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Residuated operations in hyperstructures: residuated multilattices

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Abstract

We initiate the exploration of the residuated operations in the framework of hyperstructures. We focus on the case of a multilattice as underlying algebraic structure, introduce the notion of residuated multilattice and study some of its properties, among which we have shown that the idempotency of the monoidal operation characterises the subclass of Heyting algebras.

Key words: hyperstructures, pocrim, multilattices, residuation.

1 Introduction and preliminary definitions

Residuation has a prominent role in the algebraic study of logical systems, which usually are partially ordered sets together with some operations reflecting the properties of the connectives. This work is related to the use of residuated implication in the framework of hyperstructures and fuzzy logic reasoning.

Although the most used structure in this context is that of residuated lattice, there are reasons which suggest to weaken some of its properties, leading to a more general class of algebraic structures for computation. A commonly considered algebraic structure is that of partially ordered commutative residuated integral monoid [2].

Definition 1 *A tuple* $\langle A, \to, *, \top, \leq \rangle$ *is said to be a* partially ordered commutative residuated integral monoid*, briefly a* ***pocrim****, if, for every* $a, b, c \in A$*, the following properties hold:*

- $\langle A, *, \top \rangle$ *is a commutative monoid with neutral element* $\top$
- $\langle A, \leq \rangle$ *is a partially ordered set which is compatible with* $*$ *(i.e.,* $a \leq b$ *implies* $a * c \leq b * c$*) and* $\top$ *is the maximum of* $\langle A, \leq \rangle$

- $\langle A, \leq\rangle$ *has the* residuum property*, that is* $a * c \leq b$ *if and only if* $c \leq a \to b$*.*

If some extra properties hold, we obtain other well-known structures, such as those given below:

Definition 2

- *A pocrim* $\langle A, \to, *, \top, \leq\rangle$ *is said to be a* ***residuated lattice*** *if, in addition,* $\langle A, \leq\rangle$ *is a lattice.*
- *A residuated lattice in which* $*$ *coincides with the meet operation is said to be a* ***Heyting algebra****.*

It is well-known that residuated lattices are considered to be the algebraic structures of substructural logics [8], which are logics without some of the structural rules of logic: weakening, contraction, or associativity.

We focus here on some extensions of the previously defined notions, by considering a partially-ordered set together with two non-deterministic operations which generalize the supremum and the infimum by weakening the restrictions imposed on a (complete) lattice, namely, the "existence of least upper bounds and greatest lower bounds" is relaxed to the "existence of *minimal* upper bounds and *maximal* lower bounds". Specifically, a *multisupremum of* a and b is defined as a minimal element of the set of upper bounds of a and b, we write $a \sqcup b$ to refer to *the set of all the multi-suprema of* a *and* b; the notion of multiinfimum $a \sqcap b$ is introduced similarly. Now, we can proceed with the formal definition of multilattice and related structures.

Definition 3

- *A poset* $(M, \leq)$ *is said to be a* ***multilattice*** *if for all* $a, b, x \in M$ *with* $a \leq x$ *and* $b \leq x$*, there exists*[1] $z \in a \sqcup b$*, such that* $z \leq x$*; and, similarly, for all* $a, b, x \in M$ *with* $a \geq x$ *and* $b \geq x$*, there exists* $z \in a \sqcap b$*, such that* $z \geq x$*.*
- *A multilattice is said to be* ***full*** *if* $a \sqcup b \neq \varnothing$ *and* $a \sqcap b \neq \varnothing$ *for all* $a, b \in M$*.*

The notion of multilattice was introduced originally by Benado [1], and further studied by Hansen [4], who proposed an algebraic equivalent definition of multilattice. More recently, another algebraic formalisation of the notion of multilattice was introduced in [5, 6] as a theoretical tool to deal with some problems in the theory of mechanised deduction in temporal logics. Multilattices arise as well in other research areas, such as fuzzy extensions of logic programming [7]: for instance, one of the hypotheses of the main termination result for sorted multi-adjoint logic programs [3] can be weakened only when the underlying set of truth-values is a multilattice (the question of providing a counter-example on a lattice remains open).

[1]Note that the definition is consistent with the existence of two incomparable elements *without* any multisupremum. In other words, $a \sqcup b$, and also $a \sqcap b$, can be empty.

Definition 4 *A* ***residuated multilattice*** *is a pocrim whose underlying poset is a multilattice. If, in addition, there exists a bottom element, we say that the residuated multilattice is* ***bounded****.*

It is convenient to remark that any finite poset is actually a multilattice, hence the only proper examples of pocrims not multilattices have to be infinite. The following example, taken from [9], shows a proper residuated multilattice, in that its carrier is not a lattice.

Example 1 *Let $\mathbb{Z}$, $\mathbb{Z}^-$ and $\mathbb{Z}^+$ denote, respectively, the sets of all integers, of all non-positive integers, and of all non-negative integers. Given $\bot, \top \notin \mathbb{Z}$, a pocrim A with carrier*

$$A = \Big(\{\bot\} \times \mathbb{Z}^+\Big) \cup \Big(\mathbb{Z}^+ \times \mathbb{Z}\Big) \cup \Big(\{\top\} \times \mathbb{Z}^-\Big)$$

Let $\leq$ be the partial ordering on A depicted in Figure 1 and note that

$$\langle \alpha, i\rangle \leq \langle \beta, j\rangle \quad \textit{iff} \quad i + |\alpha - \beta| \leq j$$

The operation $*$ on A is defined as follows:

$$\begin{aligned}
&x * y = y * x \\
&\langle \top, i\rangle * \langle \top, j\rangle = \langle \top, i + j\rangle && (i, j \leq 0) \\
&\langle \top, i\rangle * \langle \alpha, j\rangle = \langle \alpha, i + j\rangle && (i \leq 0) \\
&\langle \top, i\rangle * \langle \bot, j\rangle = \langle \bot, \max\{0, i + j\}\rangle && (i \leq 0 \leq j) \\
&\langle \alpha, i\rangle * \langle \beta, j\rangle = \langle \bot, \max\{0, i + j + |\alpha - \beta|\}\rangle \\
&\langle \alpha, i\rangle * \langle \bot, j\rangle = \langle \bot, k\rangle * \langle \bot, j\rangle = \langle \bot, 0\rangle && (0 \leq j, k)
\end{aligned}$$

This makes $(A; *, \langle \top, 0\rangle)$ to be residuated multilattice when considering the following residuate implication.

$$\begin{aligned}
&x \leq y \quad \textit{iff} \quad x \to y = \langle \top, 0\rangle \\
&\langle \top, i\rangle \to \langle \top, j\rangle = \langle \top, \min\{0, j - i\}\rangle && (i, j \leq 0) \\
&\langle \top, i\rangle \to \langle \alpha, j\rangle = \langle \alpha, j - i\rangle && (i \leq 0) \\
&\langle \top, i\rangle \to \langle \bot, j\rangle = \langle \bot, j - i\rangle && (i \leq 0 \leq j) \\
&\langle \alpha, i\rangle \to \langle \beta, j\rangle = \langle \top, \min\{0, j - i - |\alpha - \beta|\}\rangle \\
&\langle \alpha, i\rangle \to \langle \bot, j\rangle = \langle \alpha, j - i\rangle && (0 \leq j) \\
&\langle \bot, i\rangle \to \langle \bot, j\rangle = \langle \top, \min\{0, j - i\}\rangle && (0 \leq i, j)
\end{aligned}$$

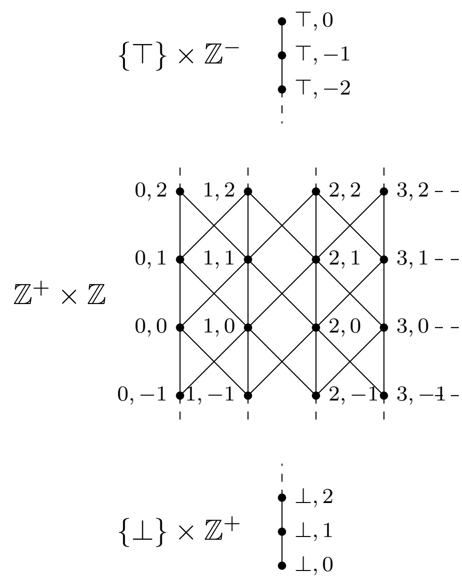

Figure 1: Hasse Diagram of $\langle A; \leq \rangle$

2 Algebraic properties of residuated multilattices

We study here some properties of the structures defined above.

Lemma 1 *Every residuated multilattice is full.*

Proof: For all $a, b \in M$ we have that $a, b \leq \top$ and, therefore, $a \sqcup b \neq \varnothing$. Furthermore, $a * b \leq a$, and $a * b \leq b$, hence $a \sqcap b \neq \varnothing$. $\square$

Lemma 2 *Let M be a residuated multilattice, then the following items hold:*

1. $a * b \sqcup a * c = minimals\{a * (b \sqcup c)\}$ *for all $a, b, c \in M$.*
2. $a * (b \sqcap c) \subseteq (a * b \sqcap a * c){\downarrow}$ *for all $a, b, c \in M$.*
3. *There exists $c \in a \sqcap b$ such that $a * (a \to b) \leq c$, for all $a, b \in M$.*
4. *There exists $c \in a \sqcap b$ such that $a * b \leq c$, for all $a, b \in M$.*

Proof: For item 1, we firstly prove that $a*b \sqcup a*c \subseteq a*(b \sqcup c)$. Let $x \in a*b \sqcup a*c$. Since $a * b, a * c \leq x$, then $b, c \leq a \to x$ and, hence, there exists $y \in b \sqcup c$ such that $y \leq a \to x$ and, thus, $a * y \leq x$. Moreover, by monotonicity of $*$, we have that $a * b \leq a * y$ and $a * c \leq a * y$ and, by definition of $\sqcup$, $x = a * y \in a * (b \sqcup c)$.

Finally, since any element in $a * (b \sqcup c)$ is an upper bound of $a * b$ and $a * c$, the equality $a * b \sqcup a * c = minimals\{a * (b \sqcup c)\}$ holds.

Items 2, 3 and 4 are immediate consequence of basic properties of pocrims and the definition of multilattice.

$\square$

Example 2 *The previous example illustrates the fact that we cannot get rid of the computation of the minimals in the first item of Lemma 2, but* $a*b\sqcup a*c\neq a*(b\sqcup c)$ *because, for instance,*

$$\langle 0,0\rangle \sqcup \langle 1,0\rangle = \{\langle 0,1\rangle, \langle 1,1\rangle\}$$

$$\langle 2,0\rangle * (\langle 0,0\rangle \sqcup \langle 1,0\rangle) = \{\langle 2,0\rangle * \langle 0,1\rangle, \langle 2,0\rangle * \langle 1,1\rangle\} = \{\langle \bot,3\rangle, \langle \bot,2\rangle\}$$

$$\langle 2,0\rangle * \langle 0,0\rangle \sqcup \langle 2,0\rangle * \langle 1,0\rangle = \langle \bot,2\rangle \sqcup \langle \bot,1\rangle = \langle \bot,2\rangle$$

Proposition 1 *Let* M *be a residuated multilattice such that* $a*b\in a\sqcap b$ *for all* $a,b\in M$*, then* M *is a Heyting algebra.*

Proof: Given $x\in a\sqcap b$, since $x\leq a$, then $x=a\sqcap x=a*x$ and the same for b. Thus $a*b*x=a*x=x$ which implies that $x\leq a*b$. As $x, a*b\in a\sqcap b$, then $x=a*b$. We have obtained that, for all $a,b\in M$, $a*b=a\sqcap b$, in particular, there exists the infimum for all a and b. Being M full (see Lemma 1), there also exists the supremum of a and b, by [5,6]. □

Lemma 3 *Let* M *be a residuated multilattice with idempotent product, then, for all* $a,b\in M$*,*

1. *If* $x\in a\sqcup b$*, then* $a*x=a$.
2. $a\leq b$ *if and only if* $a*b=a$.
3. $a*b\in a\sqcap b$

Proof:

1. Observe that $a=a\sqcup a*b=a*a\sqcup a*b=minimals\{a*(a\sqcup b)\}$. If $x\in a\sqcup b$, then $a\leq a*x$. Since, by monotonicity of $*$, $a*x\leq a$, we have $a*x=a$.
2. By monotonicity of the product, if $a\leq b$, then $a*b\leq a*\top=a$ and $a=a*a\leq a*b$ and, hence, $a*b=a$. On the other hand, if $a*b=a$, then $\top=a\to a=a\to a*b\leq a\to b$ which implies $a\leq b$.
3. By item 4 of Lemma 2, there exists $c\in a\sqcap b$ such that $a*b\leq c$, and so $a*b*c=a*b$. On the other hand, from item 2, since $c\leq b$ and $c\leq a$, we have that $a*b*c=a*c=c$. Therefore, $a*b=c\in a\sqcap b$.

□

Theorem 1 *Any idempotent residuated multilattice is a Heyting algebra.*

Proof: It is a direct consequence of the previous lemma and proposition.

□

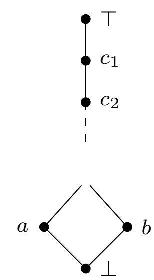

Figure 2: Hasse Diagram of $\langle A; \leq \rangle$

Sometimes, in connection to an algebraic structure with a binary operation $*$, the following relation so-called *natural preordering* has been considered:

$$a \sqsubseteq b \quad \text{if and only if} \quad a * b = a$$

In the framework of residuated multilattices, the operation $*$ is assumed to be both associative and commutative, and this implies anti-symmetry and transitivity of $\sqsubseteq$. Moreover, this relation is included in $\leq$. That is, $a \sqsubseteq b$ implies $a \leq b$ (it is due to item 4 in Lemma 2). Note, finally, that $\sqsubseteq$ is reflexive if and only if the product is idempotent. Specifically, $\sqsubseteq$ is a partial ordering relation (in a residuated multilattice) exactly in the subclass of Heyting algebras.

Example 3 *Let us consider the meet-semilattice $\langle A; \leq \rangle$ depicted in Figure 2, the product being the meet operator and the residuated implication $\to$ defined by*

$$\begin{array}{ll} x \to y = \top & \textit{iff } x \leq y \\ c_i \to x = x & \textit{for all } x \leq c_i \\ a \to \bot = a \to b = b & \\ b \to \bot = b \to a = a & \end{array}$$

then $\langle A, \to, *, \top, \leq \rangle$ is an idempotent pocrim, but it is not a lattice (elements a and b do not have a supremum) and, hence, is not a Heyting algebra.

Note that this example shows that, in general, the presence of idempotency in a pocrim is not a sufficient condition to guarantee the structure of Heyting algebra.

3 Conclusions and future work

The algebraic structure of residuated multilattice has been defined between those of partially ordered commutative residuated integral monoids (pocrims) and residuated lattices. All finite pocrims are trivial examples of residuated multilattices, an instance of an infinite pocrim not being a residuated multilattice has been shown.

Preliminary algebraic properties of this new structure have been studied and, specifically, we have shown that the idempotency of the monoidal operation characterises the subclass of Heyting algebras.

Future work will focus on the study of the ideals and filters, which turn out to be specially important in relation to the algebraic semantics of logical systems.

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