \leq is antisymmetric and, hence, $z = z' \in bw$. \square

In order to prove the converse result, we need to introduce the following definition.

Definition 3.4 *An nd -operation \cdot in a set A is said to be **m -distributive** when, for all $a, b, c \in A$, if $a \leq b$ and $w \in ac$, then $bw \cap bc \neq \emptyset$.*

The justification of this name is that a multilattice (A, \vee, \wedge) in which both operations are m -distributive satisfies the following property: for all $a, b \in A$ with $a \leq b$ and $c \in A$:

1. $(a \wedge b) \vee c \subseteq (a \vee c) \wedge (b \vee c)$
2. $(a \vee b) \wedge c \subseteq (a \wedge c) \vee (b \wedge c)$

In fact, the two latter conditions are equivalent to the m -distributivity of \wedge and \vee , as the following result shows.

Proposition 3.5 *Let (A, \vee, \wedge) be a multilattice and $a, b, c \in A$. The following conditions are equivalent:*

1. *If $a \leq b$ and $w \in a \vee c$, then $b \vee w \cap b \vee c \neq \emptyset$.*
2. *If $a \leq b$, then $(a \wedge b) \vee c \subseteq (a \vee c) \wedge (b \vee c)$.*

Proof

- (1 \Rightarrow 2) If $w \in (a \wedge b) \vee c = a \vee c$ then, by hypothesis 1, there exists $u \in b \vee w \cap b \vee c$. Since $u \in b \vee w$, by property C_1 , we have $w \leq u$ and, therefore, $w = w \wedge u \subseteq (a \vee c) \wedge (b \vee c)$.
- (2 \Rightarrow 1) Conversely, assume $w \in a \vee c$, since $a \leq b$, we have $a = a \wedge b$ and, by hypothesis 2 we have $w \in (a \wedge b) \vee c \subseteq (a \vee c) \wedge (b \vee c)$. Thus, there exists an element $u \in a \vee c$ and $v \in b \vee c$ such that $w \in u \wedge v$. Moreover,¹ by C_1 , $w \leq v$ and $b \leq v$. By m -associativity we have now that $v = b \vee v = b \vee (w \vee v) \subseteq (b \vee w) \vee v$. This means that there exists $v' \in b \vee w$ such that $v = v' \vee v$ (that is, $v' \leq v$). Finally, we will prove that $v \leq v'$ and thus $v = v' \in b \vee w \cap b \vee c$ as desired.

By C_1 we have that $b \leq v'$ and $c \leq w \leq v'$ which, by transitivity, implies $c \leq v'$. This means that v' is an upper bound of b and c and, by definition of multilattice, there should exist $v'' \in b \vee c$ such that $v'' \leq v'$. But we already know that $v' \leq v$, hence $v'' \leq v' \leq v$, but as v and v'' are multi-suprema, the only consistent possibility is that $v = v' = v''$. \square

Proposition 3.6 *Let (M, \cdot) be an m -distributive nd -groupoid that satisfies C_1 and $a, b, c \in M$. If $a \leq b$ and $w \in ac$ then there exists $z \in bc$ such that $w \leq z$.*

Proof By m -distributivity, from $a \leq b$ and $w \in ac$, we obtain that there exists $z \in bw \cap bc$. Now, by C_1 $w \leq z$. \square

Notice that the properties required as hypotheses of Proposition 3.6 and Proposition 3.3 are those of a multisemilattice without idempotency. The following result, stated in terms of a multisemilattice, is a straightforward consequence of these two propositions.

¹Recall that, as stated in the definition of multilattice, the order used considered for \wedge is \geq , this explains the following use of property C_1 .

Proposition 3.7 *Let (M, \cdot) be an m -distributive multiset lattice, \equiv be a congruence relation and $a, b, c \in M$. If $a \leq b$, $a \equiv b$, $w \in ac$ and $z \in bc$ with $w \leq z$ then $w \equiv z$.*

Now, we have all the required properties and lemmas needed in order to face the main goal of this paper, namely, to prove that the set of congruences of an nd-groupoid is a complete lattice.

It is well-known that, for every set A , the set of equivalence relations on A , $Eq(A)$, with the inclusion ordering (in the powerset of $A \times A$) is a complete lattice in which the infimum is the meet and the supremum is the transitive closure of the join. This generalization of distributivity will be proved to be a sufficient condition for the set of congruence relations being a complete lattice.

Theorem 3.8 *The set of the congruence relations in an m -distributive multiset lattice M , $Con(M)$, is a sublattice of $Eq(M)$ and, moreover is a complete lattice wrt the inclusion ordering.*

Proof Let $\{\equiv_i\}_{i \in \Lambda}$ be a set of congruence relations in M , consider \equiv_{\cap} to be their intersection.

From Theorem 3.2 we have just to check that, for all $a, b, c \in M$, $a \leq b$ and $a \equiv_{\cap} b$ imply that $ac \equiv_{\cap} bc$.

From Proposition 3.3, if $z \in bc$ then there exists $w \in ac$ such that $w \leq z$ and, for all $w \in ac$ with $w \leq z$ and all $i \in \Lambda$, $w \equiv_i z$ (so $w \equiv_{\cap} z$).

Conversely, from Proposition 3.6 and Proposition 3.7, if $w \in ac$ then there exists $z \in bc$ such that $w \leq z$ and, for all $z \in bc$ with $w \leq z$ and all $i \in \Lambda$, $w \equiv_i z$ (so $w \equiv_{\cap} z$).

The proof for the transitive closure of union follows by a routine calculation. \square

4 Conclusions and future work

We have proved that the set of congruence relations on an nd-groupoid is a complete lattice which is a sublattice of the lattice its equivalence relations. The main step in the proof of this statement is the introduction of the concept of m -distributivity, which allowed for translating the ideas used in the context of congruences on multilattices.

As future work on this research line, our plan is to keep investigating new or analogue results concerning congruences on generalized algebraic structures, specially in a non-deterministic sense. Then, continuing towards our main aim, studying the computational properties of multilattices as a suitable algebraic structure on which found an extended theory of fuzzy struc-

tures, we will try to lift our results from a crisp to a fuzzy setting.

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