Multi-adjoint concept lattices from a non-commutative perspective

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Resumen

This paper shows that the recently introduced framework of multi-adjoint concept lattices naturally embeds the generalization of fuzzy concept lattices under the assumption of non-commutative conjunctors developed by Georgescu and Popescu.

Key words: Concept lattices, multi-adjoint lattice, Galois connection.

1 Introduction

When working with methods which use uncertainty, imprecise data or incomplete information one of the main tools is the formal concept analysis; the classical approach in this context is that introduced by Ganter and Wille [5]. After this, there has been many approaches trying to generalize it, like the fuzzy concept lattices presented by Burusco and Fuentes-González [3] and later developed by Pollandt [13] which are based on the lattice $[0,1]$. Other approaches emerge trying to work with non-commutative fuzzy logic and similarity in the work of Georgescu and Popescu [6]. Bělohlávek in [2] proposed a generalization of the equality relation and similarity relation inside the fuzzy concept analysis which he called L-equalities. This last approach was extended in the case of the classical equality ($L = \{0, 1\}$), by Krajči [7, 8] introducing the generalized concept lattices.

Recently, a new approach has been proposed by Medina et al in [9, 12] who introduced the multi-adjoint concept lattices, joining the multi-adjoint lattices with concept lattices. To do this the authors needed to generalize the adjoint pairs into what they called adjoint triples. This new structure directly generalizes almost all the approaches previously cited, but the one of Georgescu and Popescu.

In this paper we show that Georgescu and Popescu concept lattices can be seen as a complete sublattice of the product of two suitable multi-adjoint concept lattices, under this embedding we can prove the corresponding representation theorem by the representation theorem for multi-adjoint concept lattice, which makes the proof easier.

The plan of this paper is the following: In Section 2 we make a brief summary of the basic notions used in formal concept analysis together with a short reminder of the multi-adjoint concept lattices. Moreover, in this section, we also give several properties of the mappings involved in the fundamental theorem of multi-adjoint lattices. This result let us in Section 3, after a introduction of their approach, prove the representation theorem of this framework. The paper ends with some conclusions and some possible vias of future work.

2 Multi-adjoint concept lattice

A basic notion in formal concept analysis is that of Galois connection, since each Galois connection has an associated complete lattice, called *Galois lattice* or *concept lattice*.

Definition 1 *Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be posets, a pair $(^\uparrow, ^\downarrow)$ of mappings $^\downarrow\colon P_1 \to P_2$, $^\uparrow\colon P_2 \to P_1$ forms a* Galois connection *between P_1 and P_2 if and only if:*

1. $^\uparrow$ and $^\downarrow$ are decreasing.

2. $x \leq_1 x^{\downarrow\uparrow}$ for all $x \in P_1$.

3. $y \leq_2 y^{\uparrow\downarrow}$ for all $y \in P_2$.

If P_1 and P_2 are complete lattices then the following theorem can be established, see [4]:

Theorem 1 *Let $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ be complete lattices, $(^\uparrow, ^\downarrow)$ a Galois connection between L_1, L_2 and $\mathcal{C} = \{\langle x, y\rangle \mid x^\uparrow = y, x = y^\downarrow; x \in L_1, y \in L_2\}$ then $\mathcal{C}$ is a complete lattice, where*

$$\bigwedge_{i \in I}\langle x_i, y_i\rangle \;=\; \langle \bigwedge_{i \in I} x_i, (\bigvee_{i \in I} y_i)^{\downarrow\uparrow}\rangle$$

$$\bigvee_{i \in I} \langle x_i, y_i \rangle \;=\; \langle (\bigvee_{i \in I} x_i)^{\uparrow\downarrow}, \bigwedge_{i \in I} y_i \rangle$$

In the rest of the section, a generalization of multi-adjoint lattices is introduced in order to admit different sorts, in which we allow non-commutative conjunctors as in [1, 6, 10]. To begin with, the adjoint pairs are generalized to adjoint triples, the basic blocks of multi-adjoint concept lattices, as follows:

Definition 2 *Let* $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ *be posets and* $\&\colon P_1 \times P_2 \longrightarrow P_3$, $\swarrow\colon P_3 \times P_2 \longrightarrow P_1$, $\nwarrow\colon P_3 \times P_1 \longrightarrow P_2$ *be applications, then* $(\&, \swarrow, \nwarrow)$, *is an* adjoint triple *with respect to* P_1, P_2, P_3 *if:*

- *$\&$ is increasing in both arguments.*

- *$\swarrow$ and $\nwarrow$ are increasing in the first argument and decreasing in the second.*

- *$x \leq_1 z \swarrow y$ iff $x \& y \leq_3 z$ iff $y \leq_2 z \nwarrow x$, where $x \in P_1$, $y \in P_2$ and $z \in P_3$.*

This last property is known as *adjoint property* and generalises the *modus ponens* rule in a non-commutative multi-valued setting. Notice that no boundary condition is required, in difference to the usual definition of multi-adjoint lattice [11] or implication triples [1].

An interesting result we can extract from the adjoint property and that we will use later, is the following lemma:

Lemma 1 *If* P_1, P_2, P_3 *have bottom element,* P_1, P_2 *have top element and* $(\&, \swarrow, \nwarrow)$ *is an adjoint triple, then for all* $x \in P_1, y \in P_2$ *and* $z \in P_3$ *the following properties hold:*

1. *$\perp_1 \& y = \perp_3$; $x \& \perp_2 = \perp_3$.*

2. *$z \nwarrow \perp_1 = \top_2$; $z \swarrow \perp_2 = \top_1$.*

Proof. The proof is straightforward from the adjoint property. □

In order to introduce a Galois connection generalizing that given in the classical concept lattices framework, the consideration of several adjoint triples leads to the following definition of multi-adjoint frame.

Definition 3 *A* multi-adjoint frame $\mathcal{L}$ *is a tuple*

$$(L_1, L_2, P, \preceq_1, \preceq_2, \leq, \&_1, \swarrow^1, \nwarrow_1, \ldots, \&_n, \swarrow^n, \nwarrow_n)$$

where L_i are complete lattices and P is a poset, and such that $(\&_i, \swarrow^i, \nwarrow_i)$ is an adjoint triple with respect to L_1, L_2, P for all $i = 1, \ldots, n$.

A multi-adjoint frame as above will be denoted as $(L_1, L_2, P, \&_1, \ldots, \&_n)$, for short. It is convenient to note

that, in principle, L_1, L_2 and P could be simply posets, the reason to consider complete lattices is that multi-adjoint frames will be used as the underlying lattice on which the operations will be made; hence, general joins and meets are required.

Definition 4 *Given a frame* $(L_1, L_2, P, \&_1, \ldots, \&_n)$, *a* context *is a tuple* (A, B, R, σ) *such that A and B are non-empty sets, R is a P-fuzzy relation $R\colon A \times B \to P$ and $\sigma\colon B \to \{1, \ldots, n\}$ is a mapping which associates any object in B with some particular adjoint triple in the frame.*

Following the usual terminology, A is to be considered as a set of attributes and B as a set of objects.

The fact that in a multi-adjoint context each object (or attribute) has an associated implication is interesting, in that subgroups with different degrees of preference can be established in a convenient way; however, a complete study of this possibility is outside the scope of this paper.

Now, given a frame and a context for that frame, the following mappings $\uparrow_\sigma\colon L_2^B \to L_1^A$ and $\downarrow^\sigma\colon L_1^A \to L_2^B$ can be defined:

$$\begin{aligned} g^{\uparrow_\sigma}(a) &= \inf\{R(a,b) \swarrow^{\sigma(b)} g(b) \mid b \in B\} \\ f^{\downarrow^\sigma}(b) &= \inf\{R(a,b) \nwarrow_{\sigma(b)} f(a) \mid a \in A\} \end{aligned}$$

Notice that these mappings generalise those given in [3, 8] and, as it is proved below, generate a Galois connection.

Proposition 1 *Given a frame* $(L_1, L_2, P, \&_1, \ldots, \&_n)$ *and a context* (A, B, R, σ), *the pair* $(\uparrow_\sigma, \downarrow^\sigma)$ *is a Galois connection between L_1^A and L_2^B.*

Now, a *concept* is a pair $\langle g, f \rangle$ satisfying that $g \in L_2^B$, $f \in L_1^A$ and that $g^\uparrow = f$ and $f^\downarrow = g$; with $(\uparrow, \downarrow)$ being the Galois connection defined above.

Definition 5 *The* multi-adjoint concept lattice *associated to a multi-adjoint frame* $(L_1, L_2, P, \&_1, \ldots, \&_n)$ *and a context* (A, B, R, σ) *is the set*

$$\mathcal{M} = \{\langle g, f \rangle \mid g \in L_2^B, f \in L_1^A \text{ and } g^\uparrow = f, f^\downarrow = g\}$$

with the ordering $\langle g_1, f_1 \rangle \preceq \langle g_2, f_2 \rangle$ *iff* $g_1 \preceq g_2$ *(equivalently $f_1 \preceq f_2$).*

Note that, by Theorem 1, the poset $(\mathcal{M}, \preceq)$ defined above is a complete lattice, since the arrows $(\uparrow, \downarrow)$ form a Galois connection between the complete lattices L_1^A and L_2^B.

From now on, given a multi-adjoint concept lattice, $(\mathcal{M}, \preceq)$, $(L_1, L_2, P, \&_1, \ldots, \&_n)$ will be its frame and (A, B, R, σ) its context associated.

The representation theorem for multi-adjoint concepts presented later, is similar to those given in the different theories of concept lattices. To begin with, we need to introduce some definitions and preliminary results.

Definition 6 *Given a set A, a poset P with bottom element $\perp$, and elements $a \in A$, $x \in P$, the characteristic mapping $@_a^x : A \to P$ is defined as:*

$$@_a^x(a') = \begin{cases} x, & \text{if } a' = a \\ \perp, & \text{otherwise} \end{cases}$$

The following lemma gives a technical property which will be needed later.

Lemma 2 *In the concept lattice $(\mathcal{M}, \preceq)$, given $a \in A$, $b \in B$, $x \in L_1$ and $y \in L_2$, the following equalities hold:*

$$@_a^{x\downarrow}(b') = R(a,b') \nwarrow_{\sigma(b')} x \quad \text{for all} \quad b' \in B$$

$$@_b^{y\uparrow}(a') = R(a',b) \nearrow^{\sigma(b)} y \quad \text{for all} \quad a' \in A$$

The following definitions introduce properties which will be used in the statement of Proposition 2.

Definition 7 *Given a complete lattice L, a subset $K \subseteq L$ is infimum-dense (resp. supremum-dense) if and only if for all $x \in L$ there exists $K' \subseteq K$ such that $x = \inf(K')$ (resp. $x = \sup(K')$).*

Definition 8 *Let $(\mathcal{M}, \preceq)$ be a multi-adjoint concept lattice, $(V, \sqsubseteq)$ a lattice and $\alpha : A \times L_1 \to V$, $\beta : B \times L_2 \to V$ two maps. We say that β is (V, R)-related with α if we have that:*

1a) $\alpha[A \times L_1]$ is infimum-dense;

1b) $\beta[B \times L_2]$ is supremum-dense; and

2) for each $a \in A$, $b \in B$, $x \in L_1$ and $y \in L_2$:

$$\beta(b,y) \sqsubseteq \alpha(a,x) \quad \text{iff} \quad x \&_{\sigma(b)} y \le R(a,b)$$

Proposition 2 *Given a multi-adjoint concept lattice $(\mathcal{M}, \preceq)$, a complete lattice $(V, \sqsubseteq)$ and a mapping $f \in L_1^{B}$, if there exist two applications $\beta : B \times L_2 \to V$, $\alpha : A \times L_1 \to V$, where β is (V, R)-related with α we have that:*

1. *$f^\downarrow(b) = \sup\{y \in L_2 \mid \beta(b,y) \sqsubseteq v_f\}$, where v_f denotes $\inf\{\alpha(a, f(a)) \mid a \in A\}$.*

2. *If $g_v(b)$ denotes $\sup\{y \in L_2 \mid \beta(b,y) \sqsubseteq v\}$, then $\sup\{\beta(b, g_v(b)) \mid b \in B\} = v$.*

Lemma 3 *Let $(V, \sqsubseteq)$ be a complete lattice, $(\mathcal{M}, \preceq)$ a multi-adjoint concept lattice, and applications $\alpha : A \times L_1 \to V$ and $\beta : B \times L_2 \to V$ such that β is (V, R)-related to α. Then the following mapping $\varphi : \mathcal{M} \to V$:*

$$\begin{aligned} \varphi(\langle g, f \rangle) &= \sup\{\beta(b, g(b)) \mid b \in B\} \\ &= \inf\{\alpha(a, f(a)) \mid a \in A\} \end{aligned}$$

is an isomorphism.

We can now state the representation theorem for multi-adjoint concept lattices.

Theorem 2 *Given a complete lattice $(V, \sqsubseteq)$ and a multi-adjoint concept lattice $(\mathcal{M}, \preceq)$, we have that V is isomorphic to $\mathcal{M}$ if and only if there exist applications $\alpha : A \times L_1 \to V$, $\beta : B \times L_2 \to V$ such that β is (V, R)-related to α.*

Now, we introduce some interesting properties that can be applied to the kind of maps α and β given in Theorem 2. The first one shows that the hypothesis that α is decreasing and β is increasing assumed in Krajči's basic theorem on generalized concept lattices [7] can be omitted.

Proposition 3 *Let $\langle P, \le \rangle$ be a poset, $\langle L_1, \preceq_1 \rangle$, $\langle L_2, \preceq_2 \rangle$, $\langle V, \preceq_V \rangle$ be complete lattices and a relation $R : A \times B \to P$; if there exist two applications $\beta : B \times L_2 \to V$, $\alpha : A \times L_1 \to V$, where β is (V, R)-related with α we have that:*

1. *For all indexed set $\{y_j\}_{j \in J} \subseteq L_2$ and $b \in B$, β satisfies:*

$$\beta(b, \sup\{y_j \mid j \in J\}) = \sup\{\beta(b, y_j) \mid j \in J\}$$

2. *For all indexed set $\{x_j\}_{j \in J} \subseteq L_1$ and $a \in A$, α satisfies:*

$$\alpha(a, \sup\{x_j \mid j \in J\}) = \inf\{\alpha(a, x_j) \mid j \in J\}$$

Corollary 1 *With the hypotheses of the proposition above, we have that β is increasing in the second argument and α is decreasing in the second argument.*

The next result shows some boundary properties of α and β functions.

Proposition 4 *If β is (V, R)-related with α we have that $\beta(b, \perp_2) = \perp_V$ and $\alpha(a, \perp_1) = \top_V$ for all $b \in B$ and $a \in A$.*

Proof. Given $a \in A$ let us prove that $\alpha(a, \perp_1) = \top_V$. As, by Lemma 1, $\perp_1 \& y \le R(a,b)$ for all $b \in B$ and $y \in L_2$, from Property 2 we obtain that $\beta(b,y) \sqsubseteq \alpha(a, \perp_1)$ for all $b \in B$ and $y \in L_2$.

On the other hand, as β is supremum-dense, there exists a subset of indices Λ such that $\top_V = \sup\{\beta(b_i, y_i) \mid i \in \Lambda\}$, therefore, from the comment above and the supremum property we have that:

$$\top_V = \sup\{\beta(b_i, y_i) \mid i \in \Lambda\} \sqsubseteq \alpha(a, \perp_1)$$

Hence, $\top_V = \alpha(a, \perp_1)$.

The other equality follows similarly. $\square$

This result can be used to prove that any subset of $A \times L_1$ is related to a concept; moreover, it is used in next section in order to generalize the framework introduced in [6].

Proposition 5 *Given a multi-adjoint concept lattice* $(\mathcal{M}, \preceq)$, $v \in V$, $\alpha: A \times L_1 \to V$, $\beta: B \times L_2 \to V$ *such that β is (V, R)-related with α we have that for each $K \subseteq A \times L_1$ there exists a unique concept $\langle g, f \rangle \in \mathcal{M}$ such that*

$$\inf\{\alpha(a,x) \mid (a,x) \in K\} = \sup\{\beta(b, g(b)) \mid b \in B\}$$
$$= \inf\{\alpha(a, f(a)) \mid a \in A\}$$

Proof. Given $K \subseteq A \times L_1$, let $h: A \to L_1$ be the function defined as:

$$h(a) = \sup\{x \mid (a,x) \in K\}$$

Therefore, we have:

$$
\inf\{\alpha(a,x) \mid (a,x) \in K\} \overset{(1)}{=} \inf\{\alpha(a, h(a)) \mid a \in A\}
$$
$$
\overset{(2)}{=} \sup\{\beta(b, h^\downarrow(b)) \mid b \in B\}
$$
$$
\overset{(3)}{=} \varphi(\langle h^\downarrow, h^{\downarrow\uparrow}\rangle)
$$
$$
\overset{(4)}{=} \inf\{\alpha(a, h^{\downarrow\uparrow}(a)) \mid a \in A\}
$$

where (1) is given by Propositions 4 and 3, (2) by Proposition 2 and (3), (4) by Lemma 3. Then the required concept is $\langle h^\downarrow, h^{\downarrow\uparrow}\rangle$. The uniqueness follows from Lemma 3. $\square$

3 Concept lattices for non-commutative fuzzy logics

In this section we will show how the concept lattice introduced in [6] can be embedded in the general setting of multi-adjoint concept lattices. This embedding, together with the representation theorem for multi-adjoint concept lattices, allows us to prove a more general fundamental theorem for those concept lattices.

To represent the Georgescu-Popescu concept lattices inside our framework we will consider two multi-adjoint frames $(L, L, P, \&)$ and $(L, L, P, \&^{op})$, where $\&^{op}$ is defined as $\&^{op}(x, y) = y \& x$, with contexts $\mathcal{C}_i = (A, B, R, \sigma_i)$, $i = \{1, 2\}$, respectively, where $\sigma_1(b) = \&$, $\sigma_2(b) = \&^{op}$ for all $b \in B$ (we will omit any further reference to σ because both contexts have only one adjoint triple); finally, the Galois connection associated to $\&$ is written as $(\uparrow, \downarrow)$ and to $\&_2$ is $(\uparrow_{op}, \downarrow^{op})$, and are defined as:

$$
\begin{aligned}
g^\uparrow(a) &= \inf\{R(a,b) \nearrow g(b) \mid b \in B\} \\
f^\downarrow(b) &= \inf\{R(a,b) \nwarrow f(a) \mid a \in A\} \\
g^{\uparrow op}(a) &= \inf\{R(a,b) \nwarrow g(b) \mid b \in B\} \\
f^{\downarrow op}(b) &= \inf\{R(a,b) \nearrow f(a) \mid a \in A\}
\end{aligned}
$$

where $g \in L^B$ and $f \in L^A$.

The four arrows associated to the two Galois connections naturally provide the means to reproduce adequately

Georgescu-Popescu concepts (in their notation $(\uparrow, \downarrow)$ and $(\uparrow_{op}, \downarrow^{op})$ would be written $(\uparrow, \Downarrow)$ and $(\Uparrow, \downarrow)$).

In the following definition, we introduce the term t-concept to distinguish them from the multi-adjoint based concepts.

Definition 9 *Let (A, B, R) be a context. The triple $(g, f, f') \in L^{B \times A \times A}$ is called a* t-concept *if and only if it satisfies:*

$$g^\uparrow = f_1 \; ; \; g^{\uparrow op} = f_2 \; ; \; f_1^\downarrow = g \; ; \; f_2^{\downarrow op} = g$$

If $\mathcal{M}_1$, $\mathcal{M}_2$ denote the complete lattices of concepts from the Galois connections $(\uparrow, \downarrow)$ and $(\uparrow_{op}, \downarrow^{op})$, and consider the set

$$\mathcal{L} = \{(g, f_1, f_2) \in L^{B \times A \times A} \mid (g, f_1, f_2) \text{ is a t-concept}\}$$

with the ordering $(g, f_1, f_2) \preceq (g', f_1', f_2')$ if and only if $g \leq g'$ (equivalently $f_1 \leq f_1'$ or $f_2 \leq f_2'$), then $\mathcal{L}$ is a complete lattice because it is isomorphic to the complete sublattice of $\mathcal{M}_1 \times \mathcal{M}_2$ containing the pairs $\langle g_1, f_1 \rangle \in \mathcal{M}_1$, $\langle g_2, f_2 \rangle \in \mathcal{M}_2$ satisfying that $g_1 = g_2$.

This new framework generalizes the given in [6] in that it allows to define the same concepts but avoids the requirement of $(L, \&, 1)$ to be a monoid, in the same way that multi-adjoint lattices generalize residuated lattices. The representation theorem for t-concepts is stated below as a generalization of that by Georgescu and Popescu.

Proposition 6 *A lattice $(V, \sqsubseteq)$ is isomorphic to a complete lattice of t-concepts $\mathcal{L}$ if and only if there exist two complete lattices $(V_1, \sqsubseteq_1)$ and $(V_2, \sqsubseteq_2)$ and four applications*

$$
\begin{aligned}
\alpha_1: A \times L \to V_1 &\quad ; \quad \alpha_2: A \times L \to V_2 \\
\beta_1: B \times L \to V_1 &\quad ; \quad \beta_2: B \times L \to V_2
\end{aligned}
$$

such that β_i is (V_i, R)-related with α_i, and there exists an isomorphism ν from V to the set of pairs

$$
\left(\inf_{(a,x) \in K_1} \alpha_1(a,x), \inf_{(a,x) \in K_2} \alpha_2(a,x) \right)
$$

where $K_1, K_2 \in A \times L$ and

$$
\inf_{(a,x) \in K_1} @_a^{x\downarrow} = \inf_{(a,x) \in K_2} @_a^{x\downarrow^{op}} \tag{1}
$$

Proof. Firstly, let $\psi: V \to \mathcal{L}$ be an isomorphism and $\mathcal{M}_1$, $\mathcal{M}_2$ the multi-adjoint concept lattices associated to the Galois connections $(\uparrow, \downarrow)$ and $(\uparrow_{op}, \downarrow^{op})$ respectively, then, by the representation theorem on multi-adjoint concept lattices and considering $V_1 = \mathcal{M}_1$, $V_2 = \mathcal{M}_1$, there exist $\alpha_1: A \times L \to V_1$, $\alpha_2: A \times L \to V_2$, $\beta_1: B \times L \to V_1$ and $\beta_2: B \times L \to V_2$ satisfying that β_i is (V_i, R)-related with

α_i; and two isomorphisms $\varphi_1\colon \mathcal{M}_1 \to V_1$, $\varphi_2\colon \mathcal{M}_2 \to V_2$ defined, by Lemma 3, as:

$$\varphi_1(\langle g, f\rangle) = \inf\{\alpha_1(a, f(a)) \mid a \in A\};$$
$$\varphi_2(\langle g, f\rangle) = \inf\{\alpha_2(a, f(a)) \mid a \in A\}$$

Now, if we define $\Pi_1\colon \mathcal{L} \to \mathcal{M}_1$ and $\Pi_2\colon \mathcal{L} \to \mathcal{M}_2$, as $\Pi_1\langle g, f_1, f_2\rangle = \langle g, f_1\rangle$ and $\Pi_2\langle g, f_1, f_2\rangle = \langle g, f_2\rangle$, respectively, we can consider the mapping ν defined for all $v \in V$ as

$$\nu(v) = (\varphi_1(\Pi_1(\psi(v))), \varphi_2(\Pi_2(\psi(v))))$$

whose image, for every $v \in V$, is: if $\psi(v) = \langle g, f_1, f_2\rangle$ and we consider the subsets $K_1 = \{(a, f_1(a)) \mid a \in A\}$ and $K_2 = \{(a, f_2(a)) \mid a \in A\}$ of $A \times L$, we have the equality (1), that is:

$$
\begin{aligned}
\inf_{(a,x)\in K_1} @_a^{x\downarrow} &= \inf_{a\in A}(@_a^{f_1(a)})^{\downarrow} \\
&= f_1^{\downarrow} \\
&\overset{(*)}{=} f_2^{\downarrow^{op}} \\
&= \inf_{a\in A}(@_a^{f_2(a)})^{\downarrow^{op}} \\
&= \inf_{(a,x)\in K_2} @_a^{x\downarrow^{op}}
\end{aligned}
$$

where $(*)$ is given because $\langle g, f_1, f_2\rangle$ is t-concept, moreover,

$$
\begin{aligned}
\inf_{(a,x)\in K_1} \alpha_1(a, x) &= \inf_{a\in A} \alpha_1(a, f_1(a)) \\
&= \varphi_1(\langle g, f_1\rangle) \\
&= \varphi_1(\Pi_1(\psi(v)))
\end{aligned}
$$

and similarly that $\inf_{(a,x)\in K_2} \alpha_2(a, x) = \varphi_2(\Pi_2(\psi(v)))$, so the image of ν is the required one. Furthermore, ν is homomorphism since all the mappings involved in its definition are homomorphisms. Now, we are going to prove that it is bijective.

Let $v_1, v_2 \in V$ be, if $\nu(v_1) = \nu(v_2)$ then, as φ_1 and φ_2 are isomorphisms, we obtain equivalently that:

$$(\Pi_1(\psi(v_1)), \Pi_2(\psi(v_1))) = ((\Pi_1(\psi(v_2)), \Pi_2(\psi(v_2)))$$

Now, if $\psi(v_1) = \langle g, f_1, f_2\rangle$ and $\psi(v_2) = \langle g', f_1', f_2'\rangle$ the equality above means that

$$g_1 = g_2;\ f_1 = f_2;\ f_1' = f_2'$$

Hence, as ψ is an isomorphism, $v_1 = v_2$ and ν is injective.

Let now $(\inf_{(a,x)\in K_1} \alpha_1(a, x), \inf_{(a,x)\in K_2} \alpha_2(a, x))$, with $K_1, K_2 \subseteq A \times L$ satisfying the equality (1), by Proposition 5 we have that there are (unique) concepts $\langle g_1, f_1\rangle$, $\langle g_2, f_2\rangle$ such that

$$
\begin{aligned}
(\inf_{(a,x)\in K_1} &\alpha_1(a, x), \inf_{(a,x)\in K_2} \alpha_2(a, x)) = \\
&= (\inf_{a\in A} \alpha_1(a, f_1(a)), \inf_{a\in A} \alpha_2(a, f_2(a))) \\
&= (\varphi_1(\langle g_1, f_1\rangle), \varphi_2(\langle g_2, f_2\rangle))
\end{aligned}
$$

where the last equality is given by definition above of φ_1 and φ_2, and f_1 satisfies that

$$
\begin{aligned}
f_1^{\downarrow}(b) &= \inf\{R(a, b) \searrow f_1(a) \mid a \in A\} \\
&\overset{(1)}{=} \inf\{R(a, b) \searrow \sup_{(a,x_i)\in K_1}\{x_i\} \mid a \in A\} \\
&\overset{(2)}{=} \inf\{\inf_{(a,x_i)\in K_1}\{R(a, b) \searrow x_i\} \mid a \in A\} \\
&= \inf_{(a,x)\in K_1}\{R(a, b) \searrow x\} \\
&\overset{(3)}{=} \inf_{(a,x)\in K_1}\{@_a^{x\downarrow}(b)\}
\end{aligned}
$$

where (1) is given by the definition of f_1 (Proposition 5), (2) from the adjoint property and (3) from Lemma 2. In a similar way, we can prove that $f_2^{\downarrow^{op}}(b) = \inf_{(a,x)\in K_2}\{@_a^{x\downarrow^{op}}(b)\}$ then, as (1) is satisfied, we obtain that $f_1^{\downarrow} = f_2^{\downarrow^{op}}$. Now if we take $v = \varphi(\langle g, f_1, f_2\rangle)$, where $g = f_1^{\downarrow}$, we have that

$$\nu(v) = (\inf_{(a,x)\in K_1} \alpha_1(a, x), \inf_{(a,x)\in K_2} \alpha_2(a, x))$$

Therefore ν is surjective. So we have proved that ν is an isomorphism.

Conversely, we have the maps α_i, β_i such that β_i is (V_i, R)-related with α_i, with $i \in \{1, 2\}$, and the isomorphism ν. Therefore, by the representation theorem on multi-adjoint concept lattices we have that there exist two isomorphisms $\varphi_1\colon \mathcal{M}_1 \to V_1$, $\varphi_2\colon \mathcal{M}_2 \to V_2$.

Now, if we consider the function ι from $\mathcal{L}$ to the complete sublattice of $\mathcal{M}_1 \times \mathcal{M}_2$ containing the pairs $\langle g_1, f_1\rangle \in \mathcal{M}_1$, $\langle g_2, f_2\rangle \in \mathcal{M}_2$ satisfying that $g_1 = g_2$, defined as $\iota(\langle g, f_1, f_2\rangle) = (\langle g, f_1\rangle, \langle g, f_2\rangle)$ we clearly have, from Proposition 5,

$$
\begin{aligned}
(\inf_{(a,x)\in K_1} &\alpha_1(a, x), \inf_{(a,x)\in K_2} \alpha_2(a, x)) = \\
&= (\varphi_1(\langle g, f_1\rangle), \varphi_2(\langle g, f_2\rangle))
\end{aligned}
$$

where $K_1, K_2 \subseteq A \times L$ satisfying (1).

Thus the function $\varphi\colon \mathcal{L} \to V$ defined as $\varphi = \nu^{-1} \circ (\varphi_1 \times \varphi_2) \circ \iota$ is an isomorphism. $\qquad\square$

The following corollary is the representation theorem of the framework presented in [6].

Corollary 2 *Let $I\colon B \times A \to P$ be a relation. A lattice $(V, \sqsubseteq)$ is isomorphic to $\mathcal{L}$ if and only if there exist two complete lattices $(V_1, \sqsubseteq_1)$ and $(V_2, \sqsubseteq_2)$ and five applications:*

$$
\begin{aligned}
\alpha_1\colon A \times L \to V_1 &\quad ; \quad \alpha_2\colon A \times L \to V_2 \\
\beta_1\colon B \times L \to V_1 &\quad ; \quad \beta_2\colon B \times L \to V_2 \\
\nu\colon V \to V_1 \times V_2
\end{aligned}
$$

such that:

1. $\alpha_1[A \times L]$ *is infimum dense in* V_1 *and* $\alpha_2[A \times L]$ *is infimum dense in* V_2.

2. $\beta_1[B \times L]$ *is supremum-dense in* V_1 *and* $\beta_2[B \times L]$ *is supremum-dense in* V_2.

3. *For each* $a \in A$, $b \in B$, $x, y \in L$:

$$\beta_1(b, y) \sqsubseteq_1 \alpha_1(a, x) \quad \textit{iff} \quad x \,\&\, y \preceq I(b, a)$$
$$\beta_2(b, y) \sqsubseteq_2 \alpha_2(a, x) \quad \textit{iff} \quad x \,\&^{op}\, y \preceq I(b, a)$$

4. ν *is a monomorphism join-preserving from* V *onto* $V_1 \times V_2$ *such that, for any* $v \in V$, *there exist* $K_1, K_2 \subseteq A \times L$ *satisfying that* $\inf_{(a,x) \in K_1} @_a^{x\downarrow} = \inf_{(a,x) \in K_2} @_a^{x\downarrow^{op}}$ *and such that* $\nu(v)$ *is equal to the pair:*

$$\left(\inf_{(a,x) \in K_1} (\sup_{b \in B} \beta_1(b, I(b, a) \nwarrow x)), \right.$$
$$\left. \inf_{(a,x) \in K_2} (\sup_{b \in B} \beta_2(b, I(b, a) \nwarrow x))) \right.$$

Proof. The items (1), (2) and (3) are equivalent to the properties that satisfies α_i, β_i $i \in \{1, 2\}$ of Proposition 6, considering the residuated case, the relation $R: A \times B \to P$ defined as $R(a, b) = I(b, a)$ and the definition of (V_i, R)-related.

For item (4) we will prove the following equalities:

$$\sup_{b \in B} \beta_1(b, R(a, b) \nwarrow x) = \alpha_1(a, x)$$
$$\sup_{b \in B} \beta_2(b, R(a, b) \nearrow x) = \alpha_2(a, x)$$

We will prove the first statement (the proof for the second is analogous). We have that $R(a, b) \nwarrow x = @_a^{x\downarrow}(b)$, by Lemma 2, therefore using Proposition 2 we obtain that

$$@_a^{x\downarrow}(b) = \sup\{y \in L_2 \mid \beta(b, y) \sqsubseteq v_{@_a^x}\}$$

where $v_{@_a^x} = \inf\{\alpha_1(a', @_a^x(a')) \mid a' \in A\}$.

Hence, by Proposition 2, we have that

$$\sup_{b \in B} \beta_1(b, @_a^{x\downarrow}(b)) = v_{@_a^x}$$
$$= \inf\{\alpha_1(a', @_a^x(a')) \mid a' \in A\}$$
$$= \alpha_1(a, x)$$

where the last equality is because α_1 is decreasing in the second argument and Proposition 4. $\square$

4 Conclusion and future work

We have given an extension of the multi-adjoint concept lattices which has allowed us to embed the concept lattice given by Georgescu and Popescu. This extension is based in a complete sublattice of the product of two multi-adjoint concept lattices which only have one adjoint triple.

We have proved the representation theorem for this new framework which embeds the one given in [6].

For future work we could consider the product of several multi-adjoint concept lattices (non necessary with only one adjoint triple) and give their representation theorem.

Another point to take into account could be the introduction of L-equalities relations as Bělohlávek do with the fuzzy concept lattices [2].

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